Contents

Chapter 1

Electroweak Interactions in the Standard Model

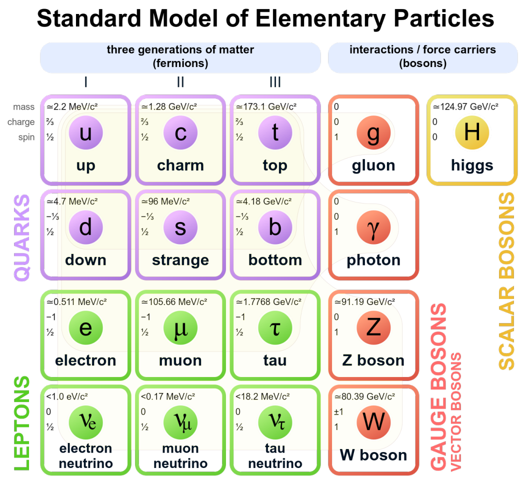


Figure 1.1. The Standard Model of particle physics.

Whenever somebody asks questions like how many particles there are? How do these particles interact with each other? What are the properties of certain particles? Are they fundamental or composite? What makes up a composite particle? How do they decay if they decay? And so on, the answer, as far as we know, can be found in a theory which is called by physicists the Standard Model (SM) of particle physics [26, 53, 60].

Loosely one can think of the SM as a Periodic Table for subatomic particles, but the reality is much more complicated. In some way this statement is not completely incorrect and, just like a Periodic Table, the Standard Model can be summarized in a table like the one in figure (1.1). However, this famous table is deceiving: it does not show all of the subatomic particles and their antiparticle! But for the purpose of the physics, particles and antiparticles are much of the same in their properties and their interactions, so it would be redundant to show them both.

Although this picture is very pretty, some physicists have a more pragmatic way of presenting what they call the SM, which is as follows: the SM is a non-Abelian Yang-Mills theory whose symmetry group is given by

$$SU(3) \times SU(2) \times U(1). \tag{1.1}$$

Equation eq. (1.1) is what we call the *gauge group of the Standard Model*. Mathematically speaking even this sentence is not very precise since what we know exactly is the Lie algebra of the SM. From the Lie algebra and the properties of the particles, we can infer the precise Lie group of the SM [8], which is somewhat different from the one in equation eq. (1.1) and is given by the following quotient

$$\mathfrak{su}(3) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1) \rightsquigarrow \frac{SU(3) \times SU(2) \times U(1)}{\mathbb{Z}_6}. \tag{1.2}$$

Yet, this is beyond the scope of this notes and we will stick to the mathematically imprecise, physicist way, of equation eq. (1.1) to avoid any confusion.

With this introduction one would be lead to believe that the SM has the possibility to answer all of our questions but, unfortunately (or fortunately, depends on who's the question being asked), this is not the case. Many questions, both theoretical and experimental, cannot be answered by a pure SM analysis, and here is when effective theories come into play. We will discuss effective theories and how they enter the game in the next chapters.

For now let us focus on what actually is the Standard Model. In this chapter we are going to give the basics upon which the Standard Model is built and how particles are described within this theory. In the first part we will focus on the $\mathfrak{su}(2) \oplus \mathfrak{u}(1)$ bit of equation eq. (1.2), the so-called *electroweak* (EW) sector or *Glashow-Weinberg-Salam* (GWS) theory [25, 54, 61], carrying the name of the physicists that theorized it and consequently won the Nobel prize for it in 1979. Then we will show how, within this theory, an incredible thing happens: the particles that make up hadrons, what we call quarks, can mix with one another in a way initially theorized by Nicola *Cabibbo* [13] and then expanded by *Kobayashi* and *Maskawa* [43] (the latter two winning the Nobel prize for it, while the former sadly did not receive it).

1.1 Basics of the Standard Model

For a more in-depth analysis of the contents of these chapters, there are many excellent books. Schwart's [55] and Sredniki's [56] texts are, in my opinion, two of the best and up to date. For a more experimental prospective, Peskin's book [50] is optimal.

In the Standard Model there are three types of particles: the matter constituents called quarks and leptons, which obey the Fermi-Dirac statistic, and the force-carrying particles, that obey the Bose-Einstein. In nature there are four main forces: gravitational, electromagnetic, weak and strong force. The bosons which carry these forces are given in table (1.1). Then the last piece is the Higgs[1] particle. This

[1] I would like to mention all the other physicists that, in some way, worked on the theory of the Higgs that are, most of the time, going unseen: Englert, Brout, Higgs, Guralnik, Hagen and Kibble.

Table 1.1. Vector and tensor boson carriers in the SM.The Graviton has not yet been discovered, it is just theorized.

Field	Boson	Spin
Grav.	Graviton	2
EM	Photon γ	1
Weak	$W^{\pm}$,Z^0	1
Strong	Gluon g	1

is again a boson, a scalar boson, whose interactions does not come from a gauge principle, that through the Higgs mechanism [31, 35, 37, 38, 42], after EW symmetry breaking gives masses to the various particles of the SM. In table (1.2) the main properties of the SM particles are given. With all the experimental properties out of the way, we can concentrate on the theoretical side of things.

The basic SM Lagrangian, before EW symmetry breaking, can be divided into five main pieces

$$\mathcal{L}_{\text{SM}} = \mathcal{L}_{\text{gauge}} + \mathcal{L}_{\text{fermions}} + \mathcal{L}_{\text{Higgs}} + \mathcal{L}_{\text{Yukawa}} + \mathcal{L}_{\text{Theta-vacuum}}, \tag{1.3}$$

where[2][3]

$$\begin{aligned}
\mathcal{L}_{\text{gauge}} &= -\frac{1}{2}\operatorname{Tr} G_{\mu\nu}G^{\mu\nu} - \frac{1}{2}\operatorname{Tr} W_{\mu\nu}W^{\mu\nu} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu} \\
&= -\frac{1}{4}\sum_{\mathrm{a}=1}^{8} G^{\mathrm{a}}_{\mu\nu}G^{\mathrm{a}\mu\nu} - \frac{1}{4}\sum_{a=1}^{3} W^{a}_{\mu\nu}W^{a\mu\nu} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu} \qquad (1.4)\\
\mathcal{L}_{\text{fermions}} &= \sum_{\substack{\text{left}\\\text{quarks}}} i\bar{Q}^i_L \not{D}^{(q)} Q^i_L + \sum_{\substack{\text{up}\\\text{quarks}}} i\bar{u}^i_R \not{D}^{(u)} u^i_R + \sum_{\substack{\text{down}\\\text{quarks}}} i\bar{d}^i_R \not{D}^{(d)} d^i_R \\
&\quad + \sum_{\substack{\text{left}\\\text{leptons}}} i\bar{L}^i \not{D}^{(\ell)} L^i + i\bar{e}^i_R \not{D}^{(e)} e^i_R \qquad (1.5)\\
\mathcal{L}_{\text{Higgs}} &= \Big|D_\mu H\Big|^2 + m^2|H|^2 - \lambda_H|H|^4 \qquad (1.6)\\
-\mathcal{L}_{\text{Yukawa}} &= Y^{ij}_D\left(\bar{Q}^i_L H d^j_R + \bar{d}^j_R H^\dagger Q^i_L\right) + Y^{ij}_U\left(\bar{Q}^i_L \tilde{H} u^j_R + \bar{u}^j_R \tilde{H}^\dagger Q^i_L\right) \\
&\quad + Y_\ell\Big[\bar{L}_i H e^i_R + \bar{e}^i_R H^\dagger L_i\Big] \qquad (1.7)\\
\mathcal{L}_{\text{Theta-vacuum}} &= -\frac{\theta}{16\pi^2}\varepsilon^{\mu\nu\rho\sigma}\operatorname{Tr} G_{\mu\nu}G_{\rho\sigma} = -\frac{\theta}{32\pi^2}\varepsilon^{\mu\nu\rho\sigma}G^{\mathrm{a}}_{\mu\nu}G^{\mathrm{a}}_{\rho\sigma}, \qquad (1.8)
\end{aligned}$$

where we used the convention that italicized indices are the $SU(2)$ indices $a = 1, 2, 3$, the non-italicized ones are for the $SU(3)$ color indices $\mathrm{a} = 1, \cdots, 8$ and the fermion indices are to label the three families of leptons and quarks $i = 1, 2, 3$. Moreover we call *up quarks* the up, charm and top quarks, while the *down quarks* are the remaining down, strange and bottom quarks.

The theta-vacuum term is a consequence of the strong CP problem [45, 49] and is given just for completion since we won't discuss the nature of this term in this notes.

[2] From hereafter the Einstein summation convention is used unless stated otherwise. Sometimes the summation will be shown for clarity's sake.

[3] We make use of the Feynman slash notation where $\gamma^\mu a_\mu = \not{a}$.

Table 1.2. Properties of the Standard Model Particles as given by the PDG [66].

	Particle	Mass (MeV)	Mean Life (s)	Charge (e)
Leptons	e^-	0.511 ± 10^{-9}	$> 6.6 \times 10^{28}$ yr	-1
	μ^-	105.7 ± 10^{-6}	2.197 ± 10^{-6}	
	τ^-			
	$\nu_e,\, \nu_\mu,\, \nu_\tau$	$< 2 \times 10^{-6}$	-	0
Quarks[a]	u	$2.2^{+0.5}_{-0.4}$	-	$\frac{2}{3}$
	c	$1.27^{+0.025}_{-0.035} \times 10^3$	-	
	t	$(173.0 \pm 0.4) \times 10^3$	-	
	d	$4.7^{+0.5}_{-0.3}$	-	$-\frac{1}{3}$
	s	95^{+9}_{-3}	-	
	b	$4.18^{+0.04}_{-0.03} \times 10^3$	-	
	Particle	Mass (GeV)	Decay Width (GeV)	Charge
Bosons	γ	$< 1 \times 10^{-24}$	Stable	$< 1 \times 10^{-35}$
	$W^\pm$	80.379 ± 0.012	2.085 ± 0.042	± 1
	Z^0	91.1876 ± 0.0021	2.4952 ± 0.0023	0
	g	0	-	-
	h^0	125.18 ± 0.16	< 0.013	0

[a] The u, d and s quark masses are estimates of so-called "current-quark masses", in a mass-independent subtraction scheme such as $\overline{\text{MS}}$ at a scale $\mu \approx 2$ GeV. The c and b quark masses are the running masses in the $\overline{\text{MS}}$ scheme. The t quark mass comes from direct measurements.

We will treat in more details equations eqs. (1.4) to (1.7) in the upcoming sections. For now we just set up our various conventions that we will use throughout the whole notes.

Starting with the gauge Lagrangian eq. (1.4), the three kinetic terms are written in terms of the field strength associated to each gauge connection of the gauge group. That is[4][5]

$$G^{\text{a}}_{\mu\nu} = \partial_{[\mu} G^{\text{a}}_{\nu]} + g_s f^{\text{abc}} G^{\text{b}}_{\mu} G^{\text{c}}_{\nu}, \tag{1.9}$$

[4]We use the notation for the antisymmetrization of the indices where $a_{[\mu} b_{\nu]} = a_\mu b_\nu - a_\nu b_\mu$. A similar notation is used for the symmetrization $a_{(\mu} b_{\nu)} = a_\mu b_\nu + a_\nu b_\mu$.

[5]More generally the field strength is defined as the curvature tensor induced by the group structure on the spacetime manifold and is therefore given by

$$F_{\mu\nu} = \frac{i}{g_F} \left[D^{(A)}_\mu, D^{(A)}_\nu \right]$$

where $D^{(A)}_\mu$ is the covariant derivative constructed from the gauge connection A.

$$W^a_{\mu\nu} = \partial_{[\mu} W^a_{\nu]} + g\epsilon^{abc} W^b_\mu W^c_\nu, \tag{1.10}$$

$$B_{\mu\nu} = \partial_{[\mu} B_{\nu]}, \tag{1.11}$$

where G^{a}_μ is the $SU(3)$ gauge connection (the gluon field), W^a_μ is the $SU(2)$ connection and B_μ is the $U(1)$ connection. The field W^a_μ and B_μ will later combine, after symmetry breaking, to make up the $W^\pm_\mu, Z^0_\mu$ bosons which are the massive vector bosons which mediate the weak interactions and the A_μ boson, the photon, which mediates the electromagnetic one. The constants f^{abc} are the structure constants of $SU(3)$ and the Levi-Civita symbol ϵ^{abc} gives the structure constants of $SU(2)$. In this notes we use the convention where the Lie algebra of the groups [23] is given by the commutators

$$\left[t^{\mathrm{a}}, t^{\mathrm{b}}\right] = i f^{\mathrm{abc}} t^{\mathrm{b}} \qquad \left[\tau^a, \tau^b\right] = i\epsilon^{abc}\tau^c, \tag{1.12}$$

where, in the fundamental representation, the eight $SU(3)$ generators t^{a} are given by the Gell-Mann matrices while the three $SU(2)$ generators are given by $\tau^a = \sigma^a/2$ where σ^a are the Pauli matrices and are both normalized as

$$\mathrm{Tr}\, t^a t^b = \frac{1}{2}\delta^{ab} \tag{1.13}$$

More complicated representations can be found in the literature or constructed using methods like the highest weight method.
The three coupling constants are g_s for the strong interactions, g for the W interactions before symmetry breaking and g' for the B interactions before SB as well.

1.2 Particles and Their Representations

Being the SM a Quantum Filed Theory, every particle belongs to some representation of the underlying symmetry group. In the case of the SM we know that the symmetry group, before symmetry breaking, is the one of equation eq. (1.1). We call the various pieces with the quantum number associated to that specific group. In particular

- $SU(3)_C$ quantum number is called *color*,
- $SU(2)_L$ quantum number is called *isospin*,
- $U(1)_Y$ quantum number is called *weak hypercharge*.

The symmetry breaking pattern of the SM induced by the non-zero vacuum expectation value (VEV) of the Higgs [20, 36, 32] is given by

$$SU(3)_C \times SU(2)_L \times U(1)_Y \to SU(3)_C \times U(1)_{\mathrm{em}}. \tag{1.14}$$

The remaining symmetry after SB is given by the same color symmetry as before plus the electromagnetic symmetry which gives to all particles a quantum number called *electric charge*.

Given this, we can give all the fields appearing in equations eqs. (1.4) to (1.7) their appropriate quantum numbers and with that it can be easily shown that every component of the Lagrangian is by itself a scalar for the SM symmetry group. All

the particles and their representation are given in table (1.3).
From the Yukawa Lagrangian eq. (1.7) there is another field which is the charge-conjugate field of the Higgs. This is given by

$$\tilde{H} = i\sigma^2 H^*, \tag{1.15}$$

where σ^2 is a Pauli matrix and H^* is the complex conjugate of the Higgs field H. This field transforms in the $(1,2)_{-1}$ representation and is needed to make all the possible scalar terms such as $\left(\bar{Q}_L^{\mathrm{a}i}\tilde{H}\right)u_R^{\mathrm{a}j}$ where i,j are flavour indices. In fact under $SU(3)$ this is a scalar since we have the following tensor product

$$\bar{\mathbf{3}} \otimes \mathbf{1} \otimes \mathbf{3} = \mathbf{1} \oplus \mathbf{8} \tag{1.16}$$

and we take the trace. Same goes for the isospin since we have

$$\mathbf{2} \otimes \mathbf{2} \otimes \mathbf{1} = \mathbf{1} \oplus \mathbf{3}, \tag{1.17}$$

and for the hypercharge

$$-\frac{1}{3} - 1 + \frac{4}{3} = 0, \tag{1.18}$$

which come out to be scalars for the entire SM symmetry group.

Table 1.3. Representations of the SM particles before symmetry breaking. The last column refers to the representation under the Lorentz group.

Field	$SU(3)_C$	$SU(2)_L$	$U(1)_Y$	$SO(1,3) \simeq SU(2) \times SU(2)^{\mathrm{a}}$
Q_L^i	3	2	$1/3$	$(1/2, 0)$
u_R^i	3	1	$4/3$	$(0, 1/2)$
d_R^i	3	1	$-2/3$	$(0, 1/2)$
L^i	1	2	-1	$(1/2, 0)$
e_R^i	1	1	-2	$(0, 1/2)$
G_μ	8	1	1	$(1/2, 1/2)$
W_μ	1	3	0	$(1/2, 1/2)$
B_μ	1	1	0	$(1/2, 1/2)$
H	1	2	1	$(0, 0)$

[a] The isomorphism is at the level of the complexified algebras $\mathfrak{so}(1;3) \hookrightarrow \mathfrak{so}(1;3)_{\mathbb{C}} \simeq \mathfrak{su}(2)_{\mathbb{C}} \oplus \mathfrak{su}(2)_{\mathbb{C}}$ but we label it with the group for simplicity.

After EW symmetry breaking, the relevant quantum numbers become the color and the electric charge. Using the remaining unbroken generator of the $SU(2)_L \times U(1)_Y$ group, we can find the relation between the isospin and hypercharge quantum numbers with the electric charge. This is the well known Gell-Mann–Nishijima formula[6] [22, 47]

$$Q = \tau^3 + \frac{Y}{2}, \tag{1.19}$$

[6]Depending on the convention used, there can be a factor of $1/2$ difference in the hypercharge giving $Q = \tau^3 + Y$. We use the convention where the factor $1/2$ is present.

where T_3 is the generator which labels the third component of the isospin and Y is the generator of the hypercharge. Just for a sanity check, one can use formula eq. (1.19) to see if the quantum numbers given in table (1.3) give back the expected electric charge. For example, take the right-handed electron which we expect to have a electric charge of -1 and, using eq. (1.19), find

$$e_R^i \to Q = 0 + (-1) = -1. \qquad \checkmark \tag{1.20}$$

1.3 Electroweak Sector

We now have sufficient knowledge to formulate the GSW theory of weak and electromagnetic interactions among leptons and quarks and to study its properties. Let us first state the starting point and the aim of our study

1. There exist charged and neutral currents.
2. The charged currents contain only couplings between left-handed fermions. This result is given by Fermi theory of weak interactions which, as we'll see, is the low energy limit of the GSW theory.
3. The bosons $W^{\pm}, Z^0$ mediating the weak interaction must be very massive.
4. Nevertheless we'll begin with massless bosons which then receive masses through the Higgs mechanism. At that point we want to simultaneously include the photon field.

Given this list of properties, we can begin to build up the first part of the SM which accounts for the electroweak sector.

1.3.1 The GWS Lagrangian and symmetry breaking

Let's begin, as we always must, to find the symmetry group of the theory. We know that at least there must be one gauge boson for the photon. Moreover there must be another two vector bosons for the $W^{\pm}$ fields. With this we need at least the $SU(2)$ symmetry group since it has 3 generators. But it turns out that this group is too small since it only accounts for left-handed interactions but we know that the electromagnetism is perfectly symmetric between left and right-handed fermions. What Glashow proposed was the following minimal group

$$SU(2)_W \otimes U(1)_Y, \tag{1.21}$$

where the reps are defined, as we have seen before, by the isospin symmetry and the hypercharge. Based on this symmetry group, the existence of a fourth gauge boson was theorized since the group has 4 generators. It will turn out that the additional gauge boson is, in fact, the Z^0 which mediates the weak neutral currents.

Since we have that the total symmetry group is the product of two groups, we need two different coupling constants g, g'. The kinetic part of the Lagrangian will be then by the second half of the Lagrangian eq. (1.4)

$$\mathcal{L} = -\frac{1}{4} W^a_{\mu\nu} W^{a\mu\nu} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu}. \tag{1.22}$$

Given the local nature of the interactions, we need to give mass to the bosons. On the other hand, the photon will be given by a linear combination of the symmetry generators which remain unbroken under the action of the Higgs mechanism.
To induce the symmetry breaking, we have a complex isospin doublet with hypercharge 1, the Higgs

$$H = \begin{pmatrix} H^+ \\ H^0 \end{pmatrix} . \tag{1.23}$$

The hypercharge is set by the Gell-Mann Nishijima formula eq. (1.19).
The Lagrangian for the Higgs field is given in equation eq. (1.6). In particular, we have to specify how the covariant derivative of equation eq. (1.6) acts on the Higgs doublet. This is easily done by exploiting the Higgs representation under the gauge group

$$D_\mu = \partial_\mu - ig'\frac{Y}{2}B_\mu - igW^a_\mu\tau^a = \partial_\mu - i\frac{g'}{2}B_\mu - igW^a_\mu\tau^a, \tag{1.24}$$

where the τ^a are the the generators of the fundamental **2** representation of $SU(2)$. The Higgs potential with the opposite mass sign, induces a VEV for H, which can be taken to be real and in the lower component without loss of generality. Thus we choose

$$H = \exp\left(\frac{i}{v}\pi^a\tau^a\right)\begin{pmatrix} 0 \\ \frac{h+v}{\sqrt{2}} \end{pmatrix}, \tag{1.25}$$

where $v = \langle 0| H |0\rangle = \mu/\sqrt{\lambda}$. Since we are going to study how the gauge bosons and the fermions gain mass through the Higgs mechanism, we will fix the gauge to the *unitary gauge* where, essentially, we set $\pi^a = 0$. With the VEV fixed it is easy to find that the broken generators, i.e. the ones for which $\tau^a\langle H\rangle \neq 0$ are given by

$$\tau^1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \tau^2 = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \tau^3 - \frac{Y}{2} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{1.26}$$

and the unbroken generator is given by

$$\tau^3 + \frac{Y}{2} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{1.27}$$

which is exactly the electric charge as given by the Gell-Mann Nishijima formula! The symmetry breaking pattern is therefore

$$SU(2)_W \otimes U(1)_Y \to U(1)_{\text{em}} \tag{1.28}$$

and we expect, thanks to Goldstone theorem and the Higgs mechanism, three out of four vector bosons to be massive while one remains massless (spoiler: the only vector boson without mass will be the photon!).

Putting the VEV in the kinetic part of the Higgs, making use of gauge freedom and choosing a gauge which "eats" the Goldstone bosons π^a called *unitary gauge*[7],

[7] One needs to be aware of which gauges are at play and the ones to choose for specific calculations.

we get

$$
\begin{aligned}
&|D_\mu H|^2 = \\
&= \frac{v^2}{8}\begin{pmatrix}0 & 1\end{pmatrix}\begin{pmatrix} g'B_\mu + gW^3_\mu & g(W^1_\mu - iW^2_\mu) \\ g(W^1_\mu + iW^2_\mu) & g'B_\mu - gW^3_\mu \end{pmatrix}\begin{pmatrix} g'B_\mu + gW^3_\mu & g(W^1_\mu - iW^2_\mu) \\ g(W^1_\mu + iW^2_\mu) & g'B_\mu - gW^3_\mu \end{pmatrix}\begin{pmatrix}0 & 1\end{pmatrix} \\
&= g^2\frac{v^2}{8}\left[\left(W^1_\mu\right)^2 + \left(W^2_\mu\right)^2 + \left(\frac{g'}{g}B_\mu - W^3_\mu\right)^2\right]
\end{aligned}
\tag{1.29}
$$

The W^1, W^2 terms are degenerate in mass

$$
M_W^2 = \frac{v^2 g^2}{4}
$$

The remaining terms are given by

$$
\frac{v^2 g^2}{4}(W^3_\mu)^2 + \frac{v^2 g'^2}{4}B^2_\mu - \frac{2gg'v^2}{4}B^\mu W^3_\mu = \frac{v^2}{4}\begin{pmatrix}B_\mu & W^3_\mu\end{pmatrix}\begin{pmatrix} g^2 & gg' \\ gg' & g'^2 \end{pmatrix}\begin{pmatrix} B^\mu \\ W^\mu_3 \end{pmatrix} \tag{1.30}
$$

it is clear that the initial basis is not the basis given by the mass eigenstates. We can therefore go to the latter by diagonalizing eq. (1.30)

$$
\begin{aligned}
\det\begin{pmatrix} g^2 - m & gg' \\ gg' & g'^2 - m \end{pmatrix} &= (g^2 - m)(g'^2 - m) - (gg')^2 = 0 \\
&= m^2 + (gg')^2 - m(g^2 + g'^2) - (gg') \\
&= m(m - g^2 - g'^2) = 0 \\
m = 0 \qquad & m = g^2 + g'^2
\end{aligned}
\tag{1.31}
$$

The two solutions give us what we wanted: a massless mode and a massive one. Looking for the eigenvectors will give us linear combinations of the B_μ and W^3_μ fields which will turn out to be the massless photon field and the massive Z^0 gauge boson field.
By means of the following reparametrization

$$
\sin\theta_W = \frac{g'}{\sqrt{g^2 + g'^2}}, \qquad \cos\theta_W = \frac{g^2}{\sqrt{g^2 + g'^2}}, \tag{1.32}
$$

where θ_W is called the *Weinberg angle* [26, 60], one can easily show that rotation based on this angle gives us indeed the linear combination that we need

$$
\begin{pmatrix} Z^0_\mu \\ A_\mu \end{pmatrix} = \begin{pmatrix} \cos\theta_W & -\sin\theta_W \\ \sin\theta_W & \cos\theta_W \end{pmatrix}\begin{pmatrix} W^3_\mu \\ B_\mu \end{pmatrix} \implies \begin{cases} Z^0_\mu = \cos\theta_W W^3_\mu - \sin\theta_W B_\mu \\ A_\mu = \sin\theta_W W^3_\mu + \sin\theta_W B_\mu. \end{cases} \tag{1.33}
$$

Now notice the following

$$
W^a_\mu \tau^a = \frac{1}{\sqrt{2}}\left(W^+_\mu \tau^+ + W^-_\mu \tau^-\right) + W^3_\mu \tau^3, \tag{1.34}
$$

where

$$\tau^{\pm} = \tau^1 \pm i\tau^2. \tag{1.35}$$

Under this definition, the charged gauge fields are given by

$$W_\mu^+ = \frac{1}{\sqrt{2}}(W_\mu^1 + iW_\mu^2) \qquad W_\mu^- = \frac{1}{\sqrt{2}}(W_\mu^1 - iW_\mu^2). \tag{1.36}$$

Therefore what we have in the hand are the following fields

$$\begin{aligned} &W_\mu^{\pm} \rightsquigarrow M_W = \frac{vg}{2}, \\ &Z_\mu^0 \rightsquigarrow m_Z = \frac{1}{2\cos\theta_W} gv = \frac{v}{2}\sqrt{g^2 + g'^2} = \frac{M_W}{\cos\theta_W}, \\ &A_\mu \rightsquigarrow m_A = 0. \end{aligned} \tag{1.37}$$

Already there's an unambiguous prediction: the W bosons should be lighter than the Z boson.
Moreover we find that, at tree level the following result should hold

$$\rho = \frac{M_W^2}{\cos^2\theta_W m_Z^2} = 1 \tag{1.38}$$

This is the result of an hidden symmetry of the Standard Model, the *custodial symmetry*[8].

1.3.2 Gauge Sector

Putting together what we found, we can write down the kinetic term in the Lagrangian for the Z and A bosons after symmetry breaking, in the unitary gauge, as

$$\mathcal{L}_{\mathrm{K}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}Z_{\mu\nu}Z^{\mu\nu} + \frac{1}{2}m_Z^2 Z_\mu Z^\mu, \tag{1.39}$$

where

$$Z_{\mu\nu} = \partial_\mu Z_\nu - \partial_\nu Z_\mu \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{1.40}$$

Since the gauge bosons transform in the adjoint rep, their interactions are given by commutators and in particular, the W_μ^3 part of the photon field gives us the known coupling

$$g[A_\mu, W_\nu^a \tau^a] = g\sin\theta_W W_\mu^3 W_\nu^a \left[\tau^3, \tau^a\right] \implies e = g\sin\theta_W = g'\cos\theta_W \tag{1.41}$$

With this in mind, the $W^{\pm}$ combinations will have ± 1 charge in units of e, which is what we want.

[8]"Turning down" the couplings to the Higgs $g, g' \to 0$ we can see that the Lagrangian eq. (1.6) has a bigger symmetry group which we label as $SU(2)_L \times SU(2)_R$. The symmetry breaking pattern then becomes $SU(2)_L \times SU(2)_R \to SU(2)_V$ which is a bigger symmetry than the one with the couplings so that only at tree level we can see it is effects.

Without giving the full calculation, one can find that the full gauge Lagrangian is

$$\begin{aligned}
\mathcal{L} =& -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}Z_{\mu\nu}Z^{\mu\nu} - \frac{1}{2}(\partial_\mu W_\nu^+ - \partial_\nu W_\mu^+)(\partial^\mu (W^-)^\nu - \partial^\nu (W^-)^\mu) \\
&+ \frac{1}{2}m_Z^2 Z_\mu Z^\mu - M_W^2 W_\mu^+ (W^-)^\mu \\
&+ ie\cot\theta_W \left[Z^{\mu\nu}W_\mu^+ W_\nu^- - (\partial_\mu W_\nu^+ - \partial_\nu W_\mu^+)Z^\mu (W^-)^\nu + (\partial_\mu W_\nu^- - \partial_\nu W_\mu^-)Z^\mu (W^+)^\nu\right] \\
&+ ie\left[F^{\mu\nu}W_\mu^+ W_\nu^- - (\partial_\mu W_\nu^+ - \partial_\nu W_\mu^+)A^\mu (W^-)^\nu + (\partial_\mu W_\nu^- - \partial_\nu W_\mu^-)A^\mu (W^+)^\nu\right] \\
&+ \frac{1}{2}\frac{e^2}{\sin^2\theta_W}\left(W_\mu^+ (W^+)^\mu W_\nu^- (W^-)^\nu - W_\mu^+ (W^-)^\mu W_\nu^+ (W^-)^\nu\right) \\
&+ e^2\left(A^\mu W_\mu^+ A^\nu W_\nu^- - A_\mu A^\mu W_\nu^+ (W^-)^\nu\right) + e^2\cot\theta_W\left(Z^\mu W_\mu^+ Z^\nu W_\nu^- - Z_\mu Z^\mu W_\nu^+ (W^-)^\nu\right) \\
&+ e^2\cot\theta_W\left(W_\mu^+ W_\nu^- A^\mu Z^\nu + W_\mu^- W_\nu^+ A^\mu Z^\nu - 2W_\mu^+ (W^-)^\mu A^\nu Z_\nu\right)
\end{aligned} \tag{1.42}$$

1.3.3 Higgs Sector

We can now return to the field h, the *Higgs Boson*. This boson remains in the spectrum of the theory even after the choice of the unitary gauge $\pi = 0$ by the Higgs mechanism.
The part of the Lagrangian which gives us the dynamics of the Higgs field is given by the expansion of the covariant derivative after symmetry breaking

$$\begin{aligned}
\mathcal{L}_\text{H} =& \frac{1}{2}(\partial_\mu h)(\partial^\mu h) - \frac{m_h^2}{2}^2 - g\frac{m_h^2}{4M_W}h^3 - \frac{g^2 m_h^2}{32M_W^2}h^4 + \\
&+ 2\frac{h}{v}\left(M_W^2 W_\mu^+ (W^-)^\mu + \frac{1}{2}m_Z^2 Z^\mu Z_\mu\right) + \frac{h^2}{v^2}\left(M_W^2 W_\mu^+ (W^-)^\mu + \frac{1}{2}m_Z^2 Z^\mu Z_\mu\right),
\end{aligned} \tag{1.43}$$

where $m_h = \sqrt{2}\,\mu$ and μ is the initial symmetry breaking parameter in the unbroken Higgs doublet Lagrangian and v is the induced VEV.

As we can see from eq. (1.43), the Higgs field interacts with itself in cubic and quartic interactions and with the other gauge bosons, again, with cubic and quartic interactions. As in all the other interaction terms, the strength of the interaction is proportional to the masses.

1.3.4 Lepton Sector

Let's study the interactions between the electroweak gauge bosons and the leptons. As seen in table (1.3) we have classified leptons into left-handed $L^i = (2,1)_{-1}$ and right-handed $e_R^i = (1,1)_{-2}$

$$L^i = \begin{pmatrix}\nu_e \\ e^-\end{pmatrix}_L, \begin{pmatrix}\nu_\mu \\ \mu^-\end{pmatrix}_L, \begin{pmatrix}\nu_\tau \\ \tau^-\end{pmatrix}_L, \qquad e_R^i = \{e_R, \mu_R, \tau_R\} \qquad i = 1,2,3. \tag{1.44}$$

We see that the left-handed field shows up as an isospin doublet, whereas the right-handed field as singlet. In equation eq. (1.44) we highlighted the three *generations* left $SU(2)$ doublets and right singlets of leptons. These all transform as a left/right-handed Weyl spinors. From now on we consider only one generation of leptons for simplicity's sake, but the argument can be easily generalized to all three.

The coupling between the leptons and the gauge boson is given by the covariant derivative in the fermion Lagrangian eq. (1.5)

$$\mathcal{L} = i\bar{L}\not{D}^{(\ell)}L + i\bar{e}_R\not{D}^{(e)}e_R, \tag{1.45}$$

where the covariant derivatives are different between the left-handed part and the right-handed one. All leptons couple to the hypercharge gauge boson as we stated in table (1.3). We denote Y_L the left-handed hypercharge and Y_R the right-handed one. So the expanded Lagrangian will be

$$\mathcal{L} = i\bar{L}\left(\not{\partial} - ig\not{W}^a\tau^a - i\frac{g'}{2}Y_L\not{B}\right)L + i\bar{e}_R\left(\not{\partial} - i\frac{g'}{2}Y_R\not{B}\right)e_R. \tag{1.46}$$

To be clear, the L or R subscript in the Lagrangian are just for convenience, since they indicate the implicit chirality of the field. But since all leptons are all left- or right-handed Weyl spinors, it would be technically correct to replace

$$\begin{aligned} \bar{L}_R\not{\partial}L &\to L^\dagger\bar{\sigma}^\mu\partial_\mu L, \\ \bar{e}_R\not{\partial}e_R &\to e_R^\dagger\sigma^\mu\partial_\mu e_R, \end{aligned} \tag{1.47}$$

where $\sigma^\mu = (\mathbb{1}, \sigma^i)$ and $\bar{\sigma}^\mu = (\mathbb{1}, -\sigma^i)$. However, since we'll almost always deal with the fields in the broken phase, where the left- and right-handed spinors combine into Dirac spinors, for simplicity we'll always write everything in the Dirac rep where[9]

$$\begin{aligned} \bar{L}\not{\partial}L &= L^\dagger\gamma^0\gamma^\mu\frac{1-\gamma_5}{2}\partial_\mu L, \\ \bar{e}_R\not{\partial}e_R &= e\gamma^0\gamma^\mu\frac{1+\gamma_5}{2}\partial_\mu e. \end{aligned} \tag{1.48}$$

As it is clear, there are still no masses for the leptons. To find them we have to build the Yukawa sector of the Lagrangian where the fields interact with the Higgs doublet. This will give mass to the leptons after symmetry breaking.
From the transformation rule of the lepton fields and the Higgs doublet, it is easy to see that the only scalar quantities we can construct are the ones in Lagrangian eq. (1.7)

$$\mathcal{L} = Y\left[\bar{L}_e H e_R + \bar{e}_R H^\dagger L_e\right]. \tag{1.49}$$

After symmetry breaking, this part will give us the mass for the electrons with the following term

$$-m_e\left(\bar{e}_L e_R + \bar{e}_R e_L\right), \qquad m_e = \frac{Y}{\sqrt{2}}v. \tag{1.50}$$

[9]Sometimes we will use the notation where the chirality projector are written as

$$P_L = \frac{1-\gamma^5}{2} \qquad P_R = \frac{1+\gamma^5}{2}$$

Again, only after electroweak symmetry breaking, together with the diagonalization of the masses for the gauge bosons, the Lagrangian eq. (1.46) becomes

$$\begin{aligned}\mathcal{L} = \bar{L}_e\left[g\tau^3\left(Z_\mu\cos\theta_W + A_\mu\sin\theta_W\right) + \frac{g'}{2}Y_L\left(-Z_\mu\sin\theta_W + A_\mu\cos\theta_W\right)\right]\gamma^\mu L_e + \\ + Y_R\frac{g'}{2}\bar{e}_R\left(-Z_\mu\sin\theta_W + A_\mu\cos\theta_W\right)\gamma^\mu e_R.\end{aligned} \tag{1.51}$$

The terms proportional to the photon field are

$$A_\mu\left[\bar{L}_e\left(g\tau^3\sin\theta_W + \frac{g'}{2}Y_L\cos\theta_W\right)L_e + \left(\frac{g'}{2}\cos\theta_W\right)Y_R\left(\bar{e}_R\gamma^\mu e_R\right)\right], \tag{1.52}$$

and using the fact that $g\sin\theta_W = g'\cos\theta_W = gg'/\sqrt{g^2+g'^2}$, we get to the expected result for the QED interaction between photons and charged leptons

$$\begin{aligned}&A_\mu g\sin\theta_W\left[\bar{L}_e\gamma^\mu\left(\tau^3 + \frac{Y_L}{2}\right)L_e + \frac{Y_R}{2}\left(\bar{e}_R\gamma^\mu e_R\right)\right]\\ &= A_\mu g\sin\theta_W\left[-\bar{e}_L\gamma^\mu e_L - \bar{e}_R\gamma^\mu e_R\right]\\ &= g\sin\theta_W A_\mu J^\mu_{\text{em}},\end{aligned}$$

where we used the unbroken generator in eq. (1.27) and

$$J^\mu_{\text{em}} = Q_e(\bar{e}\gamma^\mu e), \tag{1.53}$$

with $Q_e = e = g\sin\theta_W$. As expected, the electromagnetic interaction does not differentiate between left and right-handed chirality.

The terms proportional to the Z^0 boson are

$$\begin{aligned}&Z_\mu\left[g\cos\theta_W\bar{L}_e\gamma^\mu\tau^3 L_e - \frac{Y_L}{2}g'\sin\theta_W\bar{L}_e\gamma^\mu L_e - \frac{Y_R}{2}g'\sin\theta_W\bar{e}_R\gamma^\mu e_R\right]\\ &= Z_\mu\left[(g\cos\theta_W + g'\sin\theta_W)\bar{L}_e\gamma^\mu\tau^3 L_e - g'\sin\theta_W q J^\mu_{\text{em}}\right]\\ &= Z_\mu\frac{g}{\cos\theta_W}\left(J^\mu_3 - q\sin^2\theta_W J^\mu_{\text{em}}\right).\end{aligned} \tag{1.54}$$

Therefore the Z^0 boson not only couples to the EM current but even with an axial current

$$J^\mu_3 = \bar{L}_e\gamma^\mu\tau^3 L_e. \tag{1.55}$$

There remain only the interaction terms between the leptons and the $W^\pm$ bosons. Recalling eq. (1.34) we get, from the Lagrangian eq. (1.46)

$$gW^a_\mu\bar{L}_e\gamma^\mu\tau^a L_e = g\left[\frac{1}{\sqrt{2}}W^+_\mu\bar{L}_e\gamma^\mu\tau^- L_e + \frac{1}{\sqrt{2}}W^-_\mu\bar{L}_e\gamma^\mu\tau^+ L_e + W^3_\mu\bar{L}_e\gamma^\mu\tau^3 L_e\right] \tag{1.56}$$

and we can directly see that the charged currents are

$$\begin{aligned}J^+_\mu &= \bar{L}_e\gamma_\mu\tau^+ L_e = \begin{pmatrix}\bar{\nu}_e & \bar{e}^-\end{pmatrix}_L\gamma_\mu\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\begin{pmatrix}\nu_e\\ e^-\end{pmatrix}_L = \begin{pmatrix}\bar{\nu}_e & \bar{e}^-\end{pmatrix}_L\gamma_\mu\begin{pmatrix}e^-\\ 0\end{pmatrix}_L\\ &= \bar{\nu}_{eL}\gamma_\mu e^-_L = \bar{\nu}_e\gamma_\mu\frac{1-\gamma_5}{2}e^-\end{aligned} \tag{1.57}$$

and

$$J_\mu^- = (J_\mu^+)^\dagger = \bar{e}\gamma_\mu \frac{1-\gamma_5}{2}\nu_e. \tag{1.58}$$

The axial part for W^3 goes into the photon and Z_0 boson.

The full interaction Lagrangian between the leptons and the gauge boson after electroweak symmetry breaking becomes

$$\mathcal{L} = qeA_\mu J^\mu_{\text{em}} + \frac{g}{\cos\theta_W} Z_\mu \left(J_3^\mu - q\sin\theta_W J^\mu_{\text{em}}\right) + \frac{g}{\sqrt{2}}\left(W_\mu^+ J^{\mu-} + W_\mu^- J^{\mu+}\right). \tag{1.59}$$

1.4 Quark Mixing and CKM

Now that we talked about the electroweak sector

$$SU(2)_L \times U(1)_Y \to U(1)_{\text{em}} \tag{1.60}$$

of the Standard Model, we're ready to add one of the missing part: the quarks.
We will not talk about QCD in this section, which is the remaining $SU(3)$ of the full symmetry of the Standard Model, but only how quarks enter in the electroweak theory and how we can give masses to them with the help of the Higgs mechanism. It will turn out that whenever we try to diagonalize the mass spectrum, we'll introduce some kind of mixing between the quarks which will be mediated by the electroweak gauge bosons.

1.4.1 The Quarks

The quarks that enter in the Standard Model and their representations are summarized in table (1.3). To be more specific, quarks come in three flavours, just like leptons, and appear in the theory in their chiral basis

$$\begin{aligned} Q_L^i &= \begin{pmatrix} u \\ d \end{pmatrix}_L, \begin{pmatrix} c \\ s \end{pmatrix}_L, \begin{pmatrix} t \\ b \end{pmatrix}_L \\ u_R^i &= \{u_R, c_R, t_R\}, \qquad d_R = \{d_R, s_R, b_R\}. \end{aligned} \tag{1.61}$$

Their name are: up, down, charm, strange, top and bottom quarks. In the case of the quarks, we now consider all three families since, as we will see, they can mix with themselves, while leptons do not. Whenever using the notation u^i we'll mostly mean the up row of quarks in the SM which are up and have electric charge, in units of e, $2/3$, charm and top quarks, the others are the d^is which have electric charge $-1/3$.
Their irrep in the full SM gauge group is

$$Q_L \sim (2,3)_{\frac{1}{3}}, \quad u_R \sim (1,3)_{\frac{4}{3}}, \quad d_R \sim (1,3)_{-\frac{2}{3}}. \tag{1.62}$$

Quarks carry a lot of indices: one index for the isospin charge, one index for the color charge, one family index and a Lorentz index. We'll omit them, as per usual, since the notation would be too cluttered with them. But remember still that to construct invariant quantities all indices must be saturated in such a way to have a singlet for any of the possible symmetries.

1.4.2 Interactions and Lagrangian

In the same way as we did with the leptons, we start from the following Lagrangian given in equation eq. (1.5)[10]

$$\begin{aligned} \mathcal{L} = & i\bar{Q}_L \left(\partial\!\!\!/ - ig W\!\!\!\!/^a \tau^a - i\frac{g'}{2} Y_{Q_L} B\!\!\!/ \right) Q_L \\ & + i\bar{u}_R \left(\partial\!\!\!/ - i\frac{g'}{2} Y_{u_R} B\!\!\!/ \right) u_R + i\bar{d}_R \left(\partial\!\!\!/ - i\frac{g'}{2} Y_{d_R} B\!\!\!/ \right) d_R. \end{aligned} \tag{1.63}$$

By direct comparison with the leptons, we can easily see that the calculations will be the same and so we give directly the results for the various currents that one expects to find, coupled to the respective gauge bosons.
The full fermion currents will be the following

$$J^\mu_{\text{em}} = -\bar{e}\gamma^\mu e + \frac{2}{3}\bar{u}^a\gamma^\mu u_a - \frac{1}{3}\bar{d}^a\gamma^\mu d_a \tag{1.64}$$

is the EM current coupled to the photon[11],

$$J^3_\mu = \bar{\nu}_e\gamma_\mu \frac{1-\gamma_5}{2}\nu_e - \bar{e}\gamma_\mu\frac{1-\gamma_5}{2} e + \bar{u}^a\gamma_\mu\frac{1-\gamma_5}{2}u_a + \bar{d}^a\gamma_\mu\frac{1-\gamma_5}{2}d_a \tag{1.65}$$

is the axial current coupled to the Z^0 boson, and the charged ones

$$J^+_\mu = \bar{L}_e\gamma_\mu\tau^+ L_e + \bar{Q}^a\gamma_\mu\tau^+ Q_a, \tag{1.66}$$

$$J^-_\mu = \bar{L}_e\gamma_\mu\tau^- L_e + \bar{Q}^a\gamma_\mu\tau^- Q_a \tag{1.67}$$

which are coupled to the charged $W^\pm$ bosons.

1.4.3 Yukawa Sector

From the irreps eq. (1.62) and the Higgs we need to construct all the possible renormalizable scalar quantities. As stated in the previous sections, to do so we'll need however another form of the Higgs field since H^* won't cut it. The field we'll use is the charge conjugate of H defined by equation eq. (1.15).

Now let's see what kind of scalars we can build up. If we start from $\bar{Q}_L H$ we can easily see that this we'll be

$$\bar{Q}_L H \sim (\bar{2}, \bar{3})_{-\frac{1}{3}}(2,1)_1 = (\bar{2}\times 2, \bar{3}\times 1)_{1-\frac{1}{3}}. \tag{1.68}$$

We know that $\bar{2}\times 2$ contains a singlet state. What's missing is the hypercharge singlet since $1-\frac{1}{3} = 2/3$ and the color singlet. If we search in eq. (1.62) for a suitable quantity, we see that the d_R quark serves our purpose and so a suitable renormalizable operator for our Yukawa sector will be

$$\bar{Q}_L H d_R + \text{h.c.} = \bar{Q}_R H d_R + \bar{d}_R H^\dagger Q_L, \tag{1.69}$$

[10] Again the computations are given for only one family of quarks if not else specified. The arguments can be easily extended to three.
[11] Note the color index $a = 1, 2, 3$ on the quarks which is saturated.

where we added the Hermitian conjugate, as always, to include the reality of the Lagrangian. And this settles down the down part of the Lagrangian. For the up part we'll use the charge conjugate Higgs since, if you try, we cannot construct scalar quantities between up quarks with the normal Higgs doublet.
it is easy to see that the only renormalizable scalar quantity we can construct using u_R is

$$\bar{Q}_L \tilde{H} u_R + \bar{u}_R \tilde{H}^\dagger Q_L \tag{1.70}$$

Therefore, if we now put in all the families and the Yukawa coupling we get the Yukawa sector for quarks

$$\mathcal{L}_\text{Y} = Y_U^{ij} \left(\bar{Q}_L^i H d_R^j + \bar{d}_R^j H^\dagger Q_L^i \right) + Y_D^{ij} \left(\bar{Q}_L^i \tilde{H} u_R^j + \bar{u}_R^j \tilde{H}^\dagger Q_L^i \right) \tag{1.71}$$

which is exactly the one which was given without explanation in equation eq. (1.7). Again here we consider all three quark families since in the end quark will mix among themselves. This is not true for leptons since in the SM we are considering there are no right-handed neutrinos, which means that they will always be massless and therefore the mixing matrix can always be "rotated away" and go back to a diagonal matrix. There cannot be any mixing among neutrinos without their right-handed counterpart.

1.4.4 Symmetry Breaking

For example, given the Yukawa sector, we can use symmetry breaking and, by going in the unitary gauge[12], we get for the down quarks

$$\begin{pmatrix} \bar{u}_L & \bar{d}_L \end{pmatrix} \begin{pmatrix} 0 \\ \frac{v+h}{\sqrt{2}} \end{pmatrix} d_R = \bar{d}_L d_R \left(\frac{v+h}{\sqrt{2}} \right). \tag{1.72}$$

Thus the mass for the down quarks is given by a $SU(2)$ symmetry breaking Dirac term and, with several generations of down-like quarks, we expect

$$Y_D^{ij} \frac{v}{\sqrt{2}} \bar{d}_L^i d_R^j \implies M_D^{ij} = \frac{v}{\sqrt{2}} Y_D^{ij} \tag{1.73}$$

For the up quarks is the same but the mass matrix is given in terms of the Yukawa of the up quarks

$$M_U^{ij} = \frac{v}{\sqrt{2}} Y_U^{ij} \tag{1.74}$$

With this, we see that the mass terms in the Lagrangian for the quarks are

$$\mathcal{L} = \bar{d}_L^i M_{ij}^D d_R^j + \bar{d}_R^j M_{ij}^{D\dagger} d_L^i + \bar{u}_L^i M_{ij}^U u_R^j + \bar{u}_R^j M_{ij}^{U\dagger} u_L^i \tag{1.75}$$

Nobody assures us that the mass matrices will be diagonal, but we would like them to be diagonal since we expect the quarks, just like any other particle, to have a definite mass[13]. Since we don't have any constraint on the specific form of the mass matrix we just found, we do not know if it is possible to diagonalize it.

[12] Remember that whenever we speak about unitary gauge we're implying that we set the Goldstone boson "to zero", which is a way of saying that the gauge field eats the Goldstone boson gaining a new degree of freedom.

[13] The values of the masses of the quarks are quite difficult to define since they cannot be experimentally measured due to confinement.

1.4.5 On the Diagonalization of Matrices

We now prove that there exist a method through which we can diagonalize any matrix. This process is called *singular value decomposition* and it provides two unitary matrices L, R such that

$$L^\dagger M R = \hat{M}, \tag{1.76}$$

where we'll use the hatted matrix for the diagonal form of M.
From the generic matrix M we can construct two Hermitian matrices

$$MM^\dagger \qquad M^\dagger M \tag{1.77}$$

which in general do not commute. Provided that there is no singular eigenvalue and that the determinant is non-zero, we can easily prove that these matrices have the same eigenvalues. Indeed we gave

$$\begin{aligned} P_{MM^\dagger} &= \det\Big(MM^\dagger - \lambda\Big) = \det\{M\}\det\Big(M^\dagger - \lambda M^{-1}\Big) \\ &= \det\Big(M^\dagger - \lambda M^{-1}\Big)\det M = \det\Big(M^\dagger M - \lambda\Big) = P_{M^\dagger M}, \end{aligned} \tag{1.78}$$

and since both matrices have the same characteristic polynomial, they'll have the same eigenvalues. Being both Hermitian, we know that they can be diagonalized thanks to the spectral theorem and so there exist two matrices L and R which diagonalize the matrices to the same diagonal form since they have both the same eigenvalues

$$L(MM^\dagger)L^\dagger = \hat{D} = R(M^\dagger M)R^\dagger \tag{1.79}$$

Starting from this we define the following

$$M' = LMR^\dagger \qquad (M')^\dagger = RM^\dagger L^\dagger, \tag{1.80}$$

from which it is easy to see that

$$M'(M')^\dagger = (M')^\dagger M' = \hat{D}. \tag{1.81}$$

We know that we can always decompose a matrix into two Hermitian matrices as

$$M' = \left(\frac{M' + M'^\dagger}{2}\right) + i\left(\frac{M' - M'^\dagger}{2}\right) = H_1 + iH_2 \tag{1.82}$$

The two matrices we just defined H_1, H_2 are obviously diagonalizable since they hare Hermitian but we would like them to be diagonalizable by the same unitary matrix. From linear algebra, we know that this is possible if the two matrices commute! And it is easy to see that

$$[H_1, H_2] = \frac{1}{4i}\Big[M' + M'^\dagger, M' - M'^\dagger\Big] = \frac{1}{2i}\Big(M'M'^\dagger - M'^\dagger M'\Big) = 0 \tag{1.83}$$

Therefore there exists a unitary matrix W such that $W^\dagger M' W = \hat{M}'$ is a complex diagonal matrix and therefore, being complex diagonal we can put it in the form

$$\hat{M}' = \hat{M}\hat{U}_\varphi, \tag{1.84}$$

where $\hat{U}_\varphi = \text{diag}\left(e^{\phi_1}, e^{\phi_2}, \cdots, e^{\phi_N}\right)$ is a matrix of phases. Moreover

$$W^\dagger M' W = \hat{M}\hat{U}_\varphi = W^\dagger L^\dagger M R W \tag{1.85}$$

and therefore if we define

$$\tilde{L} = LW \qquad \tilde{R} = RW\hat{U}_\varphi^{-1} \tag{1.86}$$

we found the matrices that diagonalize M.

1.4.6 The CKM Matrix

Now that we know a way for diagonalizing any matrix, we can use it for the mass matrix for the quarks. Take the up quarks for example

$$\bar{u}_L^i \hat{m}_{ij}^u u_R^j = \bar{u}_L^{i\prime} (U_{u_L}^\dagger)_{ik} M_{kl}^U (U_{u_R})_{kj} u_R^{j\prime}, \tag{1.87}$$

where

$$\hat{m}^u = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_c & 0 \\ 0 & 0 & m_t \end{pmatrix} \tag{1.88}$$

and the new mass eigenstates are written in terms of the old ones as

$$u_L^i = (U_{u_L})_{ij} u_L^{j\prime} \qquad u_R^i = (U_{u_R})_{ij} u_R^{j\prime}. \tag{1.89}$$

To distinguish the quarks in the two basis, we put a prime on the current basis quarks.
Same thing goes for the down quarks where the diagonal form of the mass matrix will be

$$\hat{m}^d = \begin{pmatrix} m_d & 0 & 0 \\ 0 & m_s & 0 \\ 0 & 0 & m_b \end{pmatrix} \tag{1.90}$$

and the mass eigenstates

$$d_L^i = (U_{d_L})_{ij} d_L^{j\prime} \qquad d_R^i = (U_{d_R})_{ij} d_R^{j\prime}. \tag{1.91}$$

The kinetic terms do not change under this change of basis. In fact it is easy to see that, if we start from the original current base lagrangian

$$\begin{aligned} \mathcal{L} =& \left(\bar{u}_L' \quad \bar{d}_L'\right)^i \left[i\partial\!\!\!/ + \gamma_\mu \begin{pmatrix} \frac{g'}{6}B_\mu + \frac{g}{2}W_\mu^3 & \frac{g}{\sqrt{2}}W_\mu^+ \\ -\frac{g}{\sqrt{2}}W_\mu^- & \frac{g'}{6}B_\mu - \frac{g}{2}W_\mu' \end{pmatrix} \right] \begin{pmatrix} u_L' \\ d_L' \end{pmatrix}^i \\ &+ \bar{u}_R^{i\prime} \left(i\partial\!\!\!/ + g'\frac{2}{3}B\!\!\!/ \right) u_R^{i\prime} + \bar{d}_R^{i\prime} \left(i\partial\!\!\!/ - g'\frac{1}{3}B\!\!\!/ \right) d_R^{i\prime}, \end{aligned} \tag{1.92}$$

when we do the change of basis the unitarity of the transformation makes the matrices drop out since the hypercharge interactions are generation diagonal

$$i\sum_i \bar{u}_R^i D\!\!\!\!/\, u_R^i \underset{\text{flavour}}{\equiv} \bar{u}_R \mathbb{1} u_R \to \bar{u}_R \underbrace{U_{u_R}^\dagger \mathbb{1} U_{u_R}}_{\mathbb{1}} u_R = \bar{u}_R \mathbb{1} u_R \tag{1.93}$$

Moreover the same happens on the B_μ and W_μ^3 terms since these do not mix up and down-type quarks. This in turn makes the interaction with the photon unchanged. The mass term, as we expect, becomes diagonal

$$\begin{aligned}\mathcal{L}_{\text{mass}} &= \bar{d}_L^{i\prime} M_{ij}^D d_R^{j\prime} + \bar{u}_L^{i\prime} M_{ij}^U u_R^{j\prime} + h.c.\\ &= \bar{d}_L^i \left(U_{d_L} M_d U_{d_R}^\dagger\right)_{ij} d_R^j + \bar{u}_L^i \left(U_{u_L} M_u U_{u_R}^\dagger\right)_{ij} u_R^j + h.c.\\ &= \bar{d}_L^i \hat{m}_{ij}^D d_R^j + \bar{u}_L^i \hat{m}_{ij}^U u_R^j + h.c.\end{aligned} \tag{1.94}$$

The interesting bit comes out from the isospin doublet, the left part, where the two components change with different unitary matrices

$$Q_L^i = \begin{pmatrix} u_L^i \\ d_L^i \end{pmatrix} \to \begin{pmatrix} U_{u_L}^{ij} u_L^j \\ U_{d_R}^{ij} d_L^j \end{pmatrix}, \tag{1.95}$$

whenever the interaction mixes the two quark types. This happens with the $W^\pm$ couplings

$$\frac{g}{\sqrt{2}} W_\mu^+ \bar{u}_L' \gamma^\mu \mathbb{1} d_L' \underset{\text{flavour}}{\equiv} \bar{u}_L' \mathbb{1} d_L' = \bar{u}_L \underbrace{U_{u_L} \mathbb{1} U_{d_L}^\dagger}_{V_{CKM}} d_L, \tag{1.96}$$

where a new matrix in flavour space appears since we cannot use unitarity to reduce the $U_{u_L} U_{d_L}^\dagger$ term to the identity. This matrix is known as the *Cabibbo-Kobayashi-Maskawa (CKM) matrix*. The CKM matrix is a complex unitary matrix, and thus has nine real degrees of freedom, or three complex degrees of freedom. If V_{CKM} were real, it would be a $O(3)$ matrix, i.e. with three degrees of freedom. This means that out of the nine parameters of the complex CKM, three are angles and six are phases. However since the quark fields as mass eigenstates have a residual $U^6(1)$ symmetry

$$d_{L/R}^i = e^{i\alpha_i} d_{L/R}^i, \quad u_{L/R}^i = e^{i\beta_i} u_{L/R}^i. \tag{1.97}$$

Thus, we can use this freedom to set some phases to zero. Under these transformations, V_{CKM} generally transforms. However, if the two rotations are the same $\alpha_i = \beta_i$, the matrix remains unchanged. Therefore out of the 6 possible phases we could have set to zero, there remain only 5 possible combinations that effectively change the CKM matrix. Therefore there remain only one free phase in the CKM. The total remaining degrees of freedom are: three angles $\theta_{12}, \theta_{23}, \theta_{13}$, corresponding to rotations in the ij-flavour planes, and a phase δ. The angle θ_{12} is called the *Cabibbo angle* θ_C.

One possible representation of the CKM matrix is the following

$$V = \begin{pmatrix} V_{ud} & V_{uc} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}. \tag{1.98}$$

The presence of the phase reflects the CP violation of the weak charged currents. We will see other representations of the CKM matrix in later sections.

For completeness we give here the full interaction Lagrangian between the four gauge bosons γ, Z^0 and $W^\pm$. Starting from the diagonal interactions, which are the ones between the photon γ

$$\mathcal{L}_{\text{qA}} = i\sum_k \Big(\bar{u}_L^k\gamma^\mu\Big[\partial_\mu + ie\frac{2}{3}A_\mu\Big]u_L^k + \bar{u}_R^k\gamma^\mu\Big[\partial_\mu + ie\frac{2}{3}A_\mu\Big]u_R^k + \bar{d}_L^k\gamma^\mu\Big[\partial_\mu - i\frac{e}{3}A_\mu\Big]d_L^k + \bar{d}_R^k\gamma^\mu\Big[\partial_\mu - i\frac{e}{3}A_\mu\Big]d_R^k\Big) \tag{1.99}$$

and the Z^0 boson

$$\mathcal{L}_{\text{qZ}} = \sum_k \Big(- \bar{u}_L^k\gamma^\mu u_L^k\Big[\frac{eZ_\mu}{\sin 2\theta_W}\Big]\Big[1 - \frac{4}{3}\sin^2\theta_W\Big] + \bar{u}_R^k\gamma^\mu u_R^k\Big[\frac{eZ_\mu}{\sin 2\theta_W}\Big]\frac{4}{3}\sin^2\theta_W + \bar{d}_L^k\gamma^\mu d_L^k\Big[\frac{eZ_\mu}{\sin 2\theta_W}\Big]\Big[1 - \frac{2}{3}\sin^2\theta_W\Big] - \bar{d}_R^k\gamma^\mu d_R^k\Big[\frac{eZ_\mu}{\sin 2\theta_W}\Big]\frac{2}{3}\sin^2\theta_W\Big). \tag{1.100}$$

The non-diagonal interactions with the $W^\pm$ bosons are given by

$$\mathcal{L}_{\text{qW}} = -\frac{e}{\sqrt{2}\sin\theta_W}\sum_{ij}\Big(V_{CKM}^{ij}\bar{u}_{Li}\gamma^\mu d_{Lj}W_\mu^+ + \Big(V_{CKM}^{ij}\Big)^\dagger \bar{d}_{Li}\gamma^\mu u_{Lj}W_\mu^-\Big). \tag{1.101}$$

We note also that in the SM there are no *Flavour Changing Neutral Currents* (FCNC) at tree level. At loop level these are highly suppressed by the GIM mechanism [27].

1.4.7 Wolfenstein Parametrization and Standard Parametrization

The CKM matrix is usually parametrized in some specific way. The purpose of a specific parametrization is to incorporate in some way the unitarity condition of the CKM. A property that is used throughout all parametrization is the so-called *rephasing invariance* which is the possibility of changing the overall phase of any row, or any column, of the CKM matrix, without changing the physics contained in that matrix. Using this invariance, one usually sets V_{ud} and V_{us} to be real and positive[14].

As we discussed in the previous sections, we know that the CKM matrix depends on four parameters: three angles and one phase. This is, of course, independent of the specific parametrization used. One such parametrization, which does not highlight the number of free parameters, is the one given in equation eq. (1.98). There are better parametrizations that make clearer the nature of the CKM matrix. One such parametrization is the *standard parametrization*: the free parameters are three

[14]The reason for this choice has to do with a particular parameter that comes into play in the physics of the neutral kaon system.

angles $\theta_{12}, \theta_{13}, \theta_{23}$ and one complex phase δ and the CKM has the form

$$\begin{aligned} V &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_{23} & \sin\theta_{23} \\ 0 & -\sin\theta_{23} & \cos\theta_{23} \end{pmatrix} \\ &\times \begin{pmatrix} \cos\theta_{13} & 0 & \sin\theta_{13}e^{i\delta} \\ 0 & 1 & 0 \\ -\sin\theta_{13}e^{i\delta} & 0 & \cos\theta_{13} \end{pmatrix} \begin{pmatrix} \cos\theta_{12} & \sin\theta_{12} & 0 \\ -\sin\theta_{12} & \cos\theta_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix}, \end{aligned} \tag{1.102}$$

where we used the notation $s_{ij} = \sin\theta_{ij}$ and $c_{ij} = \cos\theta_{ij}$.

From the physical point of view, this matrix does not give us much more information than the original form of equation eq. (1.98). From this prospective the *Wolfenstein parametrization* is better suited. This parametrization incorporates some experimental informations from the measured moduli of the matrix elements, which we gave in equations eqs. (1.121) to (1.128). This parametrization it is based on the approximation of the various matrix elements in terms of $\lambda = \sin\theta_C \approx 0.22$ and is given, up to third order in λ, by

$$V = \begin{pmatrix} 1-\lambda^2/2 & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda & 1-\lambda^2/2 & A\lambda^2 \\ A\lambda^3(1-\rho-i\eta) & A\lambda^2 & 1 \end{pmatrix} + \mathcal{O}(\lambda^4), \tag{1.103}$$

where, with respect to the standard parametrization, we define [12, 15, 63]

$$\begin{aligned} s_{12} = \lambda = \frac{|V_{us}|}{\sqrt{|V_{ud}|^2 + |V_{us}|^2}}, \qquad s_{23} = A\lambda^2 = \lambda\left|\frac{V_{cb}}{V_{us}}\right|, \\ s_{12}e^{i\delta} = V_{ub}^* = A\lambda^3(\rho + i\eta) = \frac{A\lambda^3(\bar\rho + i\bar\eta)\sqrt{1-A^2\lambda^4}}{\sqrt{1-\lambda^2}[1 - A^2\lambda^4]\,(\bar\rho + i\bar\eta)]} \end{aligned} \tag{1.104}$$

which are called, together with λ, *Wolfenstein parameters*. These relations ensure that $\bar\rho + i\bar\eta = -(V_{ud}V_{ub}^*)/(V_{cd}V_{cb}^*)$ is phase convention independent and that the CKM matrix written in terms of the four parameters $A, \lambda, \bar\rho, \bar\eta$ is unitary to all orders in λ.

While $\lambda = 0.220658 \pm 0.00044$ and $A = 0.818 \pm 0.012$ [11] are relatively well known, the parameters ρ and η are much more uncertain. The main goal of CP-violation experiments is to over-constrain these parameters and, possibly, to find inconsistencies suggesting the existence of physics beyond the SM.

1.4.8 Unitarity Triangle

The condition on unitarity poses a strong constrain on the physics of the CKM matrix. In fact, from the unitarity condition $V^\dagger V = \mathbb{1}_{3\times 3}$ we can write six equations for the off-diagonal elements. For example, one such equation is of the form

$$\sum_{i=1}^{3} V_{id}V_{is}^* = 0. \tag{1.105}$$

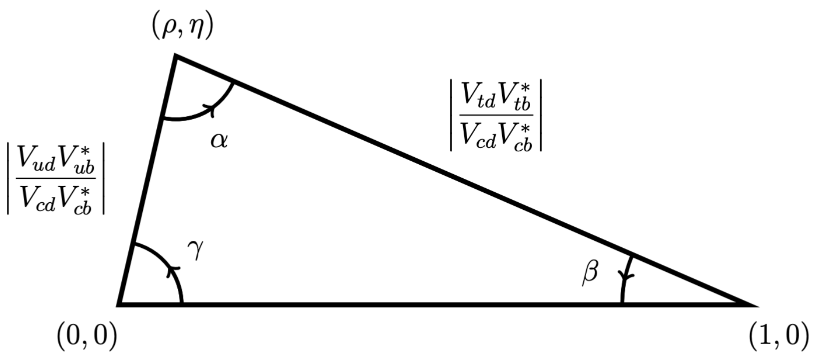
Every one of these equations define a triangle in the complex plane where each one of the legs of the triangle is one of the elements of the sum. These are called *unitarity triangles*. The Wolfenstein parametrization comes in very handy since from that of equation eq. (1.103) we can see that of the six unitary triangles, only two come with the same power of λ

$$O(\lambda^3): \quad \begin{array}{l} V_{ud}V_{ub}^* + V_{cd}V_{cb}^* + V_{td}V_{tb}^* = 0 \\ V_{td}V_{ud}^* + V_{ts}V_{us}^* + V_{tb}V_{ub}^* = 0. \end{array} \tag{1.106}$$

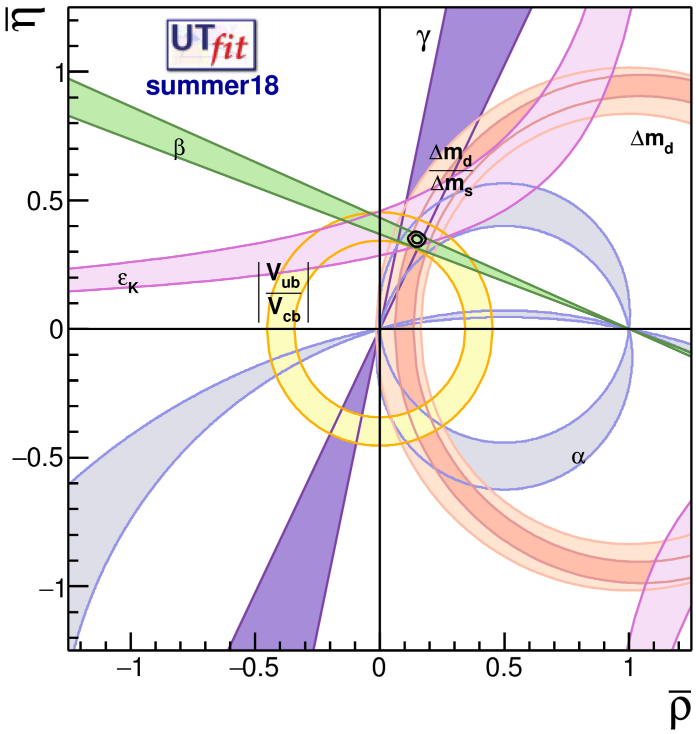
These two specific triangles are useful for the study of the B-meson decay. The other four triangles contain terms with different powers of λ and so they make up some squeezed triangles.
Above all this unitarity triangle there is *The Unitarity Triangle* (UT). This is given by the first of the relations in equation eq. (1.106) with the sides normalized by $V_{cd}V_{cb}^*$

$$\frac{V_{ud}V_{ub}^*}{V_{cd}V_{cb}^*} + 1 + \frac{V_{td}V_{tb}^*}{V_{cd}V_{cb}^*} = 0. \tag{1.107}$$

We can draw this relation in the complex plane as follows: we start from $(0,0)$ and then we move to $(1,0)$ using the second factor of eq. (1.107), then we take either one of the other two factors and go back to the origin. This process leaves us with the triangle in the complex plane in figure (1.2) alongside the experimental determination of $(\bar{\rho}, \bar{\eta})$



(a) Theoretical Unitarity Triangle.



(b) Experimental determination of $(\bar{\rho}, \bar{\eta})$, courtesy of the UTfit collaboration.

Figure 1.2. The Unitarity Triangle

The parameters $\bar{\rho}$ and $\bar{\eta}$ can be expressed in terms of the Wolfenstein parameters ρ and η by means of the following relations

$$\bar{\rho} = \rho\left(1 - \frac{\lambda^2}{2}\right) + \mathcal{O}(\lambda^4) \qquad \bar{\eta} = \eta\left(1 - \frac{\lambda^2}{2}\right) + \mathcal{O}(\lambda^4). \tag{1.108}$$

The angles in the Unitarity Triangle are defined[15] as

$$\begin{aligned} \alpha &\equiv \arg\left[-\frac{V_{td}V_{tb}^*}{V_{cd}V_{cb}^*}\right], \qquad & \beta &\equiv \arg\left[-\frac{V_{cd}V_{cb}^*}{V_{td}V_{tb}^*}\right], \\ \gamma &\equiv \arg\left[-\frac{V_{ud}V_{ub}^*}{V_{cd}V_{cb}^*}\right], \qquad & \beta_s &\equiv \arg\left[-\frac{V_{ts}V_{tb}^*}{V_{cs}V_{cb}^*}\right]. \end{aligned} \tag{1.109}$$

[15] Note this important feature: the definition of the angles are independent of any additional phase since any added phase to some quark is going to get cancelled by the ratio with the other elements. Equivalently, any CKM triangle can be rotated or scaled in the complex plane without modifying the angles that make them up.

1.5 Just a Taste: Flavour in the Standard Model

1.5.1 Global Symmetries

This peculiarity of the Yukawa interaction sparks an immediate question: what would be the full symmetry group of the SM if there wasn't any Yukawa interaction? To answer this, let us set for now all the Yukawa couplings to zero $Y_\ell, Y_U, Y_D = 0$[16]. Under this assumption, the whole global symmetry group of the SM is huge, in particular

$$G_{\text{SM}}(Y=0) = U(3)^5 = U(3)^3_{\text{q}} \times U(3)^2_\ell = SU(3)^3_{\text{q}} \times SU(3)^2_\ell \times U(1)^5, \qquad (1.110)$$

where we defined

$$U(3)^3_{\text{q}} = U(3)_{Q_L} \times U(3)_{u_R} \times U(3)_{d_R}, \qquad U(3)^2_\ell = U(3)_L \times U(3)_e. \qquad (1.111)$$

For the second equality of eq. (1.110) the isomorphism[17] $U(3) \simeq SU(3) \times U(1)$ is used, together with the definition

$$U(1)^5 = U(1)_B \times U(1)_L \times U(1)_Y \times U(1)_{\text{PQ}} \times U(1)_e. \qquad (1.112)$$

Of the residual five charges we identify the first three with the *baryon number*, the *lepton number* and the hypercharge. These are the ones that are not broken by the Yukawa interactions. The remaining two are identified by the *Peccei-Quinn symmetry*. The important thing is that the Lagrangians eqs. (1.5) and (1.6) are invariant under the flavour symmetry group $SU(3)^3_q \times SU(3)^2_\ell$. The Yukawa interactions break this symmetry, leaving us with the residual global symmetry

$$G_{\text{SM}}(Y \neq 0) = U(1)_B \times U(1)_e \times U(1)_\mu \times U(1)_\tau. \qquad (1.113)$$

Therefore the SM is not flavour invariant.

1.5.2 Counting the Physical Parameters

The discussion made until here may seem arbitrary but in fact is very useful if we want to count the number of independent parameters in the Yukawa coupling matrices.

Let us start with the Yukawa sector for the quarks: how many independent parameters does $\mathcal{L}^{\text{q}}_{\text{Y}}$ have? Consider a more general theory where the number of flavours is n. The two Yukawa matrices Y_U, Y_D are two 3×3 complex matrices, which means that they both have n^2 real parameters and n^2 imaginary ones. The kinetic part

[16] Physically we can think of this as studying the theory at an energy where the Yukawa couplings are negligible.

[17] This isomorphism is not given by the direct product. In fact there is a short exact sequence of Lie groups

$$\mathbf{1} \to SU(n) \to U(n) \xrightarrow{\det} U(1) \to \mathbf{1}$$

and therefore $U(n)$ is given by the semidirect product

$$U(n) \simeq SU(n) \rtimes U(1).$$

of the Lagrangian for the quarks has a global $U(n)^3$ symmetry that enables us to constrain $3n^2$ parameters. But the baryon number is a broken symmetry, which means that we need to remove one constrain from the global symmetry. So in the end we have

$$N_{\text{indep.}} = 2 \times 2n^2 - 3n^2 + 1 = n^2 + 1. \tag{1.114}$$

How many independent parameters are imaginary? To find them consider the limit in which $Y = 0$. In this limit, the Lagrangian is $SO(n)^3$ symmetric which implies that we can remove $3n(n-1)/2$ parameters. Therefore the number of real parameters is just

$$N_{\text{Re}} = 2n^2 - \frac{3n(n-1)}{2} = n + n + \frac{n(n-1)}{2}. \tag{1.115}$$

The division in equation eq. (1.115) is not at random: the first n real parameters are the masses of the n d-type quarks while the other n is the number of the masses for the u-type quarks. The third factor is the number of mixing angles. We can find now the number of complex phases

$$N_{\text{indep.}} = N_{\text{Re}} + \frac{(n-1)(n+1)}{2}. \tag{1.116}$$

Taking the limit of $n \to 3$ we get that the Yukawa matrices Y_D, Y_U, can be expressed in terms of 9 real parameters (three masses for the down quarks, three masses for the up quarks and three mixing angles) and one complex phase. This complex phase is crucial since it cannot be eliminated by any change of basis and, as we will see later, enables the possibility of CP violation in weak decays.

Now we do the same for the Yukawa coupling of the leptons. Here we have only one Yukawa matrix Y_ℓ that has in principle n^2 real parameters and n^2 imaginary ones. In the limit where no Yukawa interaction is present, the kinetic term for the leptons has a global $U(n)^2$ symmetry which constrains $2n^2$ parameters. In analogy with what we have done for the quarks, we can use the residual $SO(n)$ symmetry to eliminate $n(n-1)$ parameters in such a way that the number of real independent parameters becomes

$$N_{\text{Re}} = n^2 - n(n-1) = n. \tag{1.117}$$

These n parameters are exactly the n masses of the charged leptons. Since in the broken phase we still have a residual $U(1)^n$ symmetry (one for every lepton) the total number of parameters becomes

$$N_{\text{indep.}} = 2n^2 - 2n^2 + n = n = N_{\text{Re}}. \tag{1.118}$$

This means that no matter what, in the lepton sector there are *no complex phases*. This means that in the lepton sector there cannot be any CP violation and mixing!

Based on the arguments given in the previous sections, this is exactly what we expected. After symmetry breaking the leptons all got a mass matrix which was diagonal while the quarks didn't. By diagonalizing the quarks mass matrix we introduced the possibility of mixing in the weak sector and, buy the nature of the initial mass matrix, we could not require that the mixing matrix be real. By an analogous computation we found that the number of parameters in the mixing matrix was four: three angles and one complex phase. Plus we had the six masses of the quarks. The complex CKM matrix is pivotal in the analysis that we will carry in the following sections. By this mean, we can now complete the discussion with an in-depth study of the CKM matrix.

1.6 What's so Special About the CKM Matrix?

1.6.1 Interaction Vertices

Now that we have a complete theory of weak interactions we can start constructing Feynman diagrams and evaluating some measurable quantities. It turns out that whenever we have a flavour changing current we'll need now to insert in the interaction vertex one of the possible elements of the CKM matrix.

Let's take for example the pion decay. At the level of the hadrons, the decay is given by

$$\pi^+ \to \mu^+ \nu_\mu \tag{1.119}$$

which, at the level of the quarks is given at tree-level by the following Feynman diagram

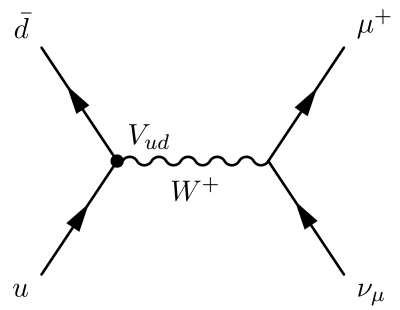


Figure 1.3. Pion decay at tree level mediated by a W^+ boson

The amplitude for such Feynman diagram is given, in the unitary gauge, by

$$\left(\frac{g}{2\sqrt{2}}\right)^2 V_{ud}^* \bar{v}_d \gamma^\mu (1-\gamma_5) u_u \frac{1}{q^2 - M_W^2} \left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{M_W^2}\right) \bar{u}_\mu \gamma^\nu (1-\gamma_5) v_\nu, \tag{1.120}$$

where an additional V_{ud} term appears with respect to the naive amplitude and $q = \sqrt{s}$. This factor can greatly suppress some processes for which the CKM matrix element is very small.

1.6.2 How to Measure the CKM Elements

The CKM matrix elements are usually measured from leptonic and semileptonic decays. With the top quarks the processes are a little bit more difficult since it is mass prevents it from forming bound states with other quarks. In that case we use hadron mixing; we won't go into much detail about it. Due to the fact that in the theoretical predictions, only the square of the amplitude for the process appears, it is not possible to measure experimentally the precise value of the CKM matrix elements but rather we can measure either the moduli of them or the difference in phases between two.

Here we give some details on how the CKM moduli are measured experimentally and their values as they appear in the PDG [66]. All of the processes governed by weak decays are proportional to some power of the Fermi constant $G_F = \frac{\sqrt{2}g_2^2}{8M_W^2}$. This is precisely measured from the decay of the muon to the electron.

$|\mathbf{V_{ud}}|$ This matrix element involves only quarks of the first generation and is thus the one which can be best determined. There are basically three ways of measuring it. The first one involves superallowed Fermi transitions, which are beta decays connecting two $J^P = 0^+$ nuclides in the same isospin multiplet. The second one is by using the neutron decay which at tree level is given by the diagram in figure (1.4).

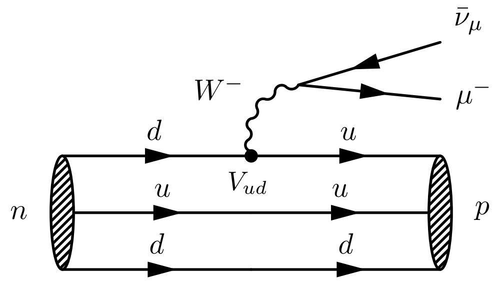


Figure 1.4. Neutron decay at tree level.

The moduli of this matrix element, as given by the PDG [33], comes out to be

$$|V_{ud}| = 0.97370 \pm 0.00014 \tag{1.121}$$

$|\mathbf{V_{us}}|$ This matrix element can be extracted from the analysis of semileptonic decays of the K-meson such as the one in figure (1.5).

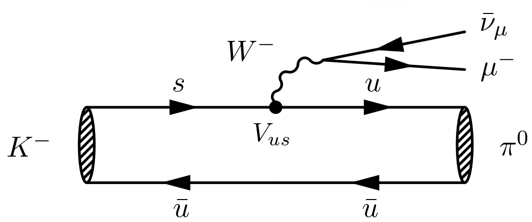


Figure 1.5. $K^- \to \pi^0 \mu^- \bar{\nu}_\mu$ at tree level

This matrix element comes out to be [6]

$$|V_{us}| = 0.2245 \pm 0.0008. \tag{1.122}$$

Another interesting way to measure the value of this matrix element is by hyperon decays [14].

$|\mathbf{V_{ub}}|$ This matrix element can be measured from the semileptonic decay of B meson such as the one in figure (1.6).

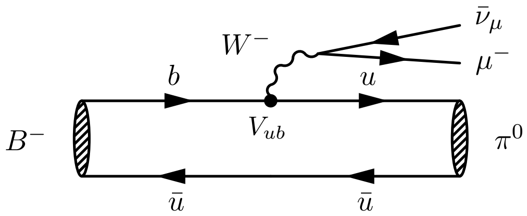


Figure 1.6. $B^- \to \pi^0 \mu^- \bar{\nu}_\mu$ at tree level

From these processes, we find a value for the moduli of this matrix element of [44]

$$|V_{ub}| = 0.00382 \pm 0.00024. \tag{1.123}$$

$|\mathbf{V_{cd}}|$ This matrix element can be measured from the semileptonic decay of charmed particles like the D meson. One such process can be found in figure (1.7).

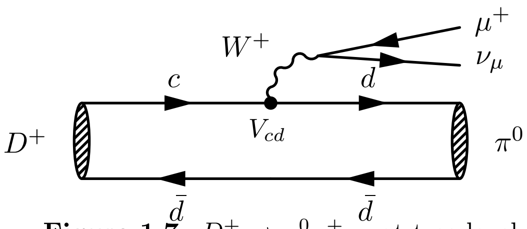


Figure 1.7. $D^+ \to \pi^0 \mu^+ \nu_\mu$ at tree level

The moduli of such matrix element come out to be [41, 24]

$$|V_{cd}| = 0.221 \pm 0.004. \tag{1.124}$$

$|\mathbf{V_{cs}}|$ This matrix element can be found by semileptonic decays of charmed particles like the D meson in which the c quark goes into a s quark. The lightest meson which contains an s quark is the K meson. Such decay is depicted at tree level in figure (1.8)

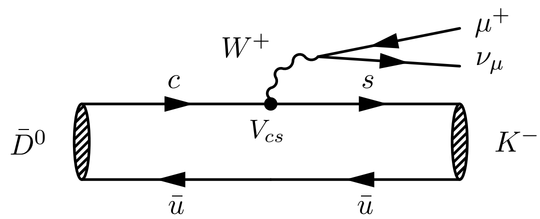


Figure 1.8. $\bar{D}^0 \to K^- \mu^+ \nu_\mu$ at tree level

This matrix element comes out to be [18, 65, 2, 3, 5]

$$|V_{cs}| = 0.978 \pm 0.011. \tag{1.125}$$

$|\mathbf{V_{cb}}|$ This matrix element is determined by the decay $B^0 \to D^{*-} l^+ \nu_l$ like the one in figure (1.9).

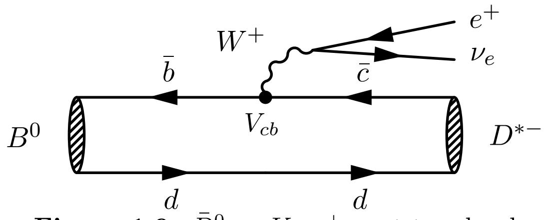


Figure 1.9. $\bar{D}^0 \to K^- \mu^+ \nu_\mu$ at tree level

For these decays, we gathered a lot of measurements that got us a value of [44]

$$|V_{cb}| = 0.041 \pm 0.0014. \tag{1.126}$$

$|\mathbf{V_{td}}|$ $|\mathbf{V_{ts}}|$ These are two of the three matrix elements which involve the top quark. These matrix elements are measured by neutral meson mixing like $B^0 - \bar{B}^0$ and $B_s - \bar{B}_s$. The diagrams describing these oscillations, which will be of fundamental importance for CP violation in later sections, are so-called *box diagrams*. One such diagram is depicted in figure (1.10).

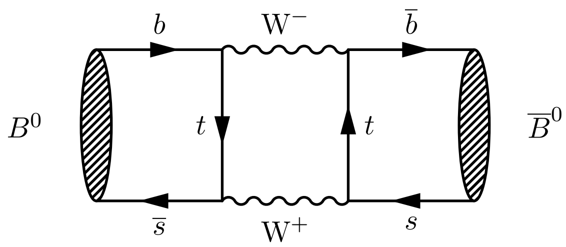


Figure 1.10. Relevant box diagram for the measurement of $|V_{td}|$ and $|V_{ts}|$.

By evaluating these processes on the lattice assuming $V_{tb} = 1$

$$|V_{td}| = 0.0080 \pm 0.0003 \qquad |V_{ts}| = 0.0388 \pm 0.0011. \tag{1.127}$$

$|\mathbf{V_{tb}}|$ This matrix element has been found by the CDF [4] and D0 [1] collaborations by measuring the branching ratio $\text{Br}(t \to Wb)/\text{Br}(t \to Wq)$ assuming three generations of quarks. The value they gave is

$$|V_{tb}| = 1.013 \pm 0.030. \tag{1.128}$$

1.6.3 CP-violation in the SM

We can now briefly discuss how the complex phase in the CKM breaks CP-invariance in the weak charged currents.

First, we need to see how the two discrete symmetries, parity and charge conjugation, act on the relevant fields for our analysis. These fields are bosonic vector fields, for the $W^\pm$, and four-spinors Ψ, for the quarks. Starting from parity, let us define a state of a single particle A with momentum p and other quantum numbers σ as $|A, p, \sigma\rangle$. The parity operator will act on this state as

$$\mathcal{P} \, |A, p, \sigma\rangle = \eta_A \, |A, -p, \sigma\rangle \tag{1.129}$$

since, at the classical level, parity just inverts the coordinates. On the antiparticle state the operator will act similarly, but with another rephasing

$$\mathcal{P}\left|\bar{A},p,\sigma\right\rangle = \eta_{\bar{A}}\left|\bar{A},-p,\sigma\right\rangle. \tag{1.130}$$

Given the creation operator $a_\sigma^\dagger$ we have

$$\mathcal{P}a_\sigma^\dagger(p)\left|0\right\rangle = a_\sigma^\dagger(-p)\left|0\right\rangle\eta_A \tag{1.131}$$

henceforth

$$\begin{aligned}&\mathcal{P}a_\sigma^\dagger(p)\mathcal{P}^{-1}\mathcal{P}\left|0\right\rangle = \mathcal{P}a_\sigma^\dagger(p)\mathcal{P}^{-1}\left|0\right\rangle = \eta_A a_\sigma^\dagger(-p)\left|0\right\rangle \\ &\implies \mathcal{P}a_\sigma^\dagger(p)\mathcal{P}^{-1} = \eta_A a_\sigma^\dagger(-p).\end{aligned} \tag{1.132}$$

Similarly for the antiparticle. Now let us consider a scalar field $\Phi(x)$ and its expansion in creation and annihilation operators, then

$$\begin{aligned}\mathcal{P}\Phi(x)\mathcal{P}^{-1} &= \int\frac{\mathrm{d}^3p}{(2\pi)^3 2E_p}\left[\mathcal{P}a_p\mathcal{P}^{-1}e^{-ipx} + \mathcal{P}b_p^\dagger\mathcal{P}^{-1}e^{ipx}\right] \\ &= \int\frac{\mathrm{d}^3p}{(2\pi)^3 2E_p}\left[\eta_A^* a_{-p}e^{-ipx} + \eta_{\bar{A}}b_{-p}^\dagger e^{ipx}\right] \doteq \eta_\Phi^*\Phi(\mathcal{P}x).\end{aligned} \tag{1.133}$$

In order for the equality to hold, we need to have $\eta_A^* = \eta_{\bar{A}} = \eta_\Phi^*$ which implies $\eta_A\eta_A = 1$. What we need to define still is the intrinsic phase η_Φ^*. Doing a similar thing for a spinor field, defining its transformation property under parity as

$$\mathcal{P}\phi(x)\mathcal{P}^{-1} \doteq \eta_A^*\gamma^0\psi(\mathcal{P}x) \tag{1.134}$$

we get a slightly different result

$$\begin{aligned}\mathcal{P}\psi(x)\mathcal{P}^{-1} &= \int\frac{\mathrm{d}^3p}{(2\pi)^3\sqrt{2E_p}}\left[\eta_A^* a_\sigma(-p)u^\sigma(p)e^{-ipx} + \eta_{\bar{A}}b_\sigma^\dagger(-p)v^\sigma(p)e^{ipx}\right] \\ &= \int\frac{\mathrm{d}^3p}{(2\pi)^3\sqrt{2E_p}}\left[\eta_A^* a_\sigma(-p)\gamma^0u^\sigma(-p)e^{-ipx} - \eta_{\bar{A}}b_\sigma^\dagger(-p)\gamma^0v^\sigma(-p)e^{ipx}\right]\end{aligned} \tag{1.135}$$

which implies $\eta_{\bar{A}} = -\eta_{\bar{A}}^*$, which depends only on the choice of $\eta_{\bar{A}}$. For a vector field we get a similar result as for the calar field $\mathcal{P}A^\mu(x)\mathcal{P}^{-1} = -\eta_A^*\mathcal{P}^\mu{}_\nu A^\nu(\mathcal{P}x)$.
Next, there is charge conjugation. This can be found by imposing that the following conditions hold: take a spinor field ψ and denote its charge conjugate as $\psi^c = \mathcal{C}\bar{\psi}^T$, then

$$\bar{\psi}^c\bar{\psi}^c = \bar{\psi}\psi \qquad \bar{\psi}^c\partial\!\!\!/\psi^c = \bar{\psi}\partial\!\!\!/\psi. \tag{1.136}$$

From the first condition, we get that

$$\begin{aligned}\left(\psi_c^\dagger\right)_\alpha\left(\gamma^0\right)_{\alpha\beta}(\psi_c)_\beta &= \psi_\gamma\left(\mathcal{C}^\dagger\right)_{\gamma\alpha}\left(\gamma^0\right)_{\alpha\beta}(\mathcal{C})_{\beta\sigma}\psi_\sigma^\dagger = -\psi_\sigma^\dagger\left(\mathcal{C}^\dagger\gamma^0\mathcal{C}\right)_{\gamma\sigma}\psi_\gamma \\ &= \psi_\sigma^\dagger\left(\gamma^0\right)_{\sigma\gamma}\psi_\gamma = \psi_\sigma^\dagger\left(\gamma^0\right)^T_{\gamma\sigma}\psi_\gamma\end{aligned}$$

from which we can extract

$$\mathcal{C}^\dagger \gamma^0 \mathcal{C} = -\gamma^0. \tag{1.137}$$

From the second condition we get

$$\begin{aligned}\left(\psi_c^\dagger\right)_\alpha \left(\gamma^0\gamma^\mu\right)_{\alpha\beta} (\partial_\mu \psi_c)_\beta &= \psi_\sigma \left(\mathcal{C}^\dagger\right)_{\sigma\alpha} \left(\gamma^0\gamma^\mu\right)_{\alpha\beta} (\mathcal{C})_{\beta\gamma} \left(\partial_\mu \psi^\dagger\right)_\gamma \\ &= \psi_\gamma^\dagger \left(\gamma^0\gamma^\mu\right)_{\gamma\sigma} (\partial_\mu \psi)_\sigma.\end{aligned} \tag{1.138}$$

By comparison, we can say

$$\mathcal{C}^\dagger \left(\gamma^0\gamma^\mu\right) \mathcal{C} = \left(\gamma^0\gamma^\mu\right)^T = (\gamma^\mu)^T \gamma^0 \tag{1.139}$$

For $\mu = 0$ it is easy to show from equation 1.139 that the operator $\mathcal{C}$ is Hermitian. Adding the unitary quantity $\gamma^0\gamma^0$, from equation 1.139 we get

$$\mathcal{C}\left(\gamma^0\gamma^\mu\gamma^0\gamma^0\right)\mathcal{C} = \mathcal{C}\gamma^{\mu\dagger}\gamma^0\mathcal{C} = (\gamma^\mu)^T\gamma^0 \tag{1.140}$$

Now we multiply both the sides by γ^0 and we exploit the anticommutation between $\mathcal{C}$ and γ^0 since the equation 1.137 is true, so we get

$$\begin{aligned}\mathcal{C}\gamma^{\mu\dagger}\gamma^0\mathcal{C}\gamma^0 &= (\gamma^\mu)^T \\ -\mathcal{C}\gamma^{\mu\dagger}\mathcal{C} &= (\gamma^\mu)^T \\ \mathcal{C}\gamma^\mu\mathcal{C} &= -\gamma^{\mu *}\end{aligned} \tag{1.141}$$

The only thing to do now is to determine the operator. We know that γ^0, γ^1, γ^3 are real and symmetric (like also γ_5) whereas γ^2 is imaginary. Since $\mathcal{C}$ anticommutes with γ^0, γ^1, γ^3 and commutes with γ^2, imposing the condition $\mathcal{C}^2 = 1$ we can define the operator up to a sign

$$\mathcal{C} = i\gamma^2. \tag{1.142}$$

With this definition of charge conjugation we get that a scalar and a pseudovector are unchanged under charge conjugation, vectors and tensors change sign.
Therefore we have

$$\begin{aligned}\mathcal{CP}\, W_\mu^+\, \mathcal{CP}^{-1} &= -e^{i\xi_W} \eta^\nu{}_\mu W_\nu^- \\ \mathcal{CP}\, W_\mu^-\, \mathcal{CP}^{-1} &= -e^{i\xi_W} \eta^\nu{}_\mu W_\nu^+ \\ \mathcal{CP}\, Z^\mu\, \mathcal{CP}^{-1} &= -\eta_Z \eta^\mu{}_\nu Z^\nu \\ \mathcal{CP}\, h\, \mathcal{CP}^{-1} &= \eta_h h \\ \mathcal{CP}\, u^i\, \mathcal{CP}^{-1} &= e^{i\phi_{u^i}} \gamma^0 \mathcal{C} \bar{u}^{iT} \\ \mathcal{CP}\, d^i\, \mathcal{CP}^{-1} &= e^{i\phi_{d^i}} \gamma^0 \mathcal{C} \bar{d}^{iT}\end{aligned} \tag{1.143}$$

The question is now: is it possible to choose the phases $\xi_W, \eta_Z, \eta_h, \phi_{u^i}, \phi_{d^i}$ in such a way that the weak Lagrangian is CP invariant? Let's go directly to the culprit, the W boson interactions with quarks. What we have is the following

$$\begin{aligned}&\mathcal{CP}\left[\frac{g}{\sqrt{2}} W_\mu^+ \bar{u}_L^i \gamma^\mu d_L^j V_{ij} + V_{ij}^* \bar{d}_L^j \gamma^\mu u_L^i W_\mu^-\right] \mathcal{CP}^{-1} \\ &= \frac{g}{\sqrt{2}} V_{ij} \left(W_\mu^- e^{-i\xi_W}\right) e^{i(\phi_d^j - \phi_u^i)} \left(-\bar{d}_L^j \gamma^\mu u_L^i\right) + \frac{g}{\sqrt{2}} V_{ij}^* (W_\mu^+ e^{i\xi_W}) e^{i(\phi_u^i - \phi_d^j)} \left(-\bar{u}_L^i \gamma^\mu d_L^j\right).\end{aligned} \tag{1.144}$$

The condition for invariance is given by

$$V_{ij}^* e^{-i(\xi_W+\phi_d^j-\phi_u^i)/2} = V_{ij} e^{i(\xi_W+\phi_d^j-\phi_u^i)/2} \tag{1.145}$$

which can only be true if we can choose the phases such that the product between the phase and the CKM matrix elements is real. But we know that the CKM matrix has a complex phase that cannot be reabsorbed in any way and that is present in every choice of basis. This means that the equality eq. (1.145) cannot hold and therefore we have CP violation.

1.6.4 The Jarlskog Invariant

Now that we're familiar with the existence of the CP-violating phase, we would like to be able to quantify it in a meaningful way that is manifestly basis-independent. What we need is some kind of invariant that identifies CP violation. Such an object exists and it is called the Jarlskog invariant, J [19, 28, 40, 39, 64]. It is defined by

$$\text{Im}\left[V_{ij}V_{kl}V_{i\ell}^*V_{kj}^*\right] = J\sum_{mn}\epsilon_{ikm}\epsilon_{j\ell n},$$

where there is no implicit sum on the left-hand side. In terms of the Wolfenstein parametrization, this corresponds to

$$J = c_{12}c_{23}c_{13}^2 s_{12}s_{23}s_{13}\sin\delta \approx \lambda^6 A^2 \eta$$

This parametrization-independent quantity measures the amount of CP violation in our model. The most remarkable observation is that it depends on every physical mixing angle. Thus if any of the mixing angles are zero, there would be no CP violation. In fact, we can see that the amount of CP violation in the Standard Model is small, but it is not small because the CP phase δ is small. Quite on the contrary, it is small because of the mixing angles. We can see this in the Wolfenstein parametrization where the Jarlskog invariant comes along with six powers of λ.
The experimental value of J is [66]

$$J = (3.0^{+0.15}_{-0.09}) \times 10^{-5}. \tag{1.146}$$

Chapter 2

Quantum Chromodynamics and Strong Interaction

The strong interaction is the missing piece of the Standard Model which is described by the $SU(3)$ symmetry group. Of the three fundamental forces that the SM can explain, the strong force is by far the most complicated. This force governs the behaviour of quarks and how they bind together to form composite particles which we call *hadrons*. Not only that, but on a larger scale, it also governs how the nuclei of different atoms bind together and their stability. At these two different scales, the particles that mediate the interactions are different: on the smaller, hadronic, scale the force mediator is called *gluon*, while at the larger scale the carriers are light mesons such as the pion. Quantum chromodynamics is the Quantum Field Theory that describes the interactions of *colored* particles. Color is the quantum number of the $SU(3)$ symmetry group in the SM.

What makes the strong force so difficult to work with and to do actual calculations, it its highly non-perturbative structure at low enough energy scales. Perturbation theory is an essential bit of mathematics that enables us to carry out specific computations and without it, unless the model is exactly solvable[1], we cannot go any further. But then, how come that we in fact do calculations which depend on strong dynamics? We use a tool called *lattice QCD*[2]. This approach was introduced by Wilson [62] and is an approximation scheme in which the continuum gauge theory is replaced by a discrete statistical mechanical system on a four-dimensional Euclidean lattice. The basic idea of lattice QCD is to employ a specific ultraviolet regulator, i.e. the lattice, on which we can do computations exactly. Of course, such a tool comes with its benefits and with its drawbacks. One such drawback is that by discretizing space-time, we lose one of the foundations of QFT which is Lorentz invariance. Moreover, a lattice calculation requires some very intensive computations which, in turn, requires a very powerful computer. Although we are going to use heavily the ideas of lattice QCD and lattice regularization, we won't go into much details on how such lattice computations are done, what is important is that even in

[1] Sadly we know far too well that most of the actual physical models are not exactly solvable by analytical methods.

[2] This is not the only tool. There are also other theories like the $1/N_c$ [57] expansion or Chiral Perturbation Theory (ChPT).

the non-perturbative regime, physicist know how to do calculations.
The fundamental property of QCD, which causes all the problems we write up until now, is *asymptotic freedom*. What this means is that, at high enough energy, QCD can be treated perturbatively: the theory becomes asymptotically free $g_s(\Lambda) \overset{\Lambda\to\infty}{\longrightarrow} 0$, where Λ is some energy scale. That's right, the coupling depends on the energy at which the process takes place. This is a general feature of any QFT and comes from the process of reabsorbing infinities in a process called *renormalization*. We will see how this process will come into play in this chapter.

2.1 The QCD Lagrangian

The QCD Lagrangian is constructed by taking the kinetic term of the relevant gauge field and the kinetic term for the fermions. This leaves us with a Lagrangian of the form

$$\mathcal{L}_{\text{QCD}} = -\frac{1}{4} G^{\text{a}}_{\mu\nu} G^{\text{a}\mu\nu} + \bar{\psi}^{\text{a}}_q \Big(i \not{D}^{\text{ab}} - m\delta^{\text{ab}} \Big) \psi^{\text{b}}_q, \tag{2.1}$$

in which the field G^{a}_μ are the gluons and the fields ψ_q are the quarks. But reality, as always, is more complicated than this. What we glossed over up until now is that gauge freedom completely breaks down our possibility to do calculations. This is due to the fact that when studying QFTs one makes use of an object called *generating functional* $Z[J]$. The generating functional is a functional[3] which is used to find, in any order in perturbation theory, the value of some Feynman diagram: it "generates" at any order, the relevant perturbative terms and is in this bit of mathematics that the problem lies. Take a general gauge field A^a_μ associated with some gauge group G. Let $S[A]$ be the action[4] functional, then the generating functional is defined in Minkowski space as

$$Z[J^\mu_a] = \int \mathcal{D}A \, \exp\left[iS[A] - i \int \mathrm{d}^4x \, A^a_\mu(x) J^\mu_a(x) \right], \tag{2.2}$$

where J^μ_a are $\dim G$ external currents. Then, green functions can be found by functionally deriving $Z[J]$ and setting the external current to zero. For example, the two-point function (propagator) of the gauge field, is found by

$$G^{ab}_{\mu\nu}(x) = \langle 0 | A^a_\mu(x) A^b_\nu(0) | 0 \rangle = \frac{1}{Z[0]} \frac{\delta}{\delta J^\mu_a} \frac{\delta}{\delta J^\nu_b} Z[J] \bigg|_{J=0}. \tag{2.3}$$

If we have some additional fields which interact with the gauge field, just like our initial QCD Lagrangian, equation like eq. (2.3), gives the complete two-point function which incorporates all the possible corrections to the free propagator. But what is the integration measure in the definition of the generating functional of equation eq. (2.2)? Broadly speaking it means that we need to integrate over all possible field configurations. But here's the catch: gauge freedom makes any configuration part of an infinite equivalence class of configurations where the physics is unchanged and we need to integrate over all of them. It is not difficult to see that even naively,

[3] Incredible!
[4] The action is defined as the integral over spacetime of the Lagrangian density.

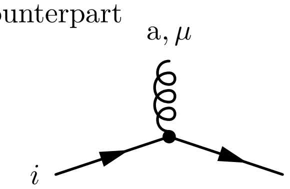
this makes the integration measure divergent. To solve this problem we need to fix the gauge in such a way that the physical result is independent of such a fix but such that makes the integration well defined. The procedure to fix the gauge in a non-Abelian theory was found by Fadeev and Popov[5] [21]. Without going into much more details, what they discovered is that by fixing the gauge there appeared some new fictitious particles which we call *Ghosts*. Ghosts are in no way physical particles, they cannot be measured, they are not real, but they are only a result of the mathematical process of fixing the gauge. This particles appear in the Lagrangian as a bosonic term but they are described by Grassman variables and so obey Fermi-Dirac statistic.
There are a whole plethora of possible gauges: covariant gauges, axial gauges, non-linear gauges, and so on. In the simplest case of the covariant gauge-fixing $\partial^\mu G^{\mathrm{a}}_\mu = 0$, the full QCD, gauge-fixed, Lagrangian in all of its glory is given by

$$\mathcal{L}_{\mathrm{QCD}} = -\frac{1}{4} G^{\mathrm{a}}_{\mu\nu} G^{\mathrm{a}\mu\nu} - \frac{1}{2\alpha}\left(\partial^\mu G^{\mathrm{a}}_\mu\right)^2 + (\partial_\mu \chi_{\mathrm{a}})^* D^{\mathrm{ab}}_\mu \chi_{\mathrm{b}} + \bar{\psi}_q (i\not{D} - m)\psi_q, \qquad (2.4)$$

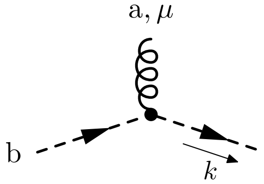
where α is the gauge parameter, χ_i are the Ghost fields and $D^{\mathrm{ab}}_\mu = \delta^{\mathrm{ab}}\partial_\mu - g_s f^{\mathrm{abc}} A^{\mathrm{c}}_\mu$ is the covariant derivative in the adjoint representation of $SU(3)$.

2.2 Perturbative QCD

We are now ready to start the analysis of QCD in the perturbative regime. To employ perturbation theory, we need to choose an energy scale such that quarks and gluons become the asymptotic states of the theory instead of the hadrons.
Given the Lagrangian eq. (2.4) we can extrapolate the various interaction vertices. The non-Abelian nature of the $SU(3)$ symmetry group adds some interesting interactions such as three- and four-gluon vertices which in a simpler theory like QED are not present. We now give a short list of the Feynman rules for the vertices and their mathematical counterpart

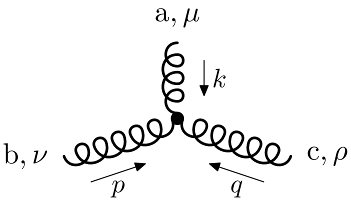


$$= i g_s \gamma^\mu t^{\mathrm{a}} \qquad (2.5)$$

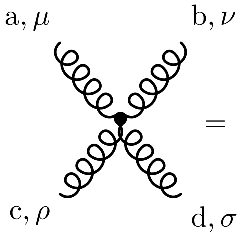


$$= -i g_s f^{\mathrm{abc}} k^\mu \qquad (2.6)$$

[5] There is still a problem with this approach since fixing an orbit in the guage configuration space can result in ambiguities, the so-called *Gribov ambiguity* [29]

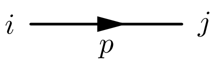


$$= g f^{\mathrm{abc}}[g^{\mu\nu}(k-p)^{\rho} + g^{\nu\rho}(p-q)^{\mu} + g^{\rho\mu}(q-k)^{\nu}] \tag{2.7}$$

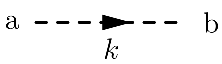


$$\begin{aligned} = \; & -ig^2 \Big[f^{\mathrm{abe}} f^{\mathrm{cde}} (g^{\mu\rho} g^{\nu\sigma} - g^{\mu\sigma} g^{\nu\rho}) \\ & + f^{\mathrm{ace}} f^{\mathrm{dbe}} (g^{\nu\mu} g^{\rho\sigma} - g^{\mu\sigma} g^{\nu\rho}) \\ & + f^{\mathrm{ade}} f^{\mathrm{bce}} (g^{\mu\nu} g^{\rho\sigma} - g^{\mu\rho} g^{\nu\sigma}) \Big] \end{aligned} \tag{2.8}$$

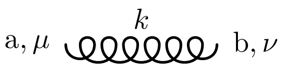
For the later chapters, we also need the free propagators of both the quarks and the gluon derived from the associated free Lagrangian and come out to be



$$= \frac{-i}{\not{p} - m}\delta_{ij} = \frac{i(\not{p} - m)}{p^2 - m^2}\delta_{ij} \tag{2.9}$$



$$= \frac{i\delta^{\mathrm{ab}}}{k^2} \tag{2.10}$$



$$= \frac{i\delta^{\mathrm{ab}}}{k^2}\left(-g_{\mu\nu} + (1-\alpha)\frac{k^{\mu}k^{\nu}}{k^2 + i\epsilon}\right), \tag{2.11}$$

where α is the covariant gauge-fixing constant appearing in the Lagrangian eq. (2.4).

2.3 Renormalization of QCD

A general feature of most QFTs is the presence of divergences in the perturbative series when evaluating loop diagrams. Since we know, experimentally, that the results should be finite we need a way to systematically eliminate the unphysical divergences. This is done by means of renormalization [17]. In order to deal with divergences that appear at the quantum[6] corrections to Green functions, the theory has to be regularized to have an explicit parametrization of the singularities and subsequently renormalized to render the Green functions finite. There are many

[6] I will use quantum and loop corrections interchangeably.

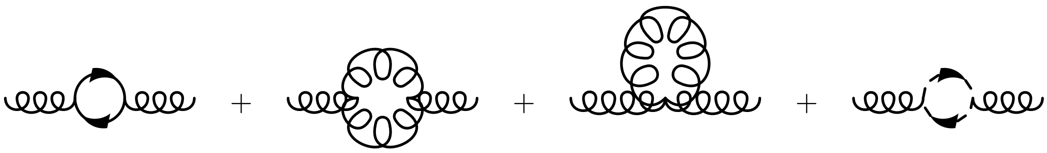
ways of regularizing Green functions, but what we will employ is Dimensional Regularization (DR)[7] [7, 10, 16, 59] by analytically continuing the spacetime dimensions to $d = 4 - 2\epsilon$; the physical limit is taken by letting $\epsilon \to 0$.
There are also many ways upon which we can subtract the singularities. For our interest, we will mostly employ the modified Minimal Subtraction, or $\overline{\text{MS}}$ for short. To eliminate the divergences we firstly need to renormalize the fields and parameters in the Lagrangian eq. (2.4) defining several renormalization constants

$$\begin{array}{lll} G^{\rm a}_{B\mu} = Z_3^{1/2} G^{\rm a}_\mu & \psi_{qB} = Z_q \psi_q & \chi^{\rm a}_B = \tilde{Z}_3^{1/2} \chi^{\rm a} \\ g_B = Z_g g \mu^\epsilon & \alpha_B = Z_3 \alpha & m_B = Z_m m \end{array} \tag{2.12}$$

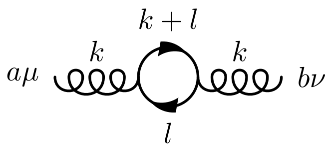
in which the index B denotes the bare quantities. Note that a scale μ has been added to the coupling constant g to make it dimensionless in $D = 4 - 2\epsilon$ spacetime dimensions.
In the next sections, we are going to evaluate the 1-loop renormalization of QCD within the given framework. In general, the Green functions are going to depend on the gauge-fixing parameter α, but the physical results are gauge-independent. Therefore we are going to use the Feynman gauge $\alpha = 1$ for the following calculations.

2.3.1 Vacuum Polarization

The first loop corrections that we are going to evaluate are the ones to the gluon propagator. At one loop there are four corrections, plus the counterterm which depends on the renormalization constants and absorbs the divergences.



We consider now a general $SU(n_f)$ Yang-Mills theory with n_f flavours. Let us start from the fermion bubble contribution: after the sum over the n_f flavours



[7] In dimensional regularization there is some liberty when defining γ_5. In our particular case we will use Naive Dimensional Regularization (NDR) where the γ_5 is such that the usual anticommutation rules hold $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$ and $\{\gamma^\mu, \gamma_5\} = 0$. Another useful definition is given by the 't Hooft-Veltmann scheme [59]

$$
\begin{aligned}
&= -n_f \int \frac{\mathrm{d}^d \ell}{(2\pi)^d} \operatorname{Tr}\left[\left(-ig\mu^\epsilon t^a_{ij}\right)\gamma^\mu \frac{i}{\not\ell + \not k}\left(-ig\mu^\epsilon t^b_{ji}\right)\gamma^\nu \frac{i}{\not\ell}\right] \\
&= -g^2\mu^{2\epsilon} n_f T_\mathrm{F}\delta^{\mathrm{ab}} \operatorname{Tr}\left\{\gamma^\mu\gamma^\alpha\gamma^\nu\gamma^\beta\right\} \int \frac{\mathrm{d}^d \ell}{(2\pi)^d} \frac{(\ell+k)^\alpha \ell^\beta}{\ell^2(\ell+k)^2} \\
&= -g^2\mu^{2\epsilon} n_f T_\mathrm{F}\delta^{\mathrm{ab}} 4\left[g^{\mu\alpha}g^{\nu\beta} - g^{\mu\nu}g^{\alpha\beta} + g^{\mu\beta}g^{\nu\alpha}\right]\left[\mathcal{B}^{\alpha\beta}(k) + k^\alpha \mathcal{B}^\beta(k)\right] \\
&= -g^2\mu^{2\epsilon} n_f T_\mathrm{F}\delta^{\mathrm{ab}} 4\left[g^{\mu\alpha}g^{\nu\beta} - g^{\mu\nu}g^{\alpha\beta} + g^{\mu\beta}g^{\nu\alpha}\right]\left[\frac{dB_0(k)}{4(d-1)}k^\alpha k^\beta - \frac{k^2 B_0(k)}{4(d-1)}g^{\alpha\beta} - \frac{B_0(k)}{2}k^\alpha k^\beta\right] \\
&= g^2\mu^{2\epsilon} n_f T_\mathrm{F}\delta^{\mathrm{ab}} \frac{B_0(k)}{(d-1)}\left[g^{\mu\alpha}g^{\nu\beta} - g^{\mu\nu}g^{\alpha\beta} + g^{\mu\beta}g^{\nu\alpha}\right]\left[k^2 g^{\alpha\beta} + (d-2)k^\alpha k^\beta\right] \\
&= g^2\mu^{2\epsilon} n_f T_\mathrm{F}\delta^{\mathrm{ab}} \frac{B_0(k)}{(d-1)} 2(d-2)\left(k^\mu k^\nu - k^2 g^{\mu\nu}\right),
\end{aligned}
\tag{2.13}
$$

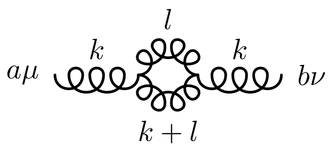
where $B_0(k)$ is the master massless loop integral given in appendix A.2 and T_F is the Dinkin label of the fundamental rep of $SU(n_f)$[8]. We also used the PV decomposition explained in appendix A.3. If we now choose $d = 4 - 2\epsilon$ and expand the scalar loop integral $B_0(k)$ around its pole in $\epsilon = 0$ we find

$$
\begin{aligned}
&= g^2 n_f T_\mathrm{F}\delta^{\mathrm{ab}} \frac{2(2-2\epsilon)}{(3-2\epsilon)} \frac{i}{16\pi^2} \frac{\Gamma^2(1-\epsilon)}{\Gamma(2-2\epsilon)}\left[\frac{1}{\epsilon} - \log\left(-\frac{p^2}{4\pi\mu^2 e^{-\gamma_E}}\right)\right]\left(k^\mu k^\nu - k^2 g^{\mu\nu}\right) \\
&= i\frac{g^2}{16\pi^2} n_f T_\mathrm{F}\delta^{\mathrm{ab}} 2(2-2\epsilon)\left(\frac{1}{3} + \frac{2}{9}\epsilon\right)\left[\frac{1}{\epsilon} - \log\left(-\frac{p^2}{4\pi\mu^2 e^{-\gamma_E}}\right)\right]\left(k^\mu k^\nu - k^2 g^{\mu\nu}\right) \\
&= i\frac{g^2}{16\pi^2} n_f T_\mathrm{F}\delta^{\mathrm{ab}} \left(\frac{4}{3} - \frac{4}{9}\epsilon\right)\left[\frac{1}{\epsilon} - \log\left(-\frac{p^2}{4\pi\mu^2 e^{-\gamma_E}}\right)\right]\left(k^\mu k^\nu - k^2 g^{\mu\nu}\right) \\
&= i\frac{g^2}{16\pi^2} n_f T_\mathrm{F}\delta^{\mathrm{ab}} \left[\frac{4}{3}\frac{1}{\epsilon} - \frac{4}{9} + \frac{4}{3}\log\frac{4\pi\mu^2 e^{-\gamma_E}}{-p^2}\right] + \mathcal{O}(\epsilon).
\end{aligned}
\tag{2.14}
$$

Since we are doing renormalization, we are only interested in the divergent part, which is

$$
i\frac{\alpha_s}{4\pi}\left(-\frac{4}{3}n_f T_\mathrm{F}\frac{1}{\epsilon}\right)(k^2 g^{\mu\nu} - k^\mu k^\nu) + \mathcal{O}(\epsilon^0). \tag{2.15}
$$

In the $\overline{\mathrm{MS}}$ scheme, besides the divergent part, the factors of $\log 4\pi - \gamma_E$ get also reabsorbed into the renormalization constant.
The second relevant diagram is the gluon bubble correction:



[8] As a convention $T_\mathrm{F} = 1/2$.

$$
\begin{aligned}
=\frac{1}{2}\int\frac{\mathrm{d}^d\ell}{(2\pi)^d}&\left(-g\mu^\epsilon f^{acd}\left[g^{\mu\rho}(k-\ell)^\sigma+g^{\rho\sigma}(2\ell+k)^\mu+g^{\sigma\mu}(-2k-\ell)^\rho\right]\right)\times\\
&\times\left(-g\mu^\epsilon f^{d'c'b}\left[g^{\sigma'\rho'}(k+2\ell)^\nu+g^{\rho'\nu}(-\ell+k)^{\sigma'}+g^{\nu\sigma'}(-2k-\ell)^{\rho'}\right]\right)\times\\
&\times\frac{-ig_{\sigma\sigma'}\delta^{dd'}}{(k+\ell)^2}\frac{-ig_{\rho\rho'}\delta^{cc'}}{\ell^2},
\end{aligned}
\tag{2.16}
$$

where the prime indices are related to the internal gluon propaghators. The 1/2 factor comes from the symmetry factor of the Feynman graph. Then, one has

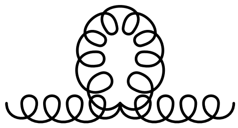
$$
\begin{aligned}
&=\frac{1}{2}g^2\mu^{2\epsilon}C_{\mathrm{A}}\delta^{\mathrm{ab}}\int\frac{\mathrm{d}^d\ell}{(2\pi)^d}\left[g^{\mu\rho}(k-\ell)^\sigma+g^{\rho\sigma}(2\ell+k)^\mu+g^{\sigma\mu}(-2k-\ell)^\rho\right]\times\\
&\qquad\times\left[g_{\sigma\rho}(k+2\ell)^\nu+g^\nu_\rho(-\ell+k)_\sigma+g^\nu_\sigma(-2k-\ell)_\rho\right]\frac{1}{\ell^2(\ell+k)^2}\\
&=g^2\mu^{2\epsilon}\frac{C_{\mathrm{A}}}{2}\delta^{\mathrm{ab}}\int\frac{\mathrm{d}^d\ell}{(2\pi)^d}\Big[(d-6)k^\mu k^\nu+5k^2g^{\mu\nu}+(2d-3)k^\mu\ell^\nu+\\
&\qquad+(2d-3)k^\nu\ell^\mu+(4d-6)\ell^\mu\ell^\nu+2g^{\mu\nu}k_\alpha\ell^\alpha+2g^{\mu\nu}\ell^2\Big]\frac{1}{\ell^2(\ell+k)^2}\\
&=g^2\mu^{2\epsilon}\frac{C_{\mathrm{A}}}{2}\delta^{\mathrm{ab}}\Big[B_0(k)\left((d-6)k^\mu k^\nu+5k^2g^{\mu\nu}\right)+(2d-3)k^\mu\mathcal{B}^\nu(k)+\\
&\qquad+(2d-3)k^\nu\mathcal{B}^\mu(k)+(4d-6)\mathcal{B}^{\mu\nu}(k)+2g^{\mu\nu}k_\alpha\mathcal{B}^\alpha(k)+2g^{\mu\nu}A_0\Big]\\
&=g^2\mu^{2\epsilon}\frac{C_{\mathrm{A}}}{2}\delta^{\mathrm{ab}}B_0(k)\Big[(d-6)k^\mu k^\nu+5k^2g^{\mu\nu}-(2d-3)k^\mu k^\nu-k^2g^{\mu\nu}+\\
&\qquad+\frac{4d-6}{4(d-1)}\left(dk^\mu k^\nu-k^2g^{\mu\nu}\right)\Big]\\
&=g^2\mu^{2\epsilon}\frac{C_{\mathrm{A}}}{2}\delta^{\mathrm{ab}}B_0(k)\left[-\frac{7d-6}{2(d-1)}k^\mu k^\nu+\frac{6d-5}{2(d-1)}k^2g^{\mu\nu}\right]\\
&=g^2\mu^{2\epsilon}C_{\mathrm{A}}\delta^{\mathrm{ab}}\frac{B_0(k)}{4(d-1)}\left[(6-7d)k^\mu k^\nu+(6d-5)k^2g^{\mu\nu}\right],
\end{aligned}
\tag{2.17}
$$

where A_0 is the massless vacuum bubble integral given in appendix A.2 and C_{A} is the Casimir for the adjoint representation[9]. Proceeding exactly as before by substituting $d = 4 - 2\epsilon$ and expanding around the pole $\epsilon = 0$ and taking only the divergent contribution, we find

$$
i\frac{\alpha_s}{4\pi}C_A\delta^{\mathrm{ab}}\frac{1}{\epsilon}\left[\frac{19}{12}k^2g^{\mu\nu}-\frac{11}{16}k^\mu k^\nu\right]+\mathcal{O}(\epsilon^0).
\tag{2.18}
$$

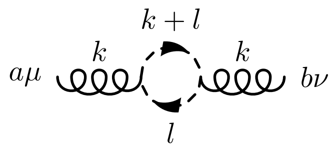
[9] For $SU(n_f)$ we have $C_{\mathrm{A}} = (n_f^2-1)/n_f$.

The gluon tadpole graph is the simplest one since it is just proportional to



$$\propto \int \frac{\mathrm{d}^d\ell}{(2\pi)^d}\frac{1}{\ell^2} = A_0 = 0 \tag{2.19}$$

since there is no scale involved.
The last contribution is from the ghost loop where we find



$$
\begin{aligned}
&= -\int \frac{\mathrm{d}^d\ell}{(2\pi)^d} g\mu^\epsilon f^{acd}(\ell+k)^\mu g\mu^\epsilon f^{bdc}\ell^\nu \frac{i}{\ell^2}\frac{i}{(\ell+k)^2} \\
&= -g^2\mu^{2\epsilon} C_\mathrm{A}\delta^{\mathrm{ab}} \int \frac{\mathrm{d}^d\ell}{(2\pi)^d}\frac{\ell^\mu\ell^\nu + k^\mu\ell^\nu}{\ell^2(\ell+k)^2} \\
&= -g^2\mu^{2\epsilon} C_\mathrm{A}\delta^{\mathrm{ab}} \left[\mathcal{B}^{\mu\nu}(k) + k^\mu\mathcal{B}^\nu(k)\right] \\
&= -g^2\mu^{2\epsilon} C_\mathrm{A}\delta^{\mathrm{ab}} B_0(k)\left[\frac{dk^\mu k^\nu}{4(d-1)} - \frac{k^2 g^{\mu\nu}}{4(d-1)} - \frac{k^\mu k^\nu}{2}\right] \\
&= g^2\mu^{2\epsilon} C_\mathrm{A}\delta^{\mathrm{ab}} \frac{B_0(k)}{4(d-1)}\left[k^2 g^{\mu\nu} + (d-2)k^\mu k^\nu\right]
\end{aligned}
$$

Same, but different, we take $d = 4 - 2\epsilon$ and expand around $\epsilon = 0$. Taking only the divergent part, we get

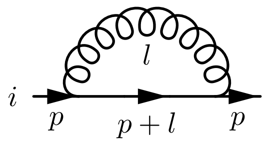
$$i\frac{\alpha_s}{4\pi} C_\mathrm{A}\delta^{\mathrm{ab}}\frac{1}{\epsilon}\left[\frac{1}{12}k^2 g^{\mu\nu} + \frac{1}{6}k^\mu k^\nu\right] + \mathcal{O}(\epsilon^0). \tag{2.20}$$

We can now define the renormalization constant Z_3 at 1-loop by summing the three divergent contributions to the gluon propagator in eqs. (2.15), (2.18) and (2.20)

$$Z_3 = 1 + \frac{\alpha_s}{4\pi}\frac{1}{\epsilon}\left(\frac{5}{3}C_\mathrm{A} - \frac{4}{3}n_f T_\mathrm{F}\right). \tag{2.21}$$

2.3.2 Quark Self-Energy

The next part in the renormalization of QCD is finding the 1-loop corrections to the quark propagator. In this case we have only one contribution from the quark loop since ghosts couple only to gluons and so they only enter in higher-order corrections to the quark propagator. So we need to evaluate the following Feynman graph



$$= \int \frac{\mathrm{d}^d\ell}{(2\pi)^d}(-ig\mu^\epsilon t^{\mathrm{a}}_{jk}\gamma^\mu)\frac{i}{\not\ell - \not p}(-ig\mu^\epsilon t^{\mathrm{a}}_{ki}\gamma_\mu)\frac{-i}{\ell^2}$$

$$\begin{aligned} &= -g^2\mu^{2\epsilon}t^{\mathrm{a}}_{jk}t^{\mathrm{a}}_{ki}\int \frac{\mathrm{d}^d\ell}{(2\pi)^d}\gamma^\mu \frac{\not\ell + \not p}{(\ell+p)^2}\gamma_\mu \frac{1}{\ell^2} \\ &= (d-2)g^2\mu^{2\epsilon}t^{\mathrm{a}}_{jk}t^{\mathrm{a}}_{ki}\int \frac{\mathrm{d}^d\ell}{(2\pi)^d}\frac{\not\ell + \not p}{(\ell+p)^2}\frac{1}{\ell^2}, \end{aligned} \tag{2.22}$$

where the identity $\gamma^\mu\gamma^\alpha\gamma_\mu = (2-d)\gamma^\alpha$ has been used. Using the properties of the $SU(n_f)$ generators and the integrals given in appendix A.2 we find

$$\begin{aligned} &= (d-2)g^2\mu^{2\epsilon}\delta_{ij}C_{\mathrm{F}}\gamma_\alpha \int \frac{\mathrm{d}^d\ell}{(2\pi)^d}\frac{\ell^\alpha + p^\alpha}{(\ell+p)^2}\frac{1}{\ell^2} \\ &= (d-2)g^2\mu^{2\epsilon}\delta_{ij}C_{\mathrm{F}}\gamma_\alpha[\mathcal{B}^\alpha(p) + p^\mu B_0(p)] \\ &= g^2\mu^{2\epsilon}\delta_{ij}C_{\mathrm{F}}\not p\left(\frac{d-2}{2}\right)B_0(p), \end{aligned} \tag{2.23}$$

where we have assumed that the quark momenta is $p^2 \neq 0$ otherwise the integral would be zero in dimensional regularization.
As before, we can now expand around the pole in $\epsilon = 0$

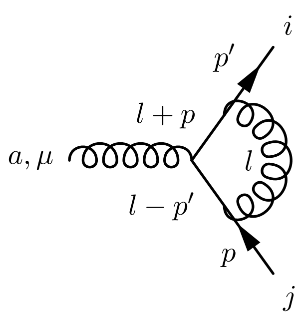
$$\begin{aligned} &= i\frac{g^2}{16\pi^2}C_{\mathrm{F}}\delta_{ij}\not p(1-\epsilon)\left[\frac{1}{\epsilon} + \log\frac{4\pi\mu^2 e^{-\gamma}}{-p^2} + \mathcal{O}(\epsilon)\right] \\ &= i\frac{g^2}{16\pi^2}C_{\mathrm{F}}\delta_{ij}\not p\left[\frac{1}{\epsilon} - 1 + \log\frac{4\pi\mu^2 e^{-\gamma}}{-p^2}\right] + \mathcal{O}(\epsilon) \\ &\simeq i\frac{\alpha_s}{4\pi}\frac{1}{\epsilon}C_{\mathrm{F}}\delta_{ij}\not p + \mathcal{O}(\epsilon^0). \end{aligned} \tag{2.24}$$

Summing the virtual correction to the bare propagator, we find that the renormalization constant involved is given by

$$Z_2 = 1 - \frac{\alpha_s}{4\pi}\frac{1}{\epsilon}C_{\mathrm{F}}. \tag{2.25}$$

2.3.3 Quark-Gluon Vertex Correction

The 1-loop corrections to the qqg vertex are two. The first one is given by the following Feynman diagram



$$
\begin{aligned}
&= \int \frac{\mathrm{d}^d \ell}{(2\pi)^d} (-ig\mu^\epsilon t^{\mathrm{b}}_{jk}\gamma^\nu) \frac{i(\not{\ell} - \not{p}')}{(\ell - p')^2} (-ig\mu^\epsilon t^{\mathrm{a}}_{kl}\gamma^\mu) \frac{i(\not{\ell} + \not{p})}{(\ell + p)^2} (-ig\mu^\epsilon t^{\mathrm{b}}_{li}\gamma_\nu) \frac{-i}{\ell^2} \\
&= -\Big(C_{\mathrm{F}} - \frac{C_{\mathrm{A}}}{2}\Big) t^{\mathrm{a}}_{ji} g^3 \mu^{3\epsilon} \int \frac{\mathrm{d}^d \ell}{(2\pi)^d} \frac{\gamma^\nu (\not{\ell} - \not{p}')\gamma^\mu(\not{\ell} + \not{p})\gamma_\nu}{\ell^2 (\ell - p')^2 (\ell + p)^2},
\end{aligned}
\tag{2.26}
$$

where the color factors come from the $SU(n_f)$ algebra[10]. To perform this integral, since we are only interested in the UV behaviour, we make the simplification $p = p' = 0$ with a caveat that we'll see later. In this limit, we have

$$
-g^3 \mu^{3\epsilon} t^{\mathrm{a}}_{ji} \Big(C_{\mathrm{F}} - \frac{C_{\mathrm{A}}}{2}\Big) \int \frac{\mathrm{d}^d \ell}{(2\pi)^d} \frac{\gamma^\nu \not{\ell} \gamma^\mu \not{\ell} \gamma_\nu}{(\ell^2)^3}. \tag{2.27}
$$

Inside the integral, the following equality is valid

$$
\ell_\alpha \ell_\beta = \frac{g_{\alpha\beta}}{d} \ell^2 \tag{2.28}
$$

which gives us

$$
-g^3 \mu^{3\epsilon} t^{\mathrm{a}}_{ji} \Big(C_{\mathrm{F}} - \frac{C_{\mathrm{A}}}{2}\Big) \int \frac{\mathrm{d}^d \ell}{(2\pi)^d} \frac{1}{d} \frac{\gamma^\nu \gamma^\alpha \gamma^\mu \gamma_\alpha \gamma_\nu}{(\ell^2)^2}. \tag{2.29}
$$

Using the Clifford algebra in d-dimensions

$$
\gamma^\nu \gamma^\alpha \gamma^\mu \gamma_\alpha \gamma_\nu = (2 - d)\gamma^\nu \gamma^\mu \gamma_\nu = (d-2)^2 \gamma^\mu. \tag{2.30}
$$

Therefore

$$
-g^3 \mu^{3\epsilon} \Big(C_{\mathrm{F}} - \frac{C_{\mathrm{A}}}{2}\Big) \frac{(d-2)^2}{d} \gamma^\mu \int \frac{\mathrm{d}^d \ell}{(2\pi)^d} \frac{1}{(\ell^2)^2}. \tag{2.31}
$$

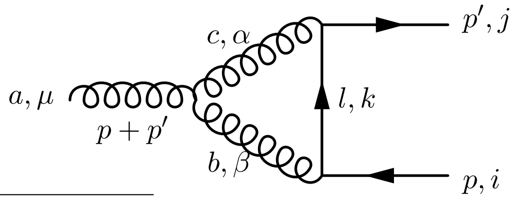
The last integral may seem to be zero, but this is only because we have taken out the relevant scales. Therefore we need to introduce a scale back in a sort of Pauli-Villard regularization

$$
\int \frac{\mathrm{d}^d \ell}{(2\pi)^d} \frac{1}{(\ell^2)^2} \to \int \frac{\mathrm{d}^d \ell}{(2\pi)^d} \frac{1}{(\ell^2 - m^2)^2} = \frac{i}{(4\pi)^{d/2}} \frac{\Gamma\big(2 - \frac{d}{2}\big)}{\Gamma(2)} (m^2)^{\frac{d}{2} - 2}. \tag{2.32}
$$

Now we can put $d = 4 - 2\epsilon$ and expand around the pole $\epsilon = 0$, taking only the divergent part

$$
-igt^{\mathrm{a}}_{ji} \gamma^\mu \frac{\alpha_s}{4\pi} \frac{1}{\epsilon} \Big(X_{\mathrm{F}} - \frac{C_{\mathrm{A}}}{2}\Big). \tag{2.33}
$$

The second diagram which contributes to the correction is given by



[10] In practice $t^{\mathrm{a}} t^{\mathrm{b}} t^{\mathrm{a}} = (C_{\mathrm{F}} - C_{\mathrm{A}}/2) t^{\mathrm{b}}$.

$$= \int \frac{d^d\ell}{(2\pi)^d}(-g\mu^\epsilon f^{abc}) \Big[g^{\mu\beta}(-2p - p' + \ell)^\alpha + g^{\beta\alpha}(-2\ell + p - p')^\mu$$
$$+ g^{\alpha\mu}(\ell + 2p' + p)^\alpha)\big] \frac{-i}{(\ell - p)^2} \frac{-i}{(\ell + p')^2}(-ig\mu^\epsilon \gamma_\alpha t^c_{jk}) \frac{i\not{\ell}}{\ell^2}(-ig\mu^\epsilon \gamma_\beta t^b_{ki}). \tag{2.34}$$

Using the color algebra equality $if^{abc}t^c t^b = C_A/2t^a$ and by, again, neglecting the momenta p and p', we get

$$-g^3\mu^{3\epsilon}\frac{C_A}{2}t^a_{ji}\int \frac{d^d\ell}{(2\pi)^d}\frac{\Big(g^{\mu\beta}\ell^\alpha - 2g^{\beta\alpha}\ell^\mu + g^{\alpha\mu}\ell^\beta\Big)\gamma_\alpha\gamma_\rho\gamma_\beta\ell^\rho}{(\ell^2)^3}. \tag{2.35}$$

With the additional Clifford algebra equality $\gamma_\mu\gamma^\mu = d$, we find that the integral becomes

$$\int \frac{d^d\ell}{(2\pi)^d}\frac{\gamma^\mu + 2\frac{(d-2)}{d}\gamma^\mu + \gamma^\mu}{(\ell^2)^2} = 4\frac{d-1}{d}\gamma^\mu \int \frac{d^d\ell}{(2\pi)^d}\frac{1}{(\ell^2)^2}. \tag{2.36}$$

The integral is the same as before, so we can proceed exactly in the same manner, obtaining

$$-igt^a_{ji}\gamma^\mu \frac{\alpha_s}{4\pi}\frac{1}{\epsilon}\frac{3}{2}C_A. \tag{2.37}$$

By summing the tree-level amplitude with the 1-loop corrections we find the renormalization constant Z_1 via

$$Z_1^{-1} = 1 + \frac{\alpha_s}{4\pi}\frac{1}{\epsilon}(C_A + C_F). \tag{2.38}$$

Let us now concentrate for a moment on the renormalized Lagrangian, and in particular on the strong coupling constant. The bare vertex reads

$$g_B\bar{\psi}_{qB}\not{G}_B\psi_{qB}, \tag{2.39}$$

where in d-dimensions, the bare coupling constant is dimensionful. If we now replace the bare fields and coupling constant with the renormalized one by virtue of eq. (2.12) we get

$$g\mu^\epsilon Z_g Z_2 Z_3^{1/2}\bar{\psi}_q\not{G}\psi_q. \tag{2.40}$$

But, from the evaluation of the loop corrections to the vertex, we know that the renormalization constant is Z_1^{-1}. Hence, since what we can actually measure can only be the vertex, we require that the divergent factors obtained by rescaling the fields must be exactly canceled by the multiplicative factor Z_1^{-1}. In this way, whenever we extract physical quantities, they will be finite. This corresponds to choosing

$$Z_1 = Z_g Z_2 Z_3^{1/2} \implies Z_g = \frac{Z_1}{Z_2 Z_3^{1/2}}. \tag{2.41}$$

Using the results we found in eqs. (2.21), (2.25) and (2.38) we find

$$Z_g = 1 - \frac{\alpha_s}{4\pi}\frac{1}{\epsilon}\Big(\frac{11}{6}C_A - \frac{4}{6}n_f T_F\Big). \tag{2.42}$$

Note that this result is gauge-independent.

2.4 Renormalization Group Equations

A general feature of renormalization is that it adds an explicit scale dependence μ on physical quantities. But it is important to note that the initial, bare, quantities did not depend on an energy scale. Thus, we have to require that the bare quantities do not depend on μ. By doing so we get the so-called *renormalization group equations* (RGE). These equations govern the dependence on the scale of relevant physical quantities like the coupling constants.
By imposing that the bare coupling constant does not depend on the renormalization scale and searching for a solution to the RGE, a peculiar thing happens: the physical parameters will depend on the energy scale. In particular, we are interested in the scale dependence of the coupling constant. We will see that this scale dependence will be such that two behaviours can arise: when the energy is high then the coupling constant is small or when the energy scale is small, then the coupling constant is small. We call theories with such behaviours *asymptotically free* in the UV for the former or in the IR for the latter. Let us be rigorous now.
Take an observable $\mathcal{O}$ calculated in the $\overline{\text{MS}}$ scheme. We have

$$\mathcal{O} = \mathcal{O}_{\overline{\text{MS}}}\left(\alpha(\mu), m(\mu), \log\frac{s}{\mu^2}, \cdots\right), \tag{2.43}$$

where $\sqrt{s}$ is the center-of-mass energy. Here $\alpha \propto g^2$ is the coupling constant and m is the physical mass of the theory. The appearance of logarithms is a general feature of regularization as clearly underlined by the previous calculations of loop corrections.
It is important to note that the physical observable $\mathcal{O}$ is μ independent assuming one works to all orders in perturbation theory. The logarithms can be large whenever $s \gg \mu$, these will be discussed on more general grounds in the following chapters.
The μ independence of the observable $\mathcal{O}$ can be expressed in the following form

$$\mu\frac{\mathrm{d}}{\mathrm{d}\mu}\mathcal{O} = \mu\frac{\mathrm{d}\alpha(\mu)}{\mathrm{d}\mu}\frac{\partial\mathcal{O}}{\partial\alpha(\mu)} + \mu\frac{\mathrm{d}m(\mu)}{\mathrm{d}\mu}\frac{\partial\mathcal{O}}{\partial m(\mu)} + \frac{\partial\mathcal{O}}{\partial\mu} = 0. \tag{2.44}$$

These equations are known as Renormalization Group Equations! In this case, we assumed that the observable $\mathcal{O}$ only depends on two quantities: the coupling constant and the mass, but a more general theory can also depend on some other parameters, but the generalization is trivial.
We define then two very important functions

$$\begin{aligned}\mu\frac{\mathrm{d}\alpha(\mu)}{\mathrm{d}\mu} &\equiv \frac{\mathrm{d}\alpha(\mu)}{\mathrm{d}\log\mu} = \beta(\alpha(\mu))\\ \mu\frac{\mathrm{d}m(\mu)}{\mathrm{d}\mu} &\equiv \frac{\mathrm{d}m(\mu)}{\mathrm{d}\log\mu} = \gamma_m(\alpha(\mu))m(\mu)\end{aligned} \tag{2.45}$$

which we call *beta function* and *mass anomalous dimension* respectively. With these definitions, eq. (2.44) becomes

$$\beta(\alpha)\frac{\partial\mathcal{O}}{\partial\alpha} + \gamma_m(\alpha)m\frac{\partial\mathcal{O}}{\partial m} + \frac{\partial\mathcal{O}}{\partial\mu} = 0. \tag{2.46}$$

2.4.1 The QCD Beta Function

Now, given the general definitions and the 1-loop result for Z_g in eq. (2.42), we are ready to find out if QCD is UV-free or IR-free. Given that

$$g_B = \mu^\epsilon Z_g(\mu) g(\mu) \implies \alpha_{sB} = \mu^{2\epsilon} Z_g^2(\mu)\alpha_s(\mu) \equiv \mu^{2\epsilon} Z_\alpha(\mu)\alpha_s(\mu), \tag{2.47}$$

where

$$Z_\alpha(\mu) = 1 - \frac{\alpha_s(\mu)}{4\pi\epsilon}\beta_0, \qquad \beta_0 = \left(\frac{11}{3}C_\text{A} - \frac{4}{3}n_f T_\text{F}\right). \tag{2.48}$$

From the fact that the bare parameters are scale-independent, we get the RGE for the running coupling

$$\frac{\mathrm{d}\alpha_{Bs}}{\mathrm{d}\log\mu} = 0 = \mu^{2\epsilon} Z_\alpha(\mu)\alpha_s(\mu)\left[2\epsilon + Z_\alpha^{-1}\frac{\mathrm{d}Z_\alpha}{\mathrm{d}\log\mu} + \frac{1}{\alpha_s}\frac{\mathrm{d}\alpha_s}{\mathrm{d}\log\mu}\right], \tag{2.49}$$

which implies

$$\frac{\mathrm{d}\alpha_s}{\mathrm{d}\log\mu} = \alpha_s\left[-2\epsilon - Z_\alpha^{-1}\frac{\mathrm{d}Z_\alpha}{\mathrm{d}\log\mu}\right] \equiv \beta(\alpha_s(\mu),\epsilon). \tag{2.50}$$

This more general β function is defined for a finite ϵ where in the limit $\epsilon \to 0$ we get the usual β function.

Consider now the fact that the renormalization constants in this scheme, only contain dependence on $1/\epsilon^n$ factors and thus the dependence on the scale μ is only through the running coupling $\alpha_s(\mu)$. Thus we can recast eq. (2.50) in the more convenient form

$$\beta(\alpha_s,\epsilon) = \alpha_s\left[-2\epsilon - \beta(\alpha_s,\epsilon)\frac{\mathrm{d}Z_\alpha}{\mathrm{d}\alpha_s}\right]. \tag{2.51}$$

To solve this equation we expand

$$\begin{aligned}\beta(\alpha_s,\epsilon) &= \beta(\alpha_s) + \sum_{k=1}^{\infty}\epsilon^k\beta^{(k)}(\alpha_s),\\ Z_\alpha &= 1 + \sum_{k=1}^{\infty}\frac{1}{\epsilon^k}Z_\alpha^{(k)}(\alpha_s).\end{aligned} \tag{2.52}$$

The solution to the equation is therefore

$$\beta(\alpha_s) = 2\alpha_s^2\frac{\mathrm{d}Z_\alpha^{(1)}(\alpha_s)}{\mathrm{d}\alpha_s}, \tag{2.53}$$

which yields

$$\beta(\alpha_s,\epsilon) = -2\epsilon\alpha_s + \beta(\alpha_s) = -2\epsilon\alpha_s + 2\alpha_s^2\frac{\mathrm{d}Z_\alpha^{(1)}}{\mathrm{d}\alpha_s}. \tag{2.54}$$

A similar relation holds for the mass anomalous dimension

$$\gamma_m(\alpha_s) = 2\alpha_s\frac{\mathrm{d}Z_m^{(1)}(\alpha_s)}{\mathrm{d}\alpha_s}. \tag{2.55}$$

These equations are really important. They state that to all orders in perturbation theory, the β-function and the anomalous dimension can be extracted from the

coefficients of the single $1/\epsilon$ pole in the renormalization constants.
In our 1-loop calculation, we find that

$$\frac{\mathrm{d}Z_\alpha^{(1)}}{\mathrm{d}\alpha_s} = -\frac{\mathrm{d}}{\mathrm{d}\alpha_s}\frac{\alpha_s}{4\pi}\beta_0 = \frac{\beta_0}{4\pi}, \tag{2.56}$$

therefore

$$\beta_0(\alpha_s) = -2\alpha_s\left(\beta_0\frac{\alpha_s}{4\pi}\right). \tag{2.57}$$

To this date, the β-function of QCD has been calculated up to 5-loop [9, 34, 46].

2.4.2 Leading-order Solution to the RGE

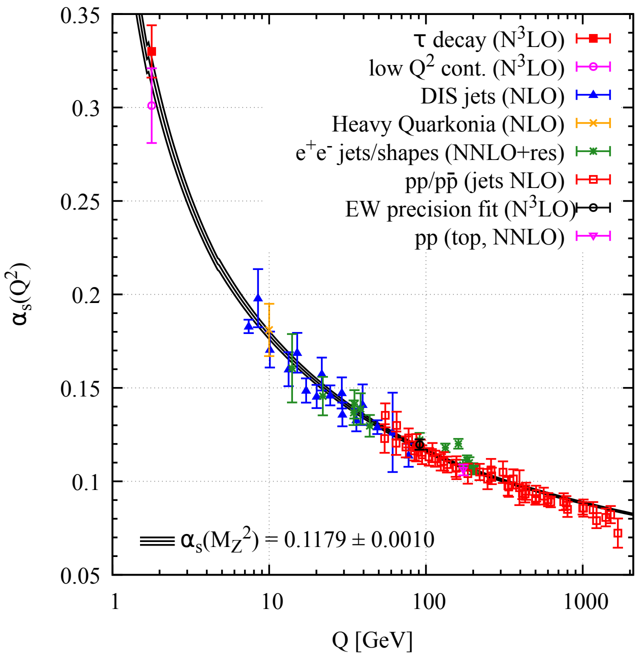


Figure 2.1. Running coupling with experimental data. The best fit value for the $\overline{\text{MS}}$ strong coupling constant is $\alpha_s(m_Z) = 0.1179 \pm 0.0010$. Image from Ref. [66].

Given the 1-loop solution for the β-function eq. (2.57), we can find the running of the strong coupling constant by means of

$$\frac{\mathrm{d}\alpha_s(\mu)}{\mathrm{d}\log\mu} = -2\beta_0\frac{\alpha_s^2(\mu)}{4\pi}. \tag{2.58}$$

This can be solved by simple separation of variables

$$-\int_{\alpha_s(\Lambda)}^{\alpha_s(\mu)}\frac{\mathrm{d}\alpha_s}{\alpha_s^2} = \frac{1}{\alpha_s(\mu)} - \frac{1}{\alpha_s(\Lambda)} = \frac{\beta_0}{4\pi}\log\frac{\mu^2}{\Lambda^2}, \tag{2.59}$$

where $\alpha_s(\Lambda)$ is the strong coupling constant measured at some energy scale Λ. Often one chooses $\Lambda = m_Z \approx 91.188$ GeV at which $\alpha_s(m_Z) = 0.1179 \pm 0.0010$ [66]. By rearranging the above equation, we find the famous expression

$$\alpha_s(\mu) = \frac{\alpha_s(\Lambda)}{1 + \frac{\alpha_s(\Lambda)}{4\pi}\beta_0 \log\frac{\mu^2}{\Lambda^2}}, \qquad \beta_0 = \frac{11}{3}C_{\rm A} - \frac{4}{3}n_f T_{\rm F}. \tag{2.60}$$

Here n_f is the effective number of active flavours below the scale μ. Now it is possible to see the fundamental behaviour of the strong coupling constant: being $\beta_0 > 0$, since $n_f < 17$, when we increase the energy scale μ, $\alpha_s(\mu)$ becomes smaller. This phenomenon is referred to as *asymptotic freedom* [30, 52].
At this point, it is important to fix an energy scale with respect to which we define the UV and IR phases. We can rearrange eq. (2.60) in a suitable way as

$$\alpha_s(\mu) = \left[\beta_0 \log\left(\frac{\mu^2}{\Lambda^2_{\rm QCD}}\right)\right]^{-1}, \tag{2.61}$$

that defines a new constant called $\Lambda_{\rm QCD}$[11]. In reality, $\Lambda_{\rm QCD}$ is not completely defined at LO. One has to evaluate at least the NLO solution to the RGE for α_s which requires the calculation of the 2-loop corrections. Without going into the specific calculations, we give here the second order contribution to the beta function which is needed for the NLO solution of the RGE for α_s

$$\beta_1 = \frac{34}{3}C_{\rm A}^2 - \frac{20}{3}C_{\rm A}T_{\rm F}n_f - 4C_{\rm F}T_{\rm F}n_f. \tag{2.62}$$

The solution to the RGE for α_s is therefore

$$\alpha_s(\mu) = \frac{4\pi}{\beta_0 \log\frac{\mu^2}{\Lambda^2}}\left[1 - \frac{\beta_1}{\beta_0^2}\frac{\log\log\frac{\mu^2}{\Lambda^2}}{\log\frac{\mu^2}{\Lambda^2}}\right]. \tag{2.63}$$

If we include higher-order corrections and measure the value of α_s at some energy scale like $\alpha_s(m_Z)$, one finds that $\Lambda_{\rm QCD} \approx 250$ MeV. At energies below $\Lambda_{\rm QCD}$, the theory becomes strongly coupled and therefore ordinary perturbation theory breaks up and we need to use non-perturbative techniques to find meaningful results like lattice QCD. In some regimes, there are other theories like Chiral Perturbation Theory (ChPT) [51] and $1/N$ expansion [58] that contains the main features of QCD but are not an exact solution like lattice QCD.
Note that in the $\overline{\rm MS}$ scheme, the slope of the curve changes whenever we cross a quark mass threshold. This is because β_0 depends on the number of active flavour below the scale μ so, if we go from $\mu \approx m_b$ to $\mu \approx m_c$, the active flavours go from $n_f = 5$ to $n_f = 4$ thus changing β_0.

[11] Note that if we started with a mass-less theory we would have gotten the same result. Hence a completely arbitrary energy scale has appeared even although the theory was scaleless to begin with. This phenomenon is known as dimensional transmutation.

Appendix A

Mathematical Tools

We give here a brief mathematical appendix on the relevant methods and integrals which can commonly be found in field theories when evaluating loop diagrams.

A.1 Feynman Parametrization

The Feynman parametrization is a very useful tool that is employed almost every time one has to isolate divergences from loop integrals.
The simplest Feynman parametrization is the following

$$\frac{1}{AB} = \int_0^1 \mathrm{d}x \, \frac{1}{[A + (B-A)x]^2} = \int \mathrm{d}x \, \mathrm{d}y \, \delta(x+y-1) \frac{1}{[xA + yB]^2} \tag{A.1}$$

which can be easily proven. The powers in the denominator can be raised by simple derivation

$$\frac{1}{AB^n} = \frac{(-1)^{n-1}}{(n-1)!} \frac{\partial^{n-1}}{\partial B^{n-1}} \frac{1}{AB} = \int_0^1 \mathrm{d}x \, \mathrm{d}y \, \delta(x+y-1) \frac{n y^{n-1}}{[xA + yB]^{n+1}}. \tag{A.2}$$

More terms in the denominator can be added by simple iteration of eq. (A.1)

$$\frac{1}{ABC} = \frac{1}{AB} \frac{1}{C} = 2 \int_0^1 \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z \, \delta(x+y+z-1) \frac{1}{[xA + yB + zC]^3}. \tag{A.3}$$

From this, one can get the most general formula which gives

$$\prod_{k=1}^{n} \frac{1}{A_k^{c_k}} = \frac{\Gamma(\sum_k c_k)}{\prod_k \Gamma(c_k)} \int_0^1 \prod_{k=1}^{n} x_k^{c_k - 1} \, \mathrm{d}x_k \, \delta\left(\sum_{k=1}^{n} x_k - 1 \right) \frac{1}{[\sum_{k=1}^{n} x_k A_k]^{\sum_k c_k}}, \tag{A.4}$$

where $\Gamma(z)$ is the Euler gamma function.

A.2 Scalar One-Loop Integrals

Here we give a list of interesting integrals which come up in this notes. The general scalar one loop integral with n external massive legs, with masses m_i, carrying momentum p_i is of the form

$$I = \int \frac{\mathrm{d}^d \ell}{(2\pi)^d} \frac{1}{[(\ell + p_1)^2 - m_1^2 + i\epsilon]\,[(\ell + p_{12})^2 - m_2^2 + i\epsilon] \cdots [(\ell + p_{12\cdots n})^2 - m_n^2 + i\epsilon]}, \tag{A.5}$$

where we employed dimensional regularization, ℓ is the loop momenta and $p_{12\cdots n} = \sum_{k=1}^n p_k$.
Using the Feynman parametrization eq. (A.4) we can write

$$I = \Gamma(n) \int_0^1 \prod_{k=1}^n \mathrm{d}x_k \int \frac{\mathrm{d}^d \ell}{(2\pi)^d} \frac{\delta(\sum_{k=1}^n x_k - 1)}{[\sum_{k=1}^n x_k A_k]^n}, \tag{A.6}$$

where

$$\begin{aligned} \sum_{k=1}^n x_k A_k &= \sum_{k=1}^n x_k \Big[(\ell + p_{12\cdots k})^2 - m_k^2 + i\epsilon\Big] \\ &= \ell^2 + 2\ell \cdot \left(\sum_{k=1}^n x_k p_{12\cdots k}\right) + \sum_{k=1}^n x_k \Big(p_{12\cdots k}^2 - m_k^2 + i\epsilon\Big) \\ &= \ell^2 + 2\ell \cdot P + K^2 + i\epsilon. \end{aligned} \tag{A.7}$$

Therefore, by substitution in the integral and translating the loop momentum integration $\ell \to \ell + P$, one gets

$$I = \Gamma(n) \int_0^1 \prod_{k=1}^n \mathrm{d}x_k\, \delta\left(\sum_{k=1}^n x_k - 1\right) \int \frac{\mathrm{d}^d \ell}{(2\pi)^d} \frac{1}{(\ell^2 - m^2 + i\epsilon)^n}, \tag{A.8}$$

where $m^2 = K^2 - P^2$. The integral over the loop momentum can be performed by Wick rotating the temporal ℓ^0 coordinate and then using polar coordinates in Euclidean space. Therefore the integral I becomes

$$\begin{aligned} I &= i\Gamma(n) \int_0^1 \prod_{k=1}^n \mathrm{d}x_k\, \delta\left(\sum_{k=1}^n x_k - 1\right) \int \frac{\mathrm{d}\ell_0^E\, \mathrm{d}^{d-1}\ell}{(2\pi)^d} \frac{1}{\Big(-(\ell_0^E)^2 - |\ell|^2 - m^2\Big)^n} \\ &= \frac{(-1)^n i\Gamma(n)}{(2\pi)^d} \int_0^1 \prod_{k=1}^n \mathrm{d}x_k\, \delta\left(\sum_{k=1}^n x_k - 1\right) \int \mathrm{d}^d\Omega\, \mathrm{d}\ell_E\, \frac{\ell_E^{d-1}}{(\ell_E^2 + m^2)^n} \\ &= \frac{(-1)^n i\Gamma(n)\Omega_d}{2(2\pi)^d} B\Big(\frac{d}{2}, n - \frac{d}{2}\Big) \int_0^1 \prod_{k=1}^n \mathrm{d}x_k\, \delta\left(\sum_{k=1}^n x_k - 1\right) (m^2)^{\frac{d}{2}-n}, \end{aligned} \tag{A.9}$$

where in the last step we changed variables $x = \frac{1}{1+\ell_E^2/m^2}$ and used the definition of the Beta function

$$B(a,b) = \int_0^1 \mathrm{d}x\, x^{a-1}(1-x)^{b-1}. \tag{A.10}$$

Ending with a few manipulations on m^2

$$\begin{aligned}
m^2 = P^2 - K^2 &= \left(\sum_{i=1}^{n} x_k p_{1\cdots i}\right)^2 - \sum_{i=1}^{n} x_k (p_{1\cdots i}^2 - m_i^2 + i\epsilon) \\
&= \sum_{i=1}^{n} \alpha_i^2 p_{1\ldots i}^2 + 2\sum_{i>j}^{n} \alpha_i\alpha_j p_{1\ldots i} p_{1\ldots j} - \sum_{i=1}^{n} \alpha_i p_{1\ldots i}^2 + \sum_{i=1}^{n} \alpha_i m_i^2 - i\epsilon \\
&= -\sum_{i=1}^{n} \alpha_i \sum_{j\neq i} \alpha_j p_{1\ldots i}^2 + 2\sum_{i>j}^{n} \alpha_i\alpha_j p_{1\ldots i} p_{1\ldots j} + \sum_{i=1}^{n} \alpha_i m_i^2 - i\epsilon \\
&= -\sum_{i>j}^{n} \alpha_i\alpha_j p_{1\ldots i}^2 - \sum_{i>j}^{n} \alpha_i\alpha_j p_{1\ldots i} p_{1\ldots j} \\
&\quad -\sum_{j>i}^{n} \alpha_i\alpha_j p_{1\ldots i}^2 - \sum_{j>i}^{n} \alpha_j\alpha_i p_{1\ldots j} p_{1\ldots i} + \sum_{i=1}^{n} \alpha_i m_i^2 - i\epsilon \\
&= -\sum_{i>j}^{n} \alpha_i\alpha_j p_{1\ldots i} p_{j+1\ldots i} + \sum_{j>i}^{n} \alpha_i\alpha_j p_{1\ldots i} p_{i+1\ldots j} + \sum_{i=1}^{n} \alpha_i m_i^2 - i\epsilon \\
&= -\sum_{i>j}^{n} \alpha_i\alpha_j p_{j+1\ldots i}^2 + \sum_{i=1}^{n} \alpha_i m_i^2 - i\epsilon = \Delta.
\end{aligned} \tag{A.11}$$

In summary

$$I = (-1)^n \frac{i}{(4\pi)^{d/2}} \Gamma\left(n - \frac{d}{2}\right) \int_0^1 \prod_{k=1}^{n} \mathrm{d}x_k\, \delta\left(\sum_{k=1}^{n} x_k - 1\right) \frac{1}{\Delta^{n-d/2}} \tag{A.12}$$

A.2.1 One-point Green Function

The integral for the one point Green function, which appears in tadpole diagrams, is given by

$$\begin{aligned}
A_0(m^2) &= \int \frac{\mathrm{d}^d\ell}{(2\pi)^d} \frac{1}{\ell^2 - m^2 + i\epsilon} = \frac{-i\Gamma\left(\frac{2-d}{2}\right)}{(4\pi)^{d/2}} \int_0^1 \mathrm{d}x\, \delta(x-1)(xm^2 - i\epsilon)^{\frac{d-2}{2}} \\
&= \frac{i\Gamma\left(\frac{2-d}{2}\right)}{(4\pi)^{d/2}} (m^2 - i\epsilon)^{\frac{2-d}{2}}.
\end{aligned} \tag{A.13}$$

When the mass of the particle propagating in the loop is zero we get that $A_0 = 0$ which is what we expect since there are no dimensionful variables that carry the dimension of A_0 after integrating.
When $m \neq 0$, defining as usual $d = 4 - 2\epsilon$, the integral diverges as $1/\epsilon$ as $\epsilon \to 0$, in fact

$$A_0(m^2) = \frac{i}{16\pi^2} \frac{(4\pi)^\epsilon \Gamma(1+\epsilon)}{\epsilon(1-\epsilon)} (m^2)^{1-\epsilon}. \tag{A.14}$$

A.2.2 Two-point Massless Green Function

The two point function is more interesting in our case, in particular when the particles in the loop are considered massless. If the particles in the loop are massless $m_1 = m_2 = 0$, the external particles need to carry a non zero momentum $p^2 \neq 0$ otherwise the whole integral would be zero just like A_0. The integral is given by

$$\begin{aligned}
B_0(p^2) &= \int \frac{\mathrm{d}^d \ell}{(2\pi)^d} \frac{1}{[\ell^2 + i\epsilon][(\ell+p)^2 + i\eta]} \\
&= \frac{i\Gamma\left(\frac{4-d}{2}\right)}{(4\pi)^{d/2}} \int_0^1 \mathrm{d}x\, \mathrm{d}y\, \delta(x+y-1) \frac{1}{(-xyp^2 - i\eta)^{\frac{4-d}{2}}} \\
&= \frac{i\Gamma\left(\frac{4-d}{2}\right)}{(4\pi)^{d/2}} \int_0^1 \mathrm{d}x \left(x(1-x)(-p^2 - i\eta)\right)^{-\frac{4-d}{2}} \\
&= \frac{i\Gamma\left(\frac{4-d}{2}\right)}{(4\pi)^{d/2}} (-p^2 - i\eta)^{\frac{4-d}{2}} \frac{\Gamma^2\left(\frac{d-2}{2}\right)}{\Gamma(d-2)}.
\end{aligned} \tag{A.15}$$

Choosing $d = 4 - 2\epsilon$ we have

$$B_0(p^2) = \frac{i}{16\pi^2} \frac{(4\pi)^\epsilon \Gamma(\epsilon)\Gamma^2(1-\epsilon)}{\Gamma(2-2\epsilon)} (-p^2 - i\eta)^{-\epsilon}. \tag{A.16}$$

In the limit $\epsilon \to 0$, the multiplicative factors are finite and amount to a one, while we have to deal with the p^2 term

$$(-p^2 - i\eta)^{-\epsilon} = 1 - \epsilon \log\left(-p^2 - i\eta\right) + \mathcal{O}(\epsilon^2). \tag{A.17}$$

When $p^2 < 0$ the logarithm is perfectly defined while if $p^2 > 0$, then $-p^2 - i\eta$ is a complex negative number with a small imaginary part, so that is below the branch cut for the definition of the logarithm. In this case

$$(-p^2 - i\eta)^{-\epsilon} = 1 - \epsilon\left[\log\left(p^2\right) - i\pi\right] + \mathcal{O}(\epsilon^2). \tag{A.18}$$

In general

$$B_0(p^2) = \frac{i}{16\pi^2} \frac{\Gamma^2(1-\epsilon)}{\Gamma(2-2\epsilon)} \left[\frac{1}{\epsilon} - \gamma_E - \log\left(-\frac{p^2}{4\pi}\right) + \mathcal{O}(\epsilon^2)\right], \tag{A.19}$$

where the factor $(4\pi)^\epsilon$ has been absorbed into the factor $(-p^2)^{-\epsilon}$ and the $\Gamma(\epsilon)$ has been expanded

$$\Gamma(\epsilon) = \frac{1}{\epsilon} + \psi(1) + \frac{\epsilon}{2}\left[\frac{\pi^2}{6} + \psi^2(1) - \psi'(1)\right] + \mathcal{O}(\epsilon^2), \tag{A.20}$$

where $\psi(1) = -\gamma_E$ is the digamma function evaluated in one and $\psi'(1) = \pi^2/6$ is its derivative.

A.3 Passarino-Veltman Tensor Integral Decomposition

Up to now, we have only dealt with scalar integrals. For tensor integrals, we can use the Passarino-Veltmann [48] decomposition with which, in many cases, we can go back to a scalar integral times some tensor quantity that depends on the specific form and properties of the integrand. We will give now some useful examples that we will use throughout the notes.

A.3.1 Vector Two-Point Function

Let us first compute the simplest tensor integral

$$\mathcal{B}^\mu(p) = \int \frac{\mathrm{d}^d\ell}{(2\pi)^d} \frac{\ell^\mu}{\ell^2(\ell+p)^2}, \tag{A.21}$$

where $p^2 \neq 0$ and we employ dimensional regularization. We can easily see that the only relevant 4-vector on which the integral can depend is p^μ, therefore we can write

$$\mathcal{B}^\mu(p) = B_{11} p^\mu. \tag{A.22}$$

In order to find the coefficient, we can just project onto p^μ and get back to a scalar integral

$$p_\mu \mathcal{B}^\mu(p) = B_{11} p^2 = \int \frac{\mathrm{d}^d\ell}{(2\pi)^d} \frac{p\cdot\ell}{\ell^2(\ell+p)^2}. \tag{A.23}$$

Since $(\ell+p)^2 = p^2 + \ell^2 + 2p\cdot\ell$ we have that

$$p\cdot\ell = \frac{1}{2}\Big[(\ell+p)^2 - \ell^2 - p^2\Big]. \tag{A.24}$$

Using this in the integral we have

$$p^2 B_{11} = \frac{1}{2}\int \frac{\mathrm{d}^d\ell}{(2\pi)^d}\left[\frac{1}{\ell^2} - \frac{1}{(\ell+p)^2} - \frac{p^2}{\ell^2(\ell+p)^2}\right], \tag{A.25}$$

but this are just scalar one-loop integrals of the form of eq. (A.15), therefore

$$p^2 B_{11} = -\frac{p^2}{2} B_0(p^2) \implies B_{11} = -\frac{B_0(p^2)}{2}. \tag{A.26}$$

This gives us the final result

$$\mathcal{B}^\mu(p^2) = -\frac{B_0(p^2)}{2} p^\mu. \tag{A.27}$$

This computation gives the basic idea behind the Passarino-Veltmann decomposition: if we have a general p-tensor integral, we list all the possible p-tensors upon which the integral can depend. Then we project on such tensors by simply contracting the integral with them and one obtains a set of linear equations that can be solved to find the coefficients.

A.3.2 2-Tensor Two-Point Funcion

We would like to compute now the 2-tensor two point function, which is a tensor integral of the form

$$\mathcal{B}^{\mu\nu}(p) = \int \frac{\mathrm{d}^d\ell}{(2\pi)^d} \frac{\ell^\mu \ell^\nu}{\ell^2(\ell+p)^2}. \tag{A.28}$$

The only 2-tensors we can construct are $p^\mu p^\nu$ and $g^{\mu\nu}$, therefore

$$\mathcal{B}^{\mu\nu}(p) = B_{21} p^\mu p^\nu + B_{22} g^{\mu\nu}. \tag{A.29}$$

By projecting onto the two tensors

$$p_\mu \mathcal{B}^{\mu\nu}(p) = p^\nu \Big(p^2 B_{21} + B_{22} \Big) = \frac{1}{2} \int \frac{\mathrm{d}^d\ell}{(2\pi)^d} \frac{\ell^\nu}{\ell^2(\ell+p)^2} \Big[(\ell+p)^2 - \ell^2 - p^2 \Big], \tag{A.30}$$

$$g_{\mu\nu} \mathcal{B}^{\mu\nu}(p) = p^2 B_{21} + d B_{22} = \int \frac{\mathrm{d}^d\ell}{(2\pi)^d} \frac{\ell^2}{\ell^2(\ell+p)^2} = 0. \tag{A.31}$$

From eq. (A.31) one obtains that

$$B_{22} = -\frac{p^2}{d} B_{21}, \tag{A.32}$$

while from eq. (A.30)

$$p^\nu \left(p^2 - \frac{p^2}{d} \right) B_{21} = -p^\nu \frac{p^2 B_{11}}{2}, \tag{A.33}$$

which gives

$$B_{21} = \frac{d}{d-1} \frac{B_{11}}{2} = \frac{d}{d-1} \frac{B_0(p^2)}{4} \tag{A.34}$$

and consequently

$$B_{21} = -\frac{p^2}{d-1} \frac{B_0(p^2)}{4}. \tag{A.35}$$

Therefore

$$\mathcal{B}^{\mu\nu}(p) = \frac{1}{d-1} \left[\frac{d}{4} B_0(p^2) p^\mu p^\nu - \frac{p^2}{4} B_0(p^2) g^{\mu\nu} \right]. \tag{A.36}$$

Bibliography

[1] Abazov, V. M. et al. Precision measurement of the ratio B($t \to Wb$)/B($t \to Wq$) and Extraction of V_{tb}. *Phys. Rev. Lett.*, **107** (2011), 121802. arXiv:1106.5436, doi:10.1103/PhysRevLett.107.121802.

[2] Ablikim, M. et al. Measurement of the $D_s^+ \to \ell^+\nu_\ell$ branching fractions and the decay constant $f_{D_s^+}$. *Phys. Rev. D*, **94** (2016), 072004. arXiv:1608.06732, doi:10.1103/PhysRevD.94.072004.

[3] Ablikim, M. et al. Determination of the pseudoscalar decay constant $f_{D_s^+}$ via $D_s^+ \to \mu^+\nu_\mu$. *Phys. Rev. Lett.*, **122** (2019), 071802. arXiv:1811.10890, doi:10.1103/PhysRevLett.122.071802.

[4] Acosta, D. et al. Measurement of $B(t \to Wb)/B(t \to Wq)$ at the Collider Detector at Fermilab. *Phys. Rev. Lett.*, **95** (2005), 102002. arXiv:hep-ex/0505091, doi:10.1103/PhysRevLett.95.102002.

[5] Alexander, J. P. et al. Measurement of $BD_s^+ \to \ell^+\nu$ and the Decay Constant $f_{D_s^+}$ From $600/pb^{-1}$ of $e^\pm$ Annihilation Data Near 4170 MeV. *Phys. Rev. D*, **79** (2009), 052001. arXiv:0901.1216, doi:10.1103/PhysRevD.79.052001.

[6] Antonelli, M. et al. An Evaluation of $|V_{us}|$ and precise tests of the Standard Model from world data on leptonic and semileptonic kaon decays. *Eur. Phys. J. C*, **69** (2010), 399. arXiv:1005.2323, doi:10.1140/epjc/s10052-010-1406-3.

[7] Ashmore, J. F. A Method of Gauge Invariant Regularization. *Lett. Nuovo Cim.*, **4** (1972), 289. doi:10.1007/BF02824407.

[8] Baez, J. C. and Huerta, J. The Algebra of Grand Unified Theories. *Bull. Am. Math. Soc.*, **47** (2010), 483. arXiv:0904.1556, doi:10.1090/S0273-0979-10-01294-2.

[9] Baikov, P. A., Chetyrkin, K. G., and Kühn, J. H. Five-Loop Running of the QCD coupling constant. *Phys. Rev. Lett.*, **118** (2017), 082002. arXiv:1606.08659, doi:10.1103/PhysRevLett.118.082002.

[10] Bollini, C. G. and Giambiagi, J. J. Dimensional Renormalization: The Number of Dimensions as a Regularizing Parameter. *Nuovo Cim. B*, **12** (1972), 20. doi:10.1007/BF02895558.

[11] Bona, M. et al. The 2004 UTfit collaboration report on the status of the unitarity triangle in the standard model. *JHEP*, **07** (2005), 028. arXiv:hep-ph/0501199, doi:10.1088/1126-6708/2005/07/028.

[12] Buras, A. J., Lautenbacher, M. E., and Ostermaier, G. Waiting for the top quark mass, $K^+ \to \pi^+ \nu \bar{\nu}$, $B^0_{(s)} - \bar{B}^0_{(s)}$ mixing and CP asymmetries in B decays. *Phys. Rev. D*, **50** (1994), 3433. arXiv:hep-ph/9403384, doi:10.1103/PhysRevD.50.3433.

[13] Cabibbo, N. Unitary Symmetry and Leptonic Decays. *Phys. Rev. Lett.*, **10** (1963), 531. doi:10.1103/PhysRevLett.10.531.

[14] Cabibbo, N., Swallow, E. C., and Winston, R. Semileptonic hyperon decays. *Ann. Rev. Nucl. Part. Sci.*, **53** (2003), 39. arXiv:hep-ph/0307298, doi:10.1146/annurev.nucl.53.013103.155258.

[15] Charles, J., Hocker, A., Lacker, H., Laplace, S., Le Diberder, F. R., Malcles, J., Ocariz, J., Pivk, M., and Roos, L. CP violation and the CKM matrix: Assessing the impact of the asymmetric B factories. *Eur. Phys. J. C*, **41** (2005), 1. arXiv:hep-ph/0406184, doi:10.1140/epjc/s2005-02169-1.

[16] Cicuta, G. M. and Montaldi, E. Analytic renormalization via continuous space dimension. *Lett. Nuovo Cim.*, **4** (1972), 329. doi:10.1007/BF02756527.

[17] Collins, J. C. *Renormalization: An Introduction to Renormalization, The Renormalization Group, and the Operator Product Expansion*, vol. 26 of *Cambridge Monographs on Mathematical Physics*. Cambridge University Press, Cambridge (1986). ISBN 978-0-521-31177-9, 978-0-511-86739-2. doi:10.1017/CBO9780511622656.

[18] del Amo Sanchez, P. et al. Measurement of the Absolute Branching Fractions for $D_s^- \to \ell^- \bar{\nu}_\ell$ and Extraction of the Decay Constant f_{D_s}. *Phys. Rev. D*, **82** (2010), 091103. [Erratum: Phys.Rev.D 91, 019901 (2015)]. arXiv:1008.4080, doi:10.1103/PhysRevD.82.091103.

[19] Dunietz, I., Greenberg, O. W., and Wu, D.-d. A Priori Definition of Maximal CP Violation. *Phys. Rev. Lett.*, **55** (1985), 2935. doi:10.1103/PhysRevLett.55.2935.

[20] Englert, F. and Brout, R. Broken Symmetry and the Mass of Gauge Vector Mesons. *prl*, **13** (1964), 321. doi:10.1103/PhysRevLett.13.321.

[21] Faddeev, L. D. and Popov, V. N. Feynman Diagrams for the Yang-Mills Field. *Phys. Lett. B*, **25** (1967), 29. doi:10.1016/0370-2693(67)90067-6.

[22] Gell-Mann, M. The interpretation of the new particles as displaced charge multiplets. *Il Nuovo Cimento*, **4** (1956), 848. doi:10.1007/BF02748000.

[23] Georgi, H. *Lie Algebras In Particle Physics: from Isospin To Unified Theories*. Frontiers in Physics. Avalon Publishing (1999). ISBN 9780813346113.

[24] Gilman, F. J., Kleinknecht, K., and Renk, B. The Cabibbo-Kobayashi-Maskawa quark-mixing matrix. *Phys. Lett. B*, **592** (2004), 793.

[25] Glashow, S. L. The renormalizability of vector meson interactions. *Nucl. Phys.*, **10** (1959), 107. doi:10.1016/0029-5582(59)90196-8.

[26] Glashow, S. L. Partial Symmetries of Weak Interactions. *Nucl. Phys.*, **22** (1961), 579. doi:10.1016/0029-5582(61)90469-2.

[27] Glashow, S. L., Iliopoulos, J., and Maiani, L. Weak Interactions with Lepton-Hadron Symmetry. *Phys. Rev. D*, **2** (1970), 1285. doi:10.1103/PhysRevD.2.1285.

[28] Greenberg, O. W. Rephase Invariant Formulation of CP Violation in the Kobayashi-Maskawa Framework. *Phys. Rev. D*, **32** (1985), 1841. doi:10.1103/PhysRevD.32.1841.

[29] Gribov, V. N. Quantization of Nonabelian Gauge Theories. *Nucl. Phys. B*, **139** (1978), 1. doi:10.1016/0550-3213(78)90175-X.

[30] Gross, D. J. and Wilczek, F. Ultraviolet Behavior of Nonabelian Gauge Theories. *Phys. Rev. Lett.*, **30** (1973), 1343. doi:10.1103/PhysRevLett.30.1343.

[31] Guralnik, G. S., Hagen, C. R., and Kibble, T. W. B. Global Conservation Laws and Massless Particles. *Phys. Rev. Lett.*, **13** (1964), 585. doi:10.1103/PhysRevLett.13.585.

[32] Guralnik, G. S., Hagen, C. R., and Kibble, T. W. B. Global conservation laws and massless particles. *Phys. Rev. Lett.*, **13** (1964), 585. Available from: https://link.aps.org/doi/10.1103/PhysRevLett.13.585, doi:10.1103/PhysRevLett.13.585.

[33] Hardy, J. and Towner, I. S. $|V_{ud}|$ from nuclear β decays. *PoS*, **CKM2016** (2016), 028. doi:10.22323/1.291.0028.

[34] Herzog, F., Ruijl, B., Ueda, T., Vermaseren, J. A. M., and Vogt, A. The five-loop beta function of Yang-Mills theory with fermions. *JHEP*, **02** (2017), 090. arXiv:1701.01404, doi:10.1007/JHEP02(2017)090.

[35] Higgs, P. W. Broken Symmetries and the Masses of Gauge Bosons. *Phys. Rev. Lett.*, **13** (1964), 508. doi:10.1103/PhysRevLett.13.508.

[36] Higgs, P. W. Broken symmetries and the masses of gauge bosons. *Phys. Rev. Lett.*, **13** (1964), 508. Available from: https://link.aps.org/doi/10.1103/PhysRevLett.13.508, doi:10.1103/PhysRevLett.13.508.

[37] Higgs, P. W. Broken symmetries, massless particles and gauge fields. *Phys. Lett.*, **12** (1964), 132. doi:10.1016/0031-9163(64)91136-9.

[38] Higgs, P. W. Spontaneous Symmetry Breakdown without Massless Bosons. *Phys. Rev.*, **145** (1966), 1156. doi:10.1103/PhysRev.145.1156.

[39] Jarlskog, C. A Basis Independent Formulation of the Connection Between Quark Mass Matrices, CP Violation and Experiment. *Z. Phys. C*, **29** (1985), 491. doi:10.1007/BF01565198.

[40] Jarlskog, C. Commutator of the Quark Mass Matrices in the Standard Electroweak Model and a Measure of Maximal CP Violation. *Phys. Rev. Lett.*, **55** (1985), 1039. doi:10.1103/PhysRevLett.55.1039.

[41] Kayis-Topaksu, A. et al. Measurement of topological muonic branching ratios of charmed hadrons produced in neutrino-induced charged-current interactions. *Phys. Lett. B*, **626** (2005), 24. doi:10.1016/j.physletb.2005.08.082.

[42] Kibble, T. W. B. Symmetry breaking in nonAbelian gauge theories. *Phys. Rev.*, **155** (1967), 1554. doi:10.1103/PhysRev.155.1554.

[43] Kobayashi, M. and Maskawa, T. CP Violation in the Renormalizable Theory of Weak Interaction. *Prog. Theor. Phys.*, **49** (1973), 652. doi:10.1143/PTP.49.652.

[44] Kowalewski, R. and Mannel, T. Semileptonic b-Hadron Decays, Determination of V_{cb}, V_{ub} (2020). doi:10.1093/ptep/ptaa104.

[45] Leader, E. and Predazzi, E. *An Introduction to gauge theories and modern particle physics. Vol. 2: CP violation, QCD and hard processes*, vol. 4. Cambridge University Press (1996). ISBN 978-0-521-49951-4, 978-1-139-24099-4.

[46] Luthe, T., Maier, A., Marquard, P., and Schroder, Y. The five-loop Beta function for a general gauge group and anomalous dimensions beyond Feynman gauge. *JHEP*, **10** (2017), 166. arXiv:1709.07718, doi:10.1007/JHEP10(2017)166.

[47] Nishijima, K. Charge Independence Theory of V Particles. *Progress of Theoretical Physics*, **13** (1955), 285. doi:10.1143/PTP.13.285.

[48] Passarino, G. and Veltman, M. J. G. One Loop Corrections for e+ e- Annihilation Into mu+ mu- in the Weinberg Model. *Nucl. Phys. B*, **160** (1979), 151. doi:10.1016/0550-3213(79)90234-7.

[49] Peccei, R. D. The Strong CP problem and axions. *Lect. Notes Phys.*, **741** (2008), 3. arXiv:hep-ph/0607268, doi:10.1007/978-3-540-73518-2_1.

[50] Peskin, M. *Concepts of Elementary Particle Physics*. Oxford Master Series in Condensed Matter Physics Series. Oxford University Press (2019). ISBN 9780198812197.

[51] Pich, A. Chiral perturbation theory. *Rept. Prog. Phys.*, **58** (1995), 563. arXiv:hep-ph/9502366, doi:10.1088/0034-4885/58/6/001.

[52] Politzer, H. D. Reliable Perturbative Results for Strong Interactions? *Phys. Rev. Lett.*, **30** (1973), 1346. doi:10.1103/PhysRevLett.30.1346.

[53] Salam, A. Weak and Electromagnetic Interactions. *Conf. Proc. C*, **680519** (1968), 367. doi:10.1142/9789812795915_0034.

[54] Salam, A. and Ward, J. C. Weak and electromagnetic interactions. *Il Nuovo Cimento*, **11** (1959), 568. doi:10.1007/BF02726525.

[55] Schwartz, M. *Quantum Field Theory and the Standard Model*. Quantum Field Theory and the Standard Model. Cambridge University Press (2014). ISBN 9781107034730.

[56] Srednicki, M. *Quantum Field Theory*. Cambridge University Press (2007). ISBN 9781139462761.

[57] 't Hooft, G. A Planar Diagram Theory for Strong Interactions. *Nucl. Phys. B*, **72** (1974), 461. doi:10.1016/0550-3213(74)90154-0.

[58] 't Hooft, G. Large N. In *The Phenomenology of Large N(c) QCD* (2002). arXiv:hep-th/0204069, doi:10.1142/9789812776914_0001.

[59] 't Hooft, G. and Veltman, M. J. G. Regularization and Renormalization of Gauge Fields. *Nucl. Phys. B*, **44** (1972), 189. doi:10.1016/0550-3213(72)90279-9.

[60] Weinberg, S. A Model of Leptons. *Phys. Rev. Lett.*, **19** (1967), 1264. doi:10.1103/PhysRevLett.19.1264.

[61] Weinberg, S. A Model of Leptons. *prl*, **19** (1967), 1264. doi:10.1103/PhysRevLett.19.1264.

[62] Wilson, K. G. Confinement of Quarks. *Phys. Rev. D*, **10** (1974), 2445. doi:10.1103/PhysRevD.10.2445.

[63] Wolfenstein, L. Parametrization of the Kobayashi-Maskawa Matrix. *Phys. Rev. Lett.*, **51** (1983), 1945. doi:10.1103/PhysRevLett.51.1945.

[64] Wu, D.-d. The Rephasing Invariants and CP. *Phys. Rev. D*, **33** (1986), 860. doi:10.1103/PhysRevD.33.860.

[65] Zupanc, A. et al. Measurements of branching fractions of leptonic and hadronic D_s^+ meson decays and extraction of the D_s^+ meson decay constant. *JHEP*, **09** (2013), 139. arXiv:1307.6240, doi:10.1007/JHEP09(2013)139.

[66] Zyla, P. A. et al. Review of Particle Physics. *PTEP*, **2020** (2020), 083C01. doi:10.1093/ptep/ptaa104.