# CARDY MATCHES BETHE ON THE SURFACE: A TALE OF A BRANE AND A BLACK HOLE

Davide Morgante [1,2] in collaboration with A. Amariti [1], P. Glorioso [1,2], A. Zanetti [1,2]

1. INFN, Sezione di Milano, Via Celoria 16, I-20133 Milano, Italy
2. Dipartimento di Fisica, Università degli studi di Milano, Via Celoria 16, I-20133, Milano, Italy

Based on [2403.17190]

## ABSTRACT

We consider the insertion of a Gukov-Witten surface defect in SU(N) $\mathcal{N}$=4 SYM corresponding to a probe D3-brane in the holographic dual setup. The defect gives rise to a 4d-2d coupled system encoding the entropy of the dual perturbed black hole, which can be extracted from the corresponding Superconformal Index. Elaborating on previous studies, we refine the results using both a saddle-point and a Bethe-Ansatz approach. The consistency of our computation is corroborated by the complete agreement between the two results in the appropriate regime of fugacities. Eventually, the sub-leading structure, emerging from our analysis, provides a suggestive EFT interpretation for the addition of the defect to the 4d system, mirroring the behavior of the probe D3-brane in the gravity dual.

## ORDER PARAMETERS

Perturbing a Black-Hole with a Polyakov loop should provide an order parameter to detect ***confinement/deconfinement transition***, expected to be dual mechanism of Hawking-Page transition in AdS. Another ***order parameter*** given by surface defects [1], which crucially have a ***dual description as probe D3-brane*** in gravity wrapping one compact direction in $AdS_5$ and one in $S^5$. Here we explore this avenue by computing the contribution of such defect to the superconformal index and, in turn, to the entropy function of the dual Black-Hole.

## GUKOV-WITTEN DEFECTS

***Codimension-two defect*** [2] prescribed by singular behavior on surface of vector field and scalar component of chiral in $\mathcal{N}$ = 4 SYM with gauge group G.

$$A = \alpha\, \mathrm{d}\theta + \cdots, \quad \phi = \beta \frac{\mathrm{d}r}{r} - \gamma\, \mathrm{d}\theta + \cdots, \quad (\alpha,\beta,\gamma,\eta) \in (\mathbb{T}\times \mathfrak{t}\times \mathfrak{t}\times {}^{L}\mathbb{T})/\mathrm{Weyl}(\mathbb{L})$$

Conformal invariance require coefficients to be independent of distance from singular surface, BPS condition requires coefficients to be mutually commuting (Nahm's equations). Singularity preserved by ***Levi subgroup***. For SU(N), classified by partitions of N

$$\lambda = [\lambda_1,\ldots,\lambda_s] \implies \mathbb{L} = \mathrm{S}\left(\bigotimes_{i=1}^{n} \mathrm{U}(k_i)\right), \quad N = \sum_{i=1}^{n} k_i$$

Ranks are coefficients of dual partition

## THE SUPERCONFORMAL INDEX

Defined, by choosing one supercharge, as a ***refined version of the Witten index*** of the theory in radial quantization. It counts BPS states in Hilbert space of the theory on $S^3$. Allows access to ***Black-Hole (BH) microstates*** [3]. In $\mathcal{N}$ = 4 SYM defined as

$$\mathcal{I}_{4\mathrm{d}} = \mathrm{Tr}_{\mathrm{gauge}}(-1)^F \mathrm{e}^{-\beta\{\mathcal{Q},\overline{\mathcal{Q}}\}} p^{J_1} q^{J_2} (pq)^{R/2} v_1^{Q_1} v_2^{Q_2}$$

Trace over gauge singlets; charges $Q_1, Q_2$ parametrize Cartan of $\mathfrak{su}(3)\subset\mathfrak{su}(4)_R$, choosing opportune definition of charges

$$p = \mathrm{e}^{2\pi\mathrm{i}\tau},\ q = \mathrm{e}^{2\pi\mathrm{i}\sigma},\ v = \mathrm{e}^{2\pi\mathrm{i}\xi},\ \Delta_a = \rho_a(\xi) + \frac{\tau+\sigma}{2}R_a \qquad \sum_{a=1}^{3}\Delta_a = \sigma+\tau \mod 1$$

By rewriting trace as integral on gauge holonomies, the SCI is given by and elliptic hypergoniometric integral

$$\mathcal{I}(\Delta,\tau,\sigma) = \frac{(p;p)_\infty^{N-1}(q;q)_\infty^{N-1}}{N!}\prod_{a=1}^{3}\widetilde{\Gamma}(\Delta_a)^{N-1}\int\prod_{i=1}^{N-1}\mathrm{d}u_i \frac{\prod_{a=1}^{3}\prod_{i\neq j}^{N}\widetilde{\Gamma}(u_{ij}+\Delta_a)}{\prod_{i\neq j}^{N}\widetilde{\Gamma}(u_{ij})}$$

Where the various terms are given by elliptic gamma functions and q-Pochhammers

$$\Gamma(z;p,q) = \prod_{m=0}^{\infty}\prod_{n=0}^{\infty}\frac{1-p^{m+1}q^{n+1}/z}{1-p^m q^n z}, \quad \widetilde{\Gamma}(u) = \Gamma(\mathrm{e}^{2\pi\mathrm{i}u};\mathrm{e}^{2\pi\mathrm{i}\tau},\mathrm{e}^{2\pi\mathrm{i}\sigma}), \quad (z;q)_\infty = \prod_{k=0}^{\infty}(1-zq^k)$$

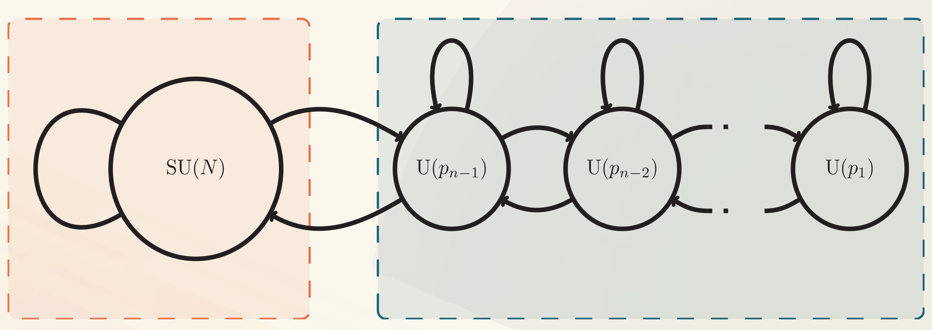
Originally defined for purely imaginary modular parameters, can be extended to whole upper-half complex plane and complex $\Delta$. Crucial for accessing BH entropy by ***relating SCI to partition function***

$$Z = \mathrm{e}^{-\beta E_0}\mathcal{I}(\sigma,\tau,\Delta_a)$$

Where phase is given by Casimir energy

## COUPLING GW DEFECT TO SYM

By choosing Levi subgroup, theory of defect described by ***2d $\mathcal{N}$ = (4,4) GLSM*** on the surface with target space $T^*(G/\mathbb{L}) = G_{\mathbb{C}}/\mathbb{L}_{\mathbb{C}}$, gauge group fixed.




$$G_{2\mathrm{d}} = \bigotimes_{i=1}^{n-1}\mathrm{U}(p_i), \quad p_i = \sum_{j=1}^{i} k_j$$

***Gauge group of 4d theory acts as flavor in 2d.*** Field content given by bi-fundamental hypers in rep ($p_i$ , $p_{i+1}$) and N fundamental hypers for U($p_{n-1}$).

## INDEX OF 2D THEORY

The relevant quantity we need is the SCI of the 4d-2d coupled system. This can be casted in the form

$$\mathcal{I}(p,q,v_i) = \int_{\mathrm{SU}(N)}\mathrm{d}u\,\mathcal{I}_{4d}(p,q,v_i;u)\mathcal{I}_{2d}(p,q,v_i;u)$$

Contribution from 2d index of theory living on the defect, wrapped along a temporal $T^2$ in $S^1\times S^3$.

For "***maximal***" GW defect $\lambda = [N-1,1]$, theory is $\mathcal{N}$ = (4,4) U(1) with one hypermultiplet, i.e. two $\mathcal{N}$ = (2,2) chirals + adjoint chiral and vector from $\mathcal{N}$ = (4,4) vector. Defect dual to ***probe D3 brane*** in gravity. Computed by ***Jeffrey-Kirwan prescription*** [4]

$$\mathcal{I}_{2d} = \frac{1}{|W|}\sum_{u_*\in\mathfrak{M}^*_{\mathrm{sing}}}\underset{u=u_*}{\text{JK-Res}}\left(\mathsf{Q}(u_*),\eta\right)Z_{1\text{-loop}} = \sum_{i=1}^{N}\prod_{j\neq i}^{N}\frac{\theta_0(u_{ij}+\zeta-2\chi;\tau)\theta_0(u_{ij}-\zeta;\tau)}{\theta_0(u_{ij};\tau)\theta_0(u_{ij}-2\chi;\tau)}$$

## CARDY-LIKE LIMIT

This limit allows to extract, at large-N, the dominant saddle configuration associated to the Black-Hole solution in the dual gravitational theory. Appropriate scaling behavior of the fugacities ***obstructs cancellations***

$$\tau = r\omega_1, \quad \sigma = r\omega_2, \quad r\in\mathbb{R};\omega_1,\omega_2\in\mathbb{H}$$
$$\Delta_a = \tilde{\Delta}_a + r\bar{\Delta}_a, \quad \tilde{\Delta}_a\in\mathbb{R}/\mathbb{Z};\ \bar{\Delta}_a\in\mathbb{R}$$

where $r$ is radius of $S^1$ and $\omega_1,\omega_2$ are related to squashing of $S^3$

Asymptotic expansion at small $S^1$ radius (high temperature limit) with integral evaluated by saddle-point approach at fixed-N. Dominant saddle for large-N in SU(N) SYM is

$$\mathcal{I}_{4d} = N\exp\left(-\frac{\pi\mathrm{i}(N^2-1)}{\sigma\tau}\prod_{a=1}^{3}\left(\{\Delta_a\}-\frac{n_0+1}{2}\right)+\mathcal{O}(r)\right)$$

$$\{\Delta_a\} = \tilde{\Delta}_a - \lfloor\tilde{\Delta}_a\rfloor + r\bar{\Delta}_a \qquad \sum_{a=1}^{3}\{\Delta_a\} = \tau+\sigma+\frac{3+n_0}{2}, \quad n_0=\pm1$$

Adding the "maximal" GW defect requires ***dictionary between fugacities*** according to superalgebra embedding

$$\mathfrak{u}(1)_A \ltimes (\mathfrak{psu}(1,1|2)\times\mathfrak{psu}(1,1|2))\ltimes\mathfrak{u}(1)_C \subset \mathfrak{psu}(2,2|4)$$

The contribution of the probe D3 is subleading in fugacities and the ***holonomy saddle is unchanged***. Final expression for coupled system is

$$\mathcal{I} = N\exp\left(-\frac{\pi\mathrm{i}(N^2-1)}{\sigma\tau}\prod_{a=1}^{3}(\{\Delta_a\}-n)+\frac{2\pi\mathrm{i}(N-1)}{\sigma}\prod_{a=2}^{3}(\{\Delta_a\}-n)\right) \qquad n = \frac{1+n_0}{2}$$

The defect gives a subleading contribution in N to the entropy function [5], as expected from the choice of defect which implements a D3-brane in ***probe regime***.

## BETHE ANSATZ APPROACH

Algebraic way to derive BH entropy at large-N [6]. Limit of ***collinear angular momenta*** considered $\tau=\sigma\equiv\omega$. Cast index in following form

$$\mathcal{I} = \kappa_N\oint_{\partial\mathcal{A}}\mathrm{d}\mathbf{u}\,\frac{\mathcal{Z}_{4d}(\mathbf{u};\omega,\Delta)}{\prod_{i=1}^{N}(1-Q_i(\mathbf{u};\omega,\Delta))} \qquad \mathcal{A} = \left\{\mathbf{u}\in\mathbb{C}^{N-1}|\mathrm{Re}\,u_i\in[0,1],\ -\mathrm{Im}\,\omega<\mathrm{Im}\,u_i<0,\forall\, i=1,\ldots,N-1\right\}$$

Contributions coming only from poles in denominator. Problem of finding solutions to set of transcendental equations: ***Bethe Ansatz Equations*** (BAEs). For $\mathcal{N}$ = 4 SYM

$$Q_i(u;\omega,\Delta) = \mathrm{e}^{2\pi\mathrm{i}(\lambda+3\sum_j u_{ij})}\prod_{j=1}^{N}\frac{\theta_0(u_{ji}+\Delta_1;\omega)\,\theta_0(u_{ji}+\Delta_2;\omega)\,\theta_0(u_{ji}-\Delta_1-\Delta_2;\omega)}{\theta_0(u_{ij}+\Delta_1;\omega)\,\theta_0(u_{ij}+\Delta_2;\omega)\,\theta_0(u_{ij}-\Delta_1-\Delta_2;\omega)} = 1$$

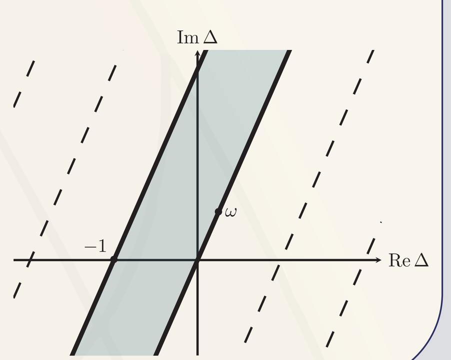
By using modular properties of BA operators, integral localizes on equivalence classes of solutions. ***Basic Solution*** reproduces leading contribution of BH saddle at large N. When adding defect, make sure that no additional poles contribute from $\mathcal{Z}_{2d}$ . Indeed, no additional poles arise in the relevant region.
Computing on the basic solution

$$\log\mathcal{I}|_{\mathrm{basic}} = -\frac{\pi\mathrm{i}}{\omega^2}N^2\prod_{a=1}^{3}(\{\Delta_a\}_\omega-n)+\frac{2\pi\mathrm{i}}{\omega}N\prod_{a=2}^{3}(\{\Delta_a\}_\omega-n)+\log N+\mathcal{O}(N^0)$$

Where $\{\Delta\}_\omega = \Delta+m$ such that $m\in\mathbb{Z}$ and $0>\mathrm{Im}\,\frac{\Delta+m}{\omega}>\mathrm{Im}\,\frac{1}{\omega}$

Fugacities $\Delta$ constrained as in Cardy-like.

This reproduces the result of the Cardy-like limit in the region of small modular parameter.




## EFFECTIVE FIELD THEORY INTERPRETATION

In Cardy-like limit, an EFT interpretation arises [7]. For pure SYM, the low energy 3d theory from Kaluza-Klein reduction is a gapped Chern-Simons theory. What happens when the defect is inserted? Reduction of GW defect on $S^1$ ***introduces a line defect*** in 3d EFT

$$\frac{\mathrm{e}^{\pi\mathrm{i}(N-1)}}{N!}\int\mathrm{d}\Lambda\int\prod_{i=1}^{N}\frac{\mathrm{d}\lambda_i}{\sqrt{-\omega_1\omega_2}}\frac{\mathrm{e}^{-\frac{\pi\mathrm{i}n_0N}{\omega_1\omega_2}\sum_{i=1}^{N}\lambda_i^2+2\pi\mathrm{i}\Lambda\sum_{j=1}^{N}\lambda_j}}{\prod_{i<j}\Gamma_h(\lambda_{ij})\Gamma_h(-\lambda_{ij})}\left(\sum_{i=1}^{N}\mathrm{e}^{-2\pi\mathrm{i}N\frac{\lambda_i}{\omega_2}}\right)$$

N-wounded anti-fundamental Wilson loop on 1-cycle on $T^2$ in $S^3$. Consistent with expectation: only pN-wounded Wilson loops in 3d CS are non-vanishing. Moreover, term in 2d contribution $-\log\theta_0(u_{ij};\sigma)$ regarded as counter-term ***suppressing effect of Wilson loop***, consistent with holographic expectation of probe D3 brane.

## REFERENCES

[1] Y. Chen, et. al, *Probing supersymmetric black holes with surface defects*, [2306.05463]
[2] S. Gukov and E. Witten, *Gauge Theory, Ramification, And The Geometric Langlands Program*, hep-th/0612073
[3] S. Choi, J. Kim, S. Kim and J. Nahmgoong, *Large AdS black holes from QFT*, [1810.12067]
[4] F. Benini, et. al, *Elliptic Genera of 2d N = 2 Gauge Theories*, Commun. Math. Phys. 333 (2015) 1241.
[5] A. Cabo-Bizet, et. al, *Thermodynamics of black holes with probe D-branes*, [2312.12533]
[6] F. Benini and E. Milan, *Black Holes in 4D N =4 Super-Yang-Mills Field Theory*, Phys. Rev. X 10 (2020) 021037.
[7] D. Cassani and Z. Komargodski, *EFT and the SUSY Index on the 2nd Sheet*, SciPost Phys. 11 (2021) 004.