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Motivation

Quantum field theory (QFT) stands as one of the most remarkable mathematical frameworks for understanding fundamental particles and their interactions. Its predictive power has delivered profound insights into the behavior of particles, culminating in the Standard Model of particle physics. However, the theory encounters significant challenges when confronted with the complexities of strong coupling regimes, such as those governing phenomena like confinement and the nature of strongly interacting quarks and gluons. These intriguing phenomena remain pivotal open questions in theoretical and mathematical physics.

In this context, supersymmetry emerges as a powerful and elegant tool for exploring the dynamics of strongly coupled gauge theories. Its application to gauge theories has led to groundbreaking insights into the non-perturbative aspects of QFT, especially through the concept of dualities. Supersymmetric dualities, in fact, usually provide weakly coupled descriptions to asymptotically free gauge theories. These dualities have been checked in various fashions: 't Hooft anomaly matching, supersymmetric localization for partition functions on various manifolds, good behavior under renormalization group flows, matching of global symmetries and, as a more recent development, matching of generalized symmetries by the SymTFT approach.

String theory and its higher-dimensional counterpart, M-theory, provide an elegant and unified approach to these aspects. The dynamics of D-brane systems, as well as their embedding in geometric engineering within M-theory, usually give a more fundamental description of such dualities. Particularly interesting are non-Lagrangian theories that can be built in string/M-theory. Likewise, the AdS/CFT correspondence also provides the tools to probe the quantum dynamics of exotic gravitational systems such as black holes (BH) by using QFT tools like the superconformal index. Here, the BH Hawking-Page (HP) phase transition could be exploited to probe the confinement/deconfinement transition in four-dimensional maximally supersymmetric YM and beyond.

Motivated by this, in this thesis we will mainly analyze supersymmetric QFTs in various dimensions. Part of the thesis will be devoted to the study of supersymmetric dualities, where we will initially discuss the fundamental result of Seiberg, and then apply such reasoning to theories in three dimensions in the presence of adjoint matter. These dualities will be checked by means of matching the 3d partition functions on both the electric and magnetic sides. Then we will focus on the topic of generalized symmetries in a special class of theories which are non-Lagrangian and arise in string theory by stacking D3 branes on particular orbifold geometries. Next, we will delve into the realm of AdS/CFT by using its powerful tools to, on one hand, compute observables of two-dimensional SCFTs arising from compactifications of M5 branes on a special geometry known as a Spindle, and on the other hand to compute the microstates of black holes in the presence of probe defects from the dual field theory.

Thesis overview

This thesis is divided into two main parts: Review and Original Contributions. As the title implies, the first part is devoted to broadly reviewing the basic topics needed for the rest of the thesis. Nonetheless, some Chapters in the second part will begin by reviewing in more detail the basics needed to understand the paper.

- **Part I: Review**
    - **Review 1: Supersymmetric Quantum Field Theories (QFTs)**
        - *Supersymmetric QFTs*: This section outlines the basic concepts of supersymmetric QFTs and their significance in modern physics. It lays the foundation for understanding more complex theories discussed later in the thesis.
        - *Dynamics of SQCD*: The dynamics of Supersymmetric Quantum Chromodynamics (SQCD) is explored as a prototypical example featuring a dual description. The fundamental results and applications to SQCD are discussed in detail.
    - **Review 2: String Theory and M-Theory**
        - *The Bosonic String and the Superstring*: This section begins with a discussion of the classical bosonic string, introducing its formulation and significance in theoretical physics. It then extends to the inclusion of fermions, which leads to the development of the superstring. Finally, the quantum aspects of the string theory are explored.
        - *Extended Objects: The Branes*: This section delves into branes, extended objects that play a crucial role in string theory. Their dynamics and implications for theoretical physics are examined.
        - *M-Theory*: This Chapter introduces M-theory, an extension of string theory that unifies the five superstring theories. The importance and applications of M-theory in modern theoretical physics are discussed.
        - *The AdS/CFT Correspondence*: The AdS/CFT correspondence, a powerful duality between gravity and field theories, is explained. We give some basics of this topic by following the original result by Maldacena.

- **Part II: Original Contributions**
  - **Topic 1: Supersymmetric Dualities**
    - This chapter focuses on various supersymmetric dualities and collects the results of [36, 37]. Initially, the idea of deconfinement is presented to then give some basic results about 3d partition functions. We conclude the introduction by spelling out the classification of 3d chiral and non-chiral dualities.
    - Sections 3.2 and 3.3 give the results of the aforementioned papers.
  - **Topic 2 : Generalized Symmetries**
    - This chapter delves into the topic of generalized symmetries by studying the global structure for a special class of non-Lagrangian theories in four-dimension. These theories arise in string theory by stacking D3 branes on specific orbifold geometries, known as S-folds. The implications of these symmetries are thoroughly examined by also postulating the existence of non-invertible self-duality symmetries at certain values of the holomorphic coupling.
  - **Topic 3: AdS/CFT Correspondence and its Applications**
    - *2d SCFTs from M-theory*: Using the AdS/CFT correspondence, in this section we find the central charge of two-dimensional Super Conformal Field Theories (SCFTs) resulting from the compactification of M5 branes on a product geometry consisting of a Riemann surface and a Spindle. The Spindle is an intriguing orbifold where supersymmetry is preserved in a non-trivial manner. The computation of the central charge is done both from field theory and from supergravity, where the final result can be found without actually solving for the flow from AdS_5 to AdS_3. We conclude by matching the two results.
    - *Black Hole microstate counting*: The entropy of a black hole system coupled to a probe D3-brane is computed by considering the superconformal index of the dual field theory setup: $\mathcal{N} = 4$ SYM in the presence of a specific Gukov-Witten surface defect. The entropy is computed from the index in two ways: following the Cardy-like limit and the Bethe Ansatz approach. We then match the two results and get some conclusions about the fate of the GW defect under dimensional reduction the thermal circle by studying the 3d EFT arising from the Cardy-like procedure.

Organizational note

Throughout the thesis, references to papers and the author's contributions are color-coded, with regular citations in red and the author's papers in green.

Part I

CHAPTER 1

Supersymmetric Theories with Four Supercharges in Diverse Dimensions

Contents

During the 20th century, the development of Quantum Field Theory (QFT), a mathematical framework for understanding the subatomic world, marked a significant milestone in theoretical physics. Alongside it, the Standard Model (SM) emerged as a cornerstone, providing deep insights into the behavior and interactions of the fundamental building blocks of our universe: particles. These particles are categorized into two main groups: fermions, which exhibit half-integer statistics and constitute matter, and bosons, with integer statistics, responsible for mediating interactions.

Despite the remarkable successes of the SM, it poses several intriguing puzzles. In addition to the gravity can only be treated classically (a topic we will explore in the following Chapter), we are interested in the non-perturbative aspects of Quantum Chromodynamics (QCD) at low energies. The challenge of confinement, a phenomenon central to QFT, becomes more manageable with the introduction of supersymmetry, a space-time symmetry that associates to every fermion a bosonic partner and viceversa. Beyond its conceptual elegance, supersymmetry offers a way for studying strongly coupled Quantum Field Theories (QFTs). While our focus will not extend to all its facets, two key phenomenological benefits of supersymmetry within the SM context include addressing the hierarchy problem through the presence of supersymmetric partners influencing the Higgs mass's loop behavior and achieving the unification of all gauge interactions in the minimal supersymmetric extension of the SM, where all gauge couplings converge at very high energies.

This Chapter will explore the basics of theories with various degrees of supersymmetry determined by the number of fermionic generators, or SUSY charges. The number of

Table 1.1: Possible allowed SUSY algebras in various dimensions for theories without gravity. In two and six dimensions left and right spinors are independent and therefore the SUSY algebras are specified by the number of spinors in each chirality $(\mathcal{N}_R, \mathcal{N}_L)$.

Space-time dimensions	Possible number of (global) SUSY $\mathcal{N}$
2	$(1,0),(1,1),(2,0),(2,2),(4,0),(4,4),(8,8)$
3	$1,2,3,4,5,6,8$
4	$1,2,3,4$
5	$2,4$
6	$(2,0),(2,2)$

supercharges depends on the space-time dimension, since different dimensions feature distinct fermionic representations. Considering global supersymmetry alone imposes an upper limit on the number of supercharges, which, in turn, depends on the space-time dimension. For instance, in four dimensions, the maximum amount of supersymmetry is $\mathcal{N} = 4$. Beyond this limit, the number of supercharges introduces particles with higher spin: in four dimensions, for $\mathcal{N} \leq 8$, there is a spin 2 representation which contains a graviton. Table 1.1 provides a summary of physically allowed values of $\mathcal{N}$ in various dimensions for theories without gravity.

This Chapter primarily draws upon several excellent texts and reviews available [96, 107, 135, 300, 301, 308].

This Chapter is organized as follows. We start in Section 1.1 by summarizing the basic elements of supersymmetry. In Subsection 1.1.1, we study the possible supersymmetry algebras in general dimensions and then construct their representations for d= 4. In Subsection 1.1.2, we delve into the Lagrangian formulation of 4D supersymmetric quantum field theories (SQFTs) by constructing the supersymmetric counterparts of the vector field, needed for gauge interactions, and the chiral field, needed to add matter possibly beyond the adjoint representation.

Then, in Subsection 1.2, we examine the prototypical example of a supersymmetric gauge theory: the supersymmetric version of Quantum Chromodynamics (SQCD). We will see that for a generic number of flavors, the theory possesses a very intricate phase diagram. This theory admits a weakly coupled description in its strongly coupled regime. This behavior, denoted as ”duality”, is discussed in Subsection 1.2.1 and serves as a prototypical example for understanding later Chapters.

By the end of the Chapter, in Section 1.3, we review the theory of 3d SQFTs, particularly how they can arise from dimensional reduction from 4d and the prominent role played by monopole operators. This topic is analyzed in Subsection 1.3.2. We conclude the Chapter in Subsection 1.3.3 by briefly describing the role played by Chern-Simons interactions in these theories.

1.1 Basic Elements of Supersymmetry

1.1.1 The SUSY Algebra

In this section we are going to lay down the basics of supersymmetric theories, starting from four dimensions. As mentioned in the introduction, supersymmetry associates to every fermion in a theory a boson and vice-versa thus balancing the number of fermionic and bosonic degrees of freedom. In a generic QFT, both degrees of freedom need not be the same and in usual phenomenological models they actually are not the same.

The fundamental theoretical reason to why supersymmetry is a viable option for a QFT lies in the Coleman-Mandula theorem [155] which states that given a theory that can be described by an S-matrix which is consistent with some fundamental conditions like locality, positivity and finiteness number of particles, the only possible symmetries are those generated by the Poincarè group plus some internal symmetry group which commutes with space-time symmetry.

If there were to be additional bosonic symmetries other than those, the S-matrix would be trivial, *i.e.* all theories would be free field theories.

The only "shady" request by Coleman and Mandula is the fact that symmetry generators must be bosonic, *i.e.* that symmetries are given by Lie algebras. It was later shown [210] that this requirement can be relaxed by considering graded Lie superalgebras, thus including the possibility of having anticommuting generators known as supercharges. Supersymmetry is the only S-matrix quantum symmetry that may connect states of different spins.

A graded Lie superalgebra is a vector space with a $\mathbb{Z}$ grading obtained by

$$\mathfrak{g} = \bigoplus_{n\in\mathbb{Z}} \mathfrak{g}_n \tag{1.1}$$

such that the Lie bracket respects this gradation

$$[\mathfrak{g}_n, \mathfrak{g}_m] \subset \mathfrak{g}_{n+m}. \tag{1.2}$$

The supersymmetry algebra is given by a grade one Lie superalgebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ where $\mathfrak{g}_0$ is the Poincarè algebra plus any internal symmetry while $\mathfrak{g}_1 = (Q^A_\alpha, \bar{Q}^A_{\dot\alpha})$, with $A = 1, \ldots, \mathcal{N}$ are supercharges: a set of $2\mathcal{N}$ anti-commuting fermionic generators which, depending on the space-time dimension, transform in the minimal spinor representation of the Lorentz group. In table 1.2 we give a list of such representations.

Table 1.2: Superalgebras in diverse dimensions. The automorphism group of the SUSY algebra is usually known as R-symmetry. The tensor $B = C\Gamma^0$ where C is the charge-conjugation matrix, while J_{AB} is the usual invariant symplectic tensor. For d odd the superalgebra can be extended by a central term which we do not write explicitly. Here, we follow the conventions of [135].

Space-time d	1	2
Supercharges	Majorana	Majorana-Weyl
Reality condition	$(Q^A)^\dagger = Q^A$	$(Q^A_\pm)^\dagger = \bar{Q}_{\pm A} = Q^A_\pm$
Superalgebra	$\{Q^A, Q^B\} = H\delta^{AB}$	$\{Q^A_\pm, Q^B_\pm\} = 2\delta^{AB}P_\pm$ $\{Q^A_+, Q^B_-\} = Z^{AB}$
Automorphism	$\mathrm{SO}(\mathcal{N})$	$\mathrm{SO}(\mathcal{N}_+) \times \mathrm{SO}(\mathcal{N}_-)$
Space-time d	3	4
Supercharges	Majorana	Weyl
Reality condition	$(Q^A_\alpha)^\dagger = Q^A_\alpha$	$(Q^A_\alpha)^\dagger = \bar{Q}_{\dot\alpha A}$
Superalgebra	$\left\{Q^A_\alpha, Q^B_\beta\right\} = 2\delta^{AB}(\Gamma^\mu\Gamma^0)_{\alpha\beta}P_\mu$	$\left\{Q^A_\alpha, \bar{Q}_{\dot\beta B}\right\} = 2\delta^A_B(\sigma^\mu)_{\alpha\dot\beta}P_\mu$ $\left\{Q^A_\alpha, Q^B_\beta\right\} = \epsilon_{\alpha\beta}Z^{AB}$
Automorphism	$\mathrm{SO}(\mathcal{N})$	$\mathrm{U}(\mathcal{N})$
Space-time d	5	6
Supercharges	Symplectic-Majorana	Symplectic-Majorana-Weyl
Reality condition	$(Q^I_\alpha)^\dagger = J_{AB}B_\alpha{}^\beta Q^B_\beta$	$(Q^I_\alpha)^\dagger \equiv \bar{Q}_{\alpha A} = J_{AB}B_\alpha{}^\beta Q^B_\beta$
Superalgebra	$\left\{Q^A_\alpha, Q^B_\beta\right\} = 2J^{AB}(\Gamma^\mu C)_{\alpha\beta}P_\mu$	$\left\{Q^A_\alpha, \bar{Q}^B_\beta\right\} = 2J^{AB}(\Sigma^\mu)_{\alpha\beta}P_\mu$ $\left\{Q^A_\alpha, \bar{Q}^B_{\dot\beta}\right\} = C_{\alpha\dot\beta}Z^{AB}$
Automorphism	$\mathrm{Sp}(\mathcal{N})$	$\mathrm{Sp}(\mathcal{N}_R) \times \mathrm{Sp}(\mathcal{N}_L)$

Focusing ourselves to the case of $d = 4$ we give here the full superPoincarè algebra

$$\begin{aligned}
[M_{\mu\nu}, M^{\rho\sigma}] &= -2\delta^{[\rho}_{[\mu} M_{\nu]}{}^{\sigma]}, \\
[P_\mu, M_{\nu\rho}] &= \eta_{\mu[\nu} P_{\rho]}, \\
[P_\mu, P_\nu] &= 0, \\
\left[P_\mu, Q^A_\alpha\right] &= 0, \\
\left[P_\mu, \bar{Q}_{\dot{\alpha}A}\right] &= 0, \\
\left[M_{\mu\nu}, Q^A_\alpha\right] &= \mathrm{i}(\sigma_{\mu\nu})_\alpha{}^\beta Q^A_\beta, \\
\left[M_{\mu\nu}, \bar{Q}_{\dot{\alpha}A}\right] &= \mathrm{i}(\bar{\sigma}_{\mu\nu})_{\dot{\alpha}}{}^{\dot{\beta}} \bar{Q}_{\dot{\beta}A}, \\
\left\{Q^A_\alpha, \bar{Q}_{\dot{\beta}B}\right\} &= 2\delta^A_B (\sigma^\mu)_{\alpha\dot{\beta}} P_\mu, \\
\left\{Q^A_\alpha, Q^B_\beta\right\} &= \epsilon_{\alpha\beta} Z^{AB}, \\
\left\{\bar{Q}_{\dot{\alpha}A}, \bar{Q}_{\dot{\beta}B}\right\} &= \epsilon_{\dot{\alpha}\dot{\beta}} (Z_{AB})^*.
\end{aligned} \tag{1.3}$$

The elements Z^{AB} in (1.3) are central charges which commute with all the other symmetry generators by definition.
From (1.3) we notice that Q^A_1 and $(Q^A_2)^\dagger$ are raising operators for the z-component of the spin, making the spin increase by $\frac{1}{2}$ while Q^A_2 and $(Q^A_1)^\dagger$ are lowering operators. This is exactly the property needed for relating bosons to fermions and vice versa: in any representation of the SUSY algebra there are going to be both fermions and bosons related by powers of the relevant Qs.
A last comment is needed concerning the commutator of the supersymmetry generators and internal global symmetries. In general the Qs carry some representation of the internal symmetry group. The largest symmetry which acts non-trivially on these generators is given by the automorphism group of the SUSY algebra. In four dimensions this is given by $\mathrm{U}(\mathcal{N})$ in the absence of central terms. The SUSY algebra dictates that also the central charges are in non-trivial representations of the automorphism group and therefore, whenever these are non-zero, only a subgroup of the automorphism group remains. This is what is called R-symmetry group.
The discussion can be extended by adding another space-time symmetry: conformal invariance. Superconformal QFTs are going to be relevant for us in the coming sections being the RG-fixed point of any theory conformal, and for further details we refer the reader to the seminal paper [160].
With the SUSY algebra (1.3) at hand, we can now dive into its representations. The Poincarè group is still a subgroup of its supersymmetric counterpart, and is well known that representations of the Poincarè group can be classified by the two Casimir invariants

$$P^2 = P_\mu P^\mu \qquad W^2 = W_\mu W^\mu \tag{1.4}$$

where P_μ is the 4-momentum and $W^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} P_\nu M_{\rho\sigma}$ is the Pauli-Lubanski vector. We can divide representations into two categories based on their mass m and spin j

$$\begin{aligned} &\text{Massive rep: } P^\mu = (m,0,0,0) \qquad \begin{cases} P^2 = -m^2 \\ W^2 = m^2 j(j+1) \end{cases} \\ &\text{Massless rep: } P^\mu = (E,0,0,E) \qquad \begin{cases} P^2 = 0 \\ W^2 = 0 \end{cases} \qquad W^\mu = M_{12} P^\mu \end{aligned} \tag{1.5}$$

Thus for massless representations the two Casimir are proportional with proportionality constant $M_{12} = \pm j$ which we call helicity and representations are labelled by it as well as the energy E.

To build representations of the superPoincarè algebra, the Pauli-Lubanski vector is not a well-defined Casimir anymore since the supercharges mix particles with different spin j in the same multiplet. We call a representation of the superPoincarè algebra a supermultiplet.

Consider for now the case where the algebra (1.3) has no central extension $Z^{AB} = 0$[1]. Then the SUSY algebra tells us[2]

$$\{Q^A_\alpha, \bar{Q}_{\dot{\alpha}B}\} = \delta^A_B \sigma^\mu_{\alpha\dot{\beta}} P_\mu \qquad \{Q^A_\alpha, Q^B_\beta\} = \{Q_{\dot{\alpha}A}, _{\dot{\beta}B}\} = 0 \tag{1.6}$$

Consider first states of positive mass $m > 0$ and, going to the rest frame $P^\mu = (m,0,0,0)$, the algebra is

$$\{Q^A_\alpha, \bar{Q}_{\dot{\alpha}B}\} = \delta^A_B \delta_{\alpha\dot{\beta}} m \tag{1.7}$$

which together with the remaining anti-commutators in (1.6) is the algebra for $2\mathcal{N}$ fermionic creation and annihilation operators. So an irreducible representation has dimension $2^{2\mathcal{N}}$.

The massless case $m = 0$ we consider the frame where $P_\mu = (E,0,0,E)$ in which case the algebra is

$$\{Q^A_\alpha, \bar{Q}_{\dot{\alpha}B}\} = \delta^A_B 2E \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}_{\alpha\dot{\beta}} \tag{1.8}$$

together with the remaining zero anti-commutators. The only difference between the two algebras is that one of the eigenvalues has been turned from a one to a zero. Focusing on the case $\alpha = \dot{\beta} = 2$ we see that

$$\{Q^A_2, \bar{Q}_{2A}\} = \{Q^A_2, (Q^A_2)^\dagger\} = 0 \qquad \text{Here there is no sum over } A \tag{1.9}$$

where in the first step we used the Weyl condition[3]. Since in general the anti-commutator of any operator with its adjoint is positive definite, the only possibility is that on massless

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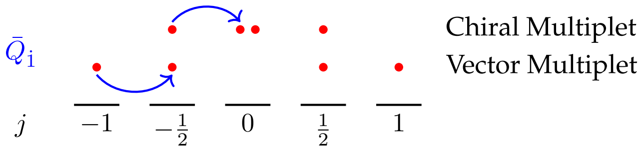
[1] This is indeed necessary for massless representations due to the positivity of the Hilbert space as we later in the Chapter.

[2] It is amusing to see that the SUSY algebra is fixed to be of this kind thanks to the Coleman-Mandula theorem. In fact, the first anti-commutator has to be proportional to the 4-momentum being it the only Lorentz vector (or possibly the special conformal transformation in the case of conformal theories). The remaning anti-commutators are more subtle since by the product of two spinors of the same kind we could get either a Lorentz scalar or a (anti-)self dual tensor. The tensor cannot be there without conformal invariance and so only scalars remain, the Z^{AB} central charges.

[3] We are always in Lorentz signature unless otherwise stated.

states the two operators Q_2^A and $\bar{Q}_{2A}$ are zero. This sets half of the Qs to zero, therefore the picture is the same as for positive mass but with only $\mathcal{N}$ creation and annihilation operator: an irreducible massless representation has dimension $2^{\mathcal{N}}$. There is an additional caveat: since in general helicities will not be distributed symmetrically around zero, to have a CPT-invariant theory, we will need to double up each multiplet.

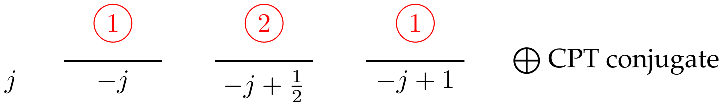
Consider now small values of $\mathcal{N}$, namely $\mathcal{N} = 1$. The massless representations are labelled by the helicity j and in $\mathcal{N} = 1$ we have only a pair of creation and annihilation operator. So in each supermultiplet we have two particles with helicity differing by $\frac{1}{2}$. Since we will only consider theories without gravity, as stated in the introduction, we will reduce ourselves to the case of representations with $|j| \leq 1$. In this case, for $m = 0$ and $\mathcal{N} = 1$ there are only two such representations



(1.10)

Both representations have 4 states. Note that the doubling of the spectrum is due to the requirement of CPT-invariance. Some comments are in order: the chiral multiplet can get a mass in a completely supersymmetric fashion while the vector multiplet cannot. This is clear since there is no state of zero helicity in a massless vector supermultiplet. To acquire a mass it needs to combine with a scalar which becomes its longitudinal.

Going now to the case of $\mathcal{N} = 2$ we have two supercharges of each kind, the massless multiplets can be found accordingly. Starting with the state of the lowest helicity $-j$



(1.11)

where the numbers give the number of states with given helicity. The case of $j = \frac{1}{2}$ is peculiar: it may or not be CPT self-conjugate. When it is, we call it half hypermultiplet and contains only the states above. If it is not, the representation is doubled and we call it hypermultiplet. It is very useful to think of $\mathcal{N} = 2$ representations in terms of $\mathcal{N} = 1$ ones. The $\mathcal{N} = 2$ vector multiplet, whose lowest helicity state is $j = 0$, is comprised of an $\mathcal{N} = 1$ vector multiplet plus a chiral multiplet both transforming in the adjoint representation of the gauge group. The case of $j = \frac{1}{2}$, the (half) hypermultiplet, comprises (one) two $\mathcal{N} = 1$ chiral multiplets with opposite chirality.

At last, consider the maximally supersymmetric $\mathcal{N} = 4$ algebra[4]. Here we have only one multiplet which in $\mathcal{N} = 1$ language contains one vector multiplet and three chiral multiplets, all transforming in the adjoint representation.

Let us readily mention massive representations, for more details one consult one of the many reviews on the subject cited in the introduction. As discussed before, massive

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[4] Without gravity.

representations are usually longer than massless ones since they have dimension $2^{2\mathcal{N}}$. Moreover, for massive representations we cannot set the central charges Z^{AB} to zero, the most we can do in 4d is to make the central charge matrix skew-diagonal by an opportune $\mathrm{U}(\mathcal{N})$ action. Without loss of generality, we consider the simpler case where $\mathcal{N}$ is an even number[5]. Then the eigenvalues of the Z^{AB} are $Z_k \geq 0$ with $k = 1, \ldots, \frac{\mathcal{N}}{2}$. For the case of $\mathcal{N} = 2$ the SUSY algebra takes the form

$$\begin{aligned} \left\{Q^A_\alpha, \bar{Q}_{\dot{\beta}B}\right\} &= 2\sigma^\mu_{\alpha\dot{\beta}} P_\mu \delta^A_B \\ \left\{Q^A_\alpha, Q^B_\beta\right\} &= \epsilon_{\alpha\beta}\epsilon^{AB} Z \\ \left\{Q_{\dot{\alpha}A}, Q_{\dot{\beta}B}\right\} &= \epsilon_{\dot{\alpha}\dot{\beta}}\epsilon_{AB} Z \end{aligned} \tag{1.12}$$

Positivity of the Hilbert space leads to a bound on the mass of the states

$$2m \geq |Z| \tag{1.13}$$

from which follows that for massless states there cannot be any central charge as mentioned before. The bound (1.13) is generalized to higher values of $\mathcal{N}$. Depending on whether this bound is saturated or not, theories with extended SUSY admit massive multiplets with different lengths

- **Long multiplets:** if $2m > |Z_k|$ $\forall k$ we have exactly $2\mathcal{N}$ creation and annihilation operators and therefore the multiplet contains $2^{2\mathcal{N}}$ states.
- **Short multiplets:** if $2m = |Z_k|$ $\forall k \neq \frac{\mathcal{N}}{2}$ some of the creation and annihilation operators are trivially realized and only $2\mathcal{N} - 2k$ remain. Therefore this multiplet contains $2^{2\mathcal{N}-2k}$ states.
- **Ultra-short multiplets:** if $2m = |Z_k|$ $\forall k$ half of annihilation and creation operators are trivially realized. Therefore, this multiplet contains $2^{\mathcal{N}}$ states, which is the same dimension as their massless counterpart.

The mass bound (1.13) is reminiscent of the Bogomol'nyi-Prasad-Sommerfeld bound for solitonic solutions in gauge theories. This is no coincidence. The origin of the central extension can be understood [317] as follows: remember that the SUSY charges $Q, \bar{Q}$, being conserved charges, are integrals of expressions in the fields. By computing anticommutators, one encounters surface terms which are normally neglected. However, in the presence of electric and magnetic charges, these surface terms are non-zero and give rise to the central charges.

Due to this connection with the Bogomol'nyi-Prasad-Sommerfeld bound, short multiplets are also called BPS multiples, and the inequality (1.13) is also known as BPS bound. For BPS states, the relationship between charge and mass is dictated by supersymmetry and does not recive perturbative nor non-perturbative contributions in the quantum theory. This is so because any modification of this relation implies that states no longer belong to short multiplets. On the other hand, quantum corrections are not expected to

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[5] When $\mathcal{N}$ is odd, the matrix Z^{AB} has additional zero eigenvalues.

generate any extra degrees of freedom needed to convert a short multiplet into a long multiplet. We conclude then that for short multiplets the condition $2m = |Z|$ cannot be modified either perturbatively nor non-perturbatively.

1.1.2 Supersymmetric QFTs

With the representation theory discussed, we now aim to understand how to construct Lagrangians that are manifestly invariant under a certain amount of supersymmetry. Generally, we focus on constructing minimal SUSY Lagrangians, since representations of higher supersymmetry can always be embedded into the minimal one, as we discussed in the previous section. Being supersymmetric, the Lagrangian must contain *on-shell* the same number of fermionic and bosonic degrees of freedom. Here, we will primarily discuss the cases of 4d $\mathcal{N} = 1$ and postpone the discussion to 3d $\mathcal{N} = 2$ to the forthcoming section. The two have the same number of supercharges, with the latter derivable from the former by dimensional reduction on S^1.

The contents of a representation of the SUSY algebra are collected into *superfields*, which depend not only on a space-time point x^μ but also on a set of anti-commuting Grassmann coordinates, usually denoted as $\theta_\alpha, \bar{\theta}_{\dot{\alpha}}$, associated with the supersymmetry generators. This construction takes us from the usual 4d Minkowski space, defined as the group coset, to the $\mathcal{N} = 1$ *superspace*

$$\mathcal{M}_{1,3} \equiv \frac{\mathrm{ISO}(1,3)}{\mathrm{SO}(1,3)} \longrightarrow \mathcal{M}_{4|1} \equiv \frac{\overline{\mathrm{Osp}(4|1)}}{\mathrm{SO}(1,3)}. \tag{1.14}$$

In this formalism, the most general superfield one can construct, based on the fact that $\theta_\alpha, \bar{\theta}_{\dot{\alpha}}$ are Grassmann variables, is the following:

$$\begin{aligned}\Phi(x^\mu, \theta_\alpha, \bar{\theta}_{\dot{\alpha}}) = f(x) &+ \theta_\alpha \psi^\alpha(x) + \bar{\theta}_{\dot{\alpha}} \bar{\chi}^{\dot{\alpha}}(x) + \theta_\alpha \theta^\alpha m(x) + \bar{\theta}_{\dot{\alpha}} \bar{\theta}^{\dot{\alpha}} n(x) \\ &+ \theta_\alpha \sigma^{\mu\alpha\dot{\beta}} \theta_{\dot{\beta}} v_\mu(x) + \theta_\alpha \theta^\alpha \bar{\theta}_{\dot{\alpha}} \bar{\lambda}^{\dot{\alpha}}(x) + \bar{\theta}_{\dot{\alpha}} \bar{\theta}^{\dot{\alpha}} \bar{\theta}_\alpha \rho^\alpha(x) + \theta_\alpha \theta^\alpha \bar{\theta}_{\dot{\alpha}} \bar{\theta}^{\dot{\alpha}} d(x).\end{aligned} \tag{1.15}$$

Essentially, each superfield is a collection of ordinary fields, both bosonic and fermionic. From (1.15) we can construct more specific superfields that realize the supermultiplets mentioned before. We won't delve into the details of the calculation and refer the reader to more in-depth texts, but we will provide some basics and see how to constrain the general superfield to encode, on-shell, only the degrees of freedom of each supermultiplet.

Consider the massless representations (1.10). The (anti-)chiral superfield Φ encodes the degrees of freedom of the chiral multiplet and is defined by the condition

$$\overline{D}_{\dot{\alpha}} \Phi = 0, \qquad D_\alpha \Phi = 0, \tag{1.16}$$

where $D_\alpha, \bar{D}_{\dot{\alpha}}$ are covariant derivatives in superspace:

$$\begin{cases} D_\alpha = \partial_\alpha + i\sigma^\mu{}_{\alpha\dot{\beta}} \bar{\theta}^{\dot{\beta}} \partial_\mu, \\ \overline{D}_{\dot{\alpha}} = \bar{\partial}_{\dot{\alpha}} + i\sigma^\mu{}_{\dot{\alpha}\beta} \theta^\beta \partial_\mu. \end{cases} \tag{1.17}$$

These conditions impose the following expansion for the field:

$$\Phi(x,\theta,\bar{\theta}) = \phi(x) + \sqrt{2}\theta\psi(x) + \mathrm{i}\theta\sigma^{\mu}\bar{\theta}\partial_{\mu}\phi(x) - \theta\theta F(x) - \frac{\mathrm{i}}{\sqrt{2}}\theta\theta\partial_{\mu}\psi(x)\sigma^{\mu}\bar{\theta} - \frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\Box\phi(x), \tag{1.18}$$

with the obvious contractions. For the anti-chiral superfield, the story is the same, just put a bar over most of the components. As one can see, on-shell, this superfield contains exactly the right degrees of freedom for a chiral multiplet. The field F is an auxiliary field that vanishes on-shell.

The vector superfield is defined by the reality condition

$$V = \bar{V}. \tag{1.19}$$

As usual with vector fields, the explicit form of the expansion depends on a specific choice of gauge. Arguably, the easiest is the Wess-Zumino gauge, where the vector superfield expansion is

$$V(x,\theta,\bar{\theta}) = \theta\sigma^{\mu}\bar{\theta}v_{\mu}(x) + \mathrm{i}\theta\theta\bar{\theta}\bar{\lambda}(x) - \mathrm{i}\bar{\theta}\bar{\theta}\theta\lambda(x) + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}D(x). \tag{1.20}$$

Here, the D-field is an auxiliary field that vanishes on-shell.

1.2 Dynamics of Super-QCD

One gauge theory that will serve as the foundation for the rest of this section is the supersymmetric version of QCD (SQCD). This theory, like its non-supersymmetric counterpart, is an $\mathrm{SU}(N)$ gauge theory coupled to N_f chiral and anti-chiral multiplets[6]. For completeness, we now present the Lagrangian for the theory

$$\begin{aligned}\mathcal{L}_{\mathrm{SQCD}} = \frac{1}{32\pi^2}\,\mathrm{Im}\left(\tau\int \mathrm{d}^2\theta\,\mathrm{Tr}\,\mathcal{W}_{\alpha}\mathcal{W}^{\alpha}\right) + 2g\sum_{A}\xi_A\int \mathrm{d}^2\theta\,\mathrm{d}^2\bar{\theta}\,V^A \\ + \int \mathrm{d}^2\theta\,\mathrm{d}^2\bar{\theta}\,\bar{\Phi}^a \mathrm{e}^{2gV}\Phi_a + \int \mathrm{d}^2\theta\,W(\Phi) + \int \mathrm{d}^2\bar{\theta}\,\overline{W}(\bar{\Phi}),\end{aligned} \tag{1.21}$$

where the first term is the gauge kinetic term, the second term, known as the Fayet-Iliopoulos term, is only present when there are abelian gauge factors, the third term is the kinetic term for the chiral fields, and the last term represents their interactions. The charges of the various fields and the symmetries of the theory are given in Table 1.3.

The R-charge of the fields is fixed by requiring anomaly cancellation, which determines R_Q to be

$$R_Q = 1 - \frac{N}{N_f}, \tag{1.22}$$

while the axial symmetry is anomalous.

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[6] Both are needed to cancel the triangle gauge anomaly.

Table 1.3: Field content of $\mathcal{N} = 1$ SQCD with gauge group $\mathrm{SU}(N)$ and their charges under the classical global symmetries of the theory. The last entry is the gaugino from the vector multiplet, which carries a non-trivial R-charge. The axial symmetry is anomalous.

	$\mathrm{SU}(N_f)_L$	$\mathrm{SU}(N_f)_R$	$\mathrm{U}(1)_A$	$\mathrm{U}(1)_B$	$\mathrm{U}(1)_R$
Q_i	$\square$	1	1	1	R_Q
$\tilde{Q}_i$	1	$\overline{\square}$	1	-1	R_Q
λ	1	1	0	0	1

Next, let us consider the situation where the theory is conformal. This occurs when the β-function of the theory is zero. At the conformal point, additional supercharges impose, for chiral fields, the relation $\Delta(Q) = \frac{3}{2}R(Q)$, where $R(Q)$ is the superconformal R-symmetry. In our case, we find that the anomalous dimension for the chiral fields in SQCD is given by

$$\Delta(Q) = \frac{3}{2} - \frac{3N}{2N_f} \implies \gamma(Q) = -\frac{3N - N_f}{2N_f}. \tag{1.23}$$

This anomalous dimension at the superconformal fixed point is exact [290]. We can infer that this theory has very interesting dynamics depending on the amount of flavor N_f. Indeed, when N_f is close to $3N$, there is a weakly-coupled superconformal fixed point. If we continue to lower N_f, we encounter another bound, the unitarity bound. Consider the gauge invariant operator $Q\tilde{Q}$. This operator has a conformal dimension of $3(1 - \frac{N}{N_f})$. Since unitarity requires that the scaling dimension of any scalar operator be greater than one, we need $N_f \geq 3N/2$ to have a conformal point. Below this bound, the operator $Q\tilde{Q}$ violates unitarity. Consequently, we expect an accidental $\mathrm{U}(1)$ symmetry to emerge along the RG flow. This accidental symmetry will mix with the R-symmetry which will therefore be not the superconformal R-symmetry. Here the IR dynamics is better understood in terms of another theory, the magnetic dual theory, which we explain in the next section.

1.2.1 Dualities

Before hitting the unitarity bound, the anomalous dimension (1.23) becomes of order one and therfore the theory is not perturbative. In this range, Seiberg found [289] that the theory is better understood from another theory whose dynamics is perturbative. Here we state the result of our first example of *duality*: the following gauge theories are dual

- $\mathrm{SU}(N)$ SQCD with N_f pairs of fundamental quarks Q^i and anti-fundamental quarks $\tilde{Q}_i$ with vainishing superpotential $W = 0$.
- $\mathrm{SU}(N_f - N)$ SQCD with N_f pairs of fundamentals q_i and anti-fundamentals $\tilde{q}^i$, a set of singlets M^i_j with superpotential $W = q_i M^i_j \tilde{q}^j$.

When saying that two theories are dual to each other, what we mean is that, when $N_f > 3N/2$, they describe the same superconformal theory in the IR. While when $N_f < 3N/2$, the dual theory can be understood as the IR limit of the original one.

Table 1.4: Mapping betwen fundamental quarks and gauge-invariant chiral operators between Seiberg dual SQCDs. The operators $B, \tilde{B}, b, \tilde{b}$ are (anti-)baryon operators.

Electric	Magnetic
Q_i^a	$\tilde{q}_i^a$
$\tilde{Q}_a^i$	q_a^i
$Q_i^a \tilde{Q}_a^j$	M_i^j
$B^{i_1,\dots,i_N} := \epsilon^{a_1,\dots a_N} Q_{a_1}^{i_1} \cdots Q_{a_N}^{i_N}$	$b_{i_1,\dots,i_{N_f-N}} := \epsilon_{a_1,\dots a_{N_f-N}} q_{i_1}^{a_1} \cdots q_{i_{N_f-N}}^{a_{N_f-N}}$
$\tilde{B}_{i_1,\dots,i_N} := \epsilon_{a_1,\dots a_N} \tilde{Q}_{i_1}^{a_1} \cdots \tilde{Q}_{i_N}^{a_N}$	$\tilde{b}^{i_1,\dots,i_{N_f-N}} := \epsilon^{a_1,\dots a_{N_f-N}} \tilde{q}_{a_1}^{i_1} \cdots \tilde{q}_{a_{N_f-N}}^{i_{N_f-N}}$

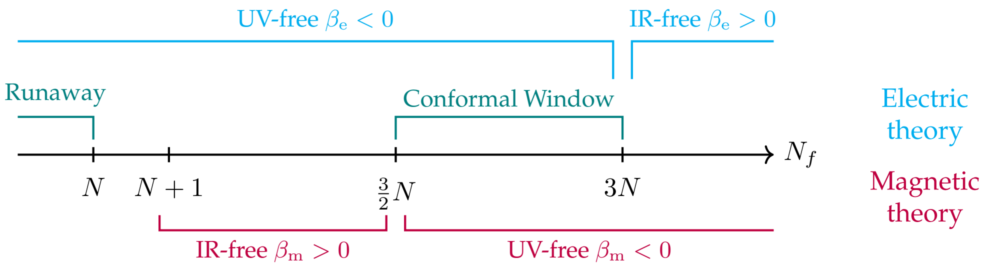

Figure 1.1: Phase diagram of both the electric and magnetic phases of SQCD as a function of the number of flavors N_f

The duality is corroborated by many non-trivial checks, which we are just going to state here. Indeed, one can prove that the two theories have the same 't Hooft anomalies and a map between gauge invariant operators can be constructed. Explicitly the mapping is given in table 1.4.

The continuous symmetries are also the same. Moreover, when integrating out flavors on the original *electric* theory, this can be proven to correspond to an Higgsing on the dual *magnetic* theory. Of course, being a duality of one takes the dual of the dual theory, this gives the initial theory.

1.2.2 Phases of SQCD

We can now discuss the various phases of SQCD and its dual in function of the number of flavors N_f. A summary is spelled out in figure 1.1.

1.2.2.1 Phase for $N < N_f < 3N$

As discussed before, when $N/N_f \sim 1/3$, the electric theory flows to a weakly-coupled SCFT, while the dual theory becomes strongly coupled[7]. As we increase N/N_f, the original theory becomes more strongly coupled, while the dual theory becomes more weakly coupled. At $N/N_f \sim 2/3$, we hit the unitarity bound for the meson operator $Q\tilde{Q}$. The

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[7] Remember that the one-loop beta function coefficient is $3N - N_f$.

region where $(3/2)N < N_f < 3N$ is known as the *conformal window*. Beyond this point, as N/N_f increases further, the meson operator violates unitarity, and thus we cannot expect the theory to be superconformal in the IR anymore. The same reasoning applies to the dual theory, where the beta function is no longer zero, and the theory cannot be conformal.

1.2.2.2 Phase for $N_f = N + 1$

We start from the case of $N_f = N+1$, in which the dual theory has rank $N' := N_f - N \geq 1$. To understand the behavior of this limiting case, we can consider starting with N_f flavors and integrating out $N' - k$ of them. This is done in the electric theory by turning on a mass deformation for $N' - k$ quarks

$$W_{\rm e} = m_i^j Q^i \tilde{Q}_j, \qquad i, j = N' - k, \dots, N_f, \tag{1.24}$$

which corresponds on the dual side to a deformation of the form

$$W_{\rm m} = qM\tilde{q} + m_i^j M_j^i. \tag{1.25}$$

This deformation on the magnetic side breaks $\mathrm{SU}(N') \to \mathrm{SU}(k)$ with $N + k$ flavors. Assuming that the final $\mathrm{SU}(k)$ theory is in the weakly-coupled regime, instanton computations, holomorphy, and symmetry completely fix the effective superpotential. In the case of interest, $k = 1$, we get a superpotential

$$W_{\rm m} = qM\tilde{q} - \det M. \tag{1.26}$$

Note that in this case $q_i \sim b_i$, and therefore the theory is described only by composite gauge-invariant operators. Hence, we conclude that the IR dynamics of $\mathrm{SU}(N)$ SQCD with $N + 1$ flavors is described by an almost free theory of gauge-invariant operators interacting with the superpotential

$$W = \frac{1}{\Lambda^{3N-(N+1)}} (B_i M_j^i \tilde{B}^j - \det M). \tag{1.27}$$

This is an example of confinement without chiral symmetry breaking, usually called *s-confinement*. This is the prototypical example for the results of [36].

1.2.2.3 Phase for $N_f = N$

We can continue to vary the values of N and N_f as long as we have $N' := N_f - N \geq 0$. The first limiting case occurs when $N' = 0$, i.e., when $N_f = N$. We can compute the dynamics of this theory by integrating out an additional quark from the theory (1.27). Again, by adding a mass term to one quark, the IR theory is described by an $\mathrm{SU}(N)$ SQCD with N flavors and superpotential

$$W = \lambda(\det M - B\tilde{B} - \Lambda^{2N}). \tag{1.28}$$

This result is quite interesting for the following reason: the equation of motion for λ imposes the expected quantum constraint on the moduli space of vacua for this theory. When $N_f = N$, the moduli space is parametrized by the solutions to the classical constraint [263]

$$\det M = B\tilde{B}, \qquad \text{where } B = \det(Q), \tilde{B} = \det(\tilde{Q}), \tag{1.29}$$

which, at the quantum level, is modified to

$$\det(M) - B\tilde{B} = \Lambda^{2N}, \tag{1.30}$$

where Λ is a holomorphic scale treated as a spurion. This is exactly the constraint arising from (1.28).

1.2.2.4 Phase for $0 < N_f < N$: Breaking SUSY

Decoupling an additional flavor as before we consider the superpotential

$$W = \lambda(\det M - B\tilde{B} - \Lambda^{2N}) + mM_N^N \tag{1.31}$$

which, after computing the equations of motion for λ and M_N^N, becomes

$$W = \frac{m\Lambda^{3N-(N-1)}}{\det M}. \tag{1.32}$$

This result was originally found by Affleck, Dine and Seiberg [2] and correctly reproduces the effective superpotential of $\mathcal{N} = 1$ pure SYM in the limit where all flavors decouple. In general, when decoupling k flavors the effective superpotential is of the form

$$W = k\left(\frac{\Lambda^{3N-(N-k)}}{\det M}\right)^{\frac{1}{k}}. \tag{1.33}$$

The potential generated by this superpotential is always non-zero as long as M is non-zero and finite. Therefore, the theory does not admit any stable vacuum at finite distance in field space so that the classical moduli space is completely lifted at the quantum level. This behavior is known as *runaway*; supersymmetry is broken at finite distances[8].
A summary of the phases of SQCD with varying number of flavors is given in table 1.5.

1.3 Elements of 3d $\mathcal{N} = 2$

In the previous sections we mainly focused on the case of 4d $\mathcal{N} = 1$ QFTs in which fundamental results like Seiberg duality were first conjectures. Now we would like to focus on 3d $\mathcal{N} = 2$ QFTs which can be obtained by the latter by dimensional reduction on S^1. In fact, as a first check, we see that the number of supercharges between 4d $\mathcal{N} = 1$ and 3d $\mathcal{N} = 2$ is the same. For a more in depth discussion on the topic the reader can consult [3, 12, 15, 112, 170, 235]. The matter content of these theories can be handily obtained

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[8] One should to be careful in treating possible non-canonical Kähler potential terms, but it can be proven that these would not generate any SUSY minima at finite distance.

Table 1.5: Summary of the phases of SQCD and its dual for varying number of flavor at fixed number of colors N.

N_f	Dual group	Behaviour
$3N-1$	$\mathrm{SU}(2N-1)$	Superconformal
$2N$	$\mathrm{SU}(N)$	Superconformal and self-dual
$(3/2)N$	$\mathrm{SU}(N/2)$	Superconformal
$(3/2)N-1$	$\mathrm{SU}(N/2-1)$	
$\downarrow$	$\downarrow$	IR-free with $W = qM\tilde{q}$
$N+2$	$\mathrm{SU}(2)$	
$N+1$	$\mathrm{SU}(1)$	S-confinement with $W = BM\tilde{B} - \det M$
N	-	Quantum deformed $W = \lambda(\det M - B\tilde{B} - \Lambda^{2N})$
$N-1$	-	ADS superpotential
0	-	Pure SYM, N vacua

by dimensional reduction of 4d $\mathcal{N}=1$ gauge theories. Thus the reduction of the chiral superfield (1.18) still comprises of a complex scalar and an auxiliarry field, whilst the Weyl fermion decomposes into two real independent Majorana fermions. Similarly, the vector superfield (1.20) decomposes as

$$V = -\mathrm{i}\theta\bar{\theta}\sigma - \theta\gamma^i\bar{\theta}A_i + \mathrm{i}\theta^2\bar{\theta}\lambda - \mathrm{i}\bar{\theta}^2\theta\lambda + \frac{1}{2}\theta^2\bar{\theta}^2 D, \tag{1.34}$$

where A_i[9] is the 3 dimensional vector field, σ is the real gauge scalar descending from A_3, λ are the gauginos and D is a real auxiliary field. Crucially the 3d vector field (1.34) differs from its 4d counterpart (1.20) by the presence of an additional scalar field σ which may acquire a vacuum (vev) thus describing a new branch of the moduli space of the theory known as *Coulomb branch*. At a generic point of the Coulomb branch, i.e. when $\langle\sigma\rangle \neq 0$, the gauge group is completely broken to its maximal torus. For example $\mathrm{SU}(N) \to \mathrm{U}(1)^{N-1}$. However, this is not the only scalar which may acquire a vev. Indeed, in 3d the vector field A_i has only one propagating degree of freedom which may be dualized to a scalar by means of Hodge duality. Take the theory at a generic point of the Coulomb branch, the 2-form $F = \mathrm{d}A$ may be dualized

$$\star F = \mathrm{d}\gamma\,, \tag{1.35}$$

where γ is a scalar known as *dual photon*. This scalar describes the only degree of freedom of the vector field A_i. Charge quantization imposes that γ takes value on S^1, which forces the Coulomb branch topology to be $\mathbb{R}\times S^1$. On this, the natural coordinate is given by the complex modulus

$$\phi = \sigma + \mathrm{i}\gamma. \tag{1.36}$$

We are going discuss further the moduli space of 3d $\mathcal{N}=2$ in the following section.

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[9] Hereafter we use the latin indices i to denote vectors of the 3d Lorentz group.

The action for $\mathcal{N} = 2$ theories can be constructed in the same way as described for the 4d ones in 1.1.2. In addition, there are two terms which arise in 3d. One can add a supersymmetric Chern-Simons (CS) term

$$S_{\text{CS}} = \int \mathrm{d}^3x \, \mathrm{Tr} \left[\epsilon^{ijk} \left(A_i \partial_j A_k + \mathrm{i}\frac{2}{3} A_i A_j A_k \right) + 2\mathcal{D}\sigma - \lambda^\dagger \lambda \right]. \tag{1.37}$$

Note that for Abelian gauge groups we can also have mixed CS terms. Indeed, if $A^{(a)}$ is the gauge field for the gauge group $\mathrm{U}(1)_a$ and $A^{(b)}$ for the gauge group $\mathrm{U}(1)_b$, we can write a term of the form

$$S_{\text{CS, mixed}} = \int \mathrm{d}^3x \, \epsilon^{ijk} A_i^{(a)} \partial_j A_k^{(b)} + D^{(a)}\sigma^{(b)} + D^{(b)}\sigma^{(a)}. \tag{1.38}$$

This terms are going to be fundamental in the forthcoming discussion for [37]. The second term we can add has a topological origin. Indeed, 3d Abelian gauge theories posses an additional topological global symmetry coming from the conservation of the current $\mathcal{J} = \star F$. Indeed, the Bianchi identity for F implies that $\mathrm{d} \star F = \mathrm{d}\mathcal{J} = 0$. In supersymmetry, any conserved current belongs to a multiplet known as linear superfield Σ satisfying $D^2\Sigma = \bar{D}^2\Sigma = 0$. Since this new symmetry $\mathrm{U}(1)_{\mathcal{J}}$ is generated by $\star F$, the vector superfield (1.34) may be described by the linear superfield Σ as

$$\Sigma = -\frac{\mathrm{i}}{2}\epsilon^{\alpha\beta}\bar{D}_\alpha D_\beta V. \tag{1.39}$$

From this one can construct a scalar superfield Φ dual to Σ whose lower scalar component is exactly the scalar ϕ in (1.36). In particular the scalar component of Φ, $\phi = \phi_R + \mathrm{i}\phi_I$, can be related to the gauge degrees of freedom as

$$\begin{aligned} \phi_R &= \frac{2\pi}{e^2}\sigma, \\ \partial_i \phi_I &= -\frac{\pi}{e^2}\mathcal{J}_i. \end{aligned} \tag{1.40}$$

1.3.1 Real Masses

In 3d there are two possible ways to generate a mass term for matter fields. The usual mass, known as *complex mass*, arises in vector-like theories by adding a relevant deformation of the form

$$W_{\mathbb{C}-\text{mass}} = m_{\mathbb{C}} Q\tilde{Q}, \tag{1.41}$$

where Q a putative matter field. The parameter $m_{\mathbb{C}}$ is a complex parameter, hence the name. But now, for 3d $\mathcal{N} = 2$ we can induce a mass on the chiral multiplet by setting $\langle\sigma\rangle \neq 0$. The kinetic term for a chiral multiplet contains in fact terms of the form

$$Qe^V Q^\dagger \supset \phi_Q^\dagger \sigma^2 \phi_Q - \mathrm{i}\psi_Q^\dagger \sigma \psi \quad \xrightarrow{\langle\sigma\rangle\neq 0} \frac{m_{\mathbb{R}}^2}{2}|\phi_Q|^2 + \mathrm{i}m_{\mathbb{R}}\epsilon^{\alpha\beta}\bar{\psi}_\alpha\psi_\beta. \tag{1.42}$$

Moreover, by weakly gauging a non-anomalous global symmetry we can induce a real mass by similar arguments. Crucially, since global symmetries match across dualities, real mass deformations can be mapped between dual theories preserving the underlying duality. This fact is going to be extensively used in chapter 3.1.4.

1.3.2 Moduli space

We can now describe the classical moduli space of 3d $\mathcal{N} = 2$ $\mathrm{SU}(N)$ gauge theories [112]. We start by analyzing theories without CS terms. The full moduli space is now described by two branches

- **Higgs branch (HB)**: This is present also in 4d and is defined by the solutions to the D-term equations. In 3d, it is still defined this way, but one also needs to impose that $\langle\sigma\rangle = 0$. The coordinates on HB are the vevs of gauge-invariant composites subject to classical constraints.
- **Coulomb branch (CB)**: This branch, present also in 4d $\mathcal{N} \geq 2$, is parameterized in 3d by the vevs of σ while setting the matter fields to zero.

Previously, we discussed how to give a vev to the scalar σ in abelian gauge theories. In non-abelian theories, the story is similar. Without any loss of generality, we can use gauge transformations to diagonalize the σ's and write the general vev as valued in the Cartan subalgebra of the gauge group. This gives rise to a mass for all matter fields, breaking the gauge group to its maximal torus. Additionally, to eliminate the leftover gauge redundancy, we impose constraints that restrict σ to a Weyl chamber

$$\langle\sigma\rangle = \mathrm{diag}(\sigma_1, \sigma_2, \ldots, \sigma_N), \tag{1.43}$$

with the ordering $\sigma_1 > \sigma_2 > \ldots > \sigma_{N-1} > \sigma_N$ while imposing the traceless condition $\sum_i \sigma_i = 0$[10]. From this analysis, we can conclude that in the absence of matter CB has topology $\mathbb{R} \times S^1$ and is described by $N - 1$ massless chiral multiplets

$$Y_i \sim e^{\Phi_i}, \tag{1.44}$$

where the Φ_i are the chiral superfields for the unbroken $\mathrm{U}(1)$'s dual to the linear superfield (1.39). It can be shown that in the full UV theory, the Y_i operators are realized as "disorder operators" that impose a unit magnetic flux on all Euclidean S^2 spheres enclosing a given point in the path integral. For this reason, the Y_i are often called *monopole operators*. In principle, we can choose the Y_i's to correspond to any basis of low-energy $\mathrm{U}(1)$'s we prefer. As we will soon see, it will prove convenient to choose our Y_i's so that their lowest (scalar) component is classically equivalent to

$$Y_i = \exp\left(\frac{2\pi}{g^2}(\sigma_i - \sigma_{i+1}) + i(a_i - a_{i+1})\right), \quad i = 1, \ldots, N-1. \tag{1.45}$$

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[10] The scalar σ is in the adjoint of the group

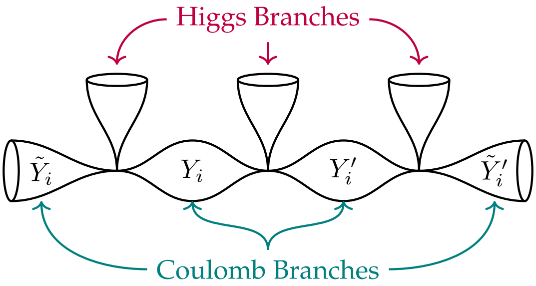

Figure 1.2: Moduli space of 3d $\mathcal{N} = 2$ in presence of matter. The Coulomb branches pinch off where they meet the Higgs branches. At these points there is additional massless matter.

The presence of matter and non-perturbative effects significantly alter this classical picture. Specifically, in the presence of matter fields, there can be regions where the HB pinches the CB. This occurs because the vev for the scalar field σ provides an effective mass to the matter fields, and this mass can vanish in certain scenarios. For fields in the fundamental representation, this happens when one or more of the σ_i vanish. Similar constraints hold for different representations. In these regions of the CB, we can turn on vev for the massless matter fields, leading to an intersection between the CB and HB. These intersection regions are referred to as ″pinches″ in the CB as shown pictorially in figure 1.2. Consequently, the CB is divided into several regions, each separated by the HB, necessitating a description that combines the relevant terms instead of solely relying on (1.44). However, this combined description is only necessary near the pinches. In the bulk of the CB, far from the pinches, all matter fields can be integrated out at a scale where gauge interactions remain perturbative.

In addition to matter fields, non-perturbative corrections also modify the classical picture. In 3d $\mathcal{N} = 2$ $\mathrm{SU}(N)$ gauge theories, there are instantons resembling the 4d ’t Hooft-Polyakov monopoles, with the scalar field σ in the 3d gauge multiplet acting like the adjoint Higgs scalar in 4d. Therefore, these instantonic configurations are also referred to as monopoles. There are $N - 1$ such configurations, each corresponding to a specific embedding $\mathrm{SU}(2) \hookrightarrow \mathrm{SU}(N)$. This establishes a one-to-one correspondence between these monopole instantons and the monopole operators defined in (1.45).

If a charged fermion under a $\mathrm{U}(1)$ symmetry has zero modes in this instanton/monopole background, then Y_i acquires charge under such $\mathrm{U}(1)$. Since the zero mode counting varies across different regions of the CB, the charge of Y_i also changes. This discontinuity necessitates two independent operators, Y_i and $\tilde{Y}_i$, to describe each side of the CB. In a pure $\mathrm{SU}(N)$ theory, the monopole instanton configurations generate the low-energy superpotential [3]

$$W_{\text{monopoles}} = \sum_i Y_i^{-1}, \tag{1.46}$$

which entirely lifts the CB. This picture changes in the presence of matter. These fields may have zero modes, according to the Callias index theorem [131]. These zero modes can suppress some of the monopole contributions to the potential (1.46), leaving parts of the CB unlifted [12]. This does not always happen: each instanton/monopole is associated with an $\mathrm{SU}(2)$ embedded in $\mathrm{SU}(N)$, and this subgroup can be broken by the vev of the matter field, according to the decomposition of its representation under this subgroup. This vev sets the monopole scale, which for the monopole corresponding to the operator Y_i is

$$\frac{1}{\rho_i} = \frac{1}{2}|\sigma_i - \sigma_{i+1}|. \tag{1.47}$$

We have to compare this scale with the induced mass of some of the fermions that transform under the $\mathrm{SU}(2)$ embedding corresponding to the monopole. If this mass is larger than the monopole scale, these fermions will be integrated out before they can affect the effective superpotential (1.46). Conversely, the fermions contribute to the zero mode counting, with a number of zero modes depending on the representation. The first scenario always realizes in the bulk of the CB; that is to say, every configuration in the bulk of the CB can be taken to a configuration where a superpotential is generated without having to go through a pinch. Even for these, it is often possible to make the mass of some matter fields arbitrarily larger than some of the monopole scales, so that only a very restricted part of the pinch regions will actually survive. So, for a theory with only fields in the fundamental and antifundamental representation, the non-perturbative effects leave unlifted the part of the CB corresponding to the non-vanishing semi-classical configuration [291]

$$\langle\sigma\rangle = \mathrm{diag}(\sigma, 0, \ldots, 0, -\sigma). \tag{1.48}$$

This direction is parametrized by the low-energy monopole

$$Y = \prod_{i=1}^{N-1} Y_i = \exp\left(\frac{2\pi}{g}(\sigma_1 - \sigma_N) + i(a_1 - a_N)\right). \tag{1.49}$$

1.3.3 One-loop CS terms

Up to now, the discussion has been done in absence of CS interactions. But, we know that the low-energy behavior of 3d $\mathcal{N} = 2$ theories depends on these terms, so let us explain how they can be dynamically generated and how they change our treatment. We know that in the 3d, in the IR two-point functions between conserved current arise conformally invariant contact terms. The UV description of these contact terms is given by CS interactions for the background gauge fields coupling to the conserved current [150, 151, 314]. The Lagrangian description of these terms has the following form

$$\mathcal{L}_{\mathrm{CS}} \sim k\mathrm{Tr}(A \wedge \mathrm{d}A) = k\mathrm{Tr}(A \wedge F). \tag{1.50}$$

This additional interaction modifies the Gauss' law[11], introducing a "magnetic" contribution to the electric charge when $k \neq 0$

$$-\frac{1}{e^2}\partial_i F_{0i} = \rho_{\text{matter}} - \frac{k}{2\pi}F_{12}, \tag{1.51}$$

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[11] This is nothing but the A_0 equation of motion

where $\rho_{\text{matter}} = \delta\mathcal{L}_{\text{matter}}/\delta A_0$ is the contribution from matter fields. In fact, by imposing Coulomb gauge to preserve canonical quantization, we make A_0 become a non-dynamical field. As a consequence we have to impose Gauss' law as a constraint and this implies that any charged field under a $\mathrm{U}(1)_J$ symmetry acquires an electric charge

$$q_{\text{elec}}^{\text{CS}} = -kq_J. \tag{1.52}$$

The story does not end here, even if a theory as a vanishing tree-level CS term, i.e. $k = 0$, one can be generated perturbately when UV divergences are regulated in a gauge-invariant way [64, 285]. Let us see how this works. As we saw earlier a non-vanishing vev for the σ's generate a real mass for the matter fields $m_i(\sigma) = n_i\langle\sigma\rangle$, where n_i is the $\mathrm{U}(1)$ charge of the fields. So, if all real masses are set to zero, when we explore directions of the CB when $\sigma \neq 0$, we have to integrate out the heavy fields in order to get an effective action for the light degrees of freedom. Considering the vacuum polarization diagram which generates one-loop Chern-Simons terms for the gauge field A_i, where in the loop circulate the heavy fermion, we have this effective action

$$S_{\text{eff}}[A] = \frac{1}{2}n_i^2 \int \frac{\mathrm{d}^3q}{(2\pi)^3}\frac{\mathrm{d}^3p}{(2\pi)^3} A_l(-p)A_m(-q)[-i\Pi^{lm}(p)], \tag{1.53}$$

where (we suppress the index i in $m_i(\sigma)$)

$$\Pi^{lm}(p) = -\mathrm{i}e^2 \int \frac{\mathrm{d}^3k}{(2\pi)^3}\mathrm{Tr}\left[\gamma^l \frac{-\mathrm{i}}{(p+k)-m(\sigma)}\gamma^m \frac{-\mathrm{i}}{k-m(\sigma)}\right]. \tag{1.54}$$

For large m,

$$\Pi^{lm} \to -\mathrm{i}\epsilon^{lmk}p_k\text{sign}[m(\sigma)] + \ldots \tag{1.55}$$

Where the dots represent divergent term which need to be regularized. Substituting (1.55) in (1.53), we can see that each charged heavy fermion, with charge n_i, generates a one-loop CS term with coefficient $k_{\text{eff,i}} = n_i^2\text{sign}(n_i\sigma)$. By integrating out all the massive fermions along the CB we obtain

$$k_{\text{eff}}(\sigma) = k + \frac{1}{2}\sum_i n_i^2\text{sign}(n_i\sigma). \tag{1.56}$$

Note that for non-abelian symmetries the following holds

$$k_{\text{eff}} = k + \frac{1}{2}\sum_i T_2(R_{\psi_i})\text{sign}(n_i\sigma), \tag{1.57}$$

where $T_2(R_{\psi_i})$ is the quadratic index of the representation R_{ψ_i} of ther fermion ψ_i. Remembering (1.52), from (1.56) we read the induced one-loop electric charge for the $\mathrm{U}(1)_{\mathcal{J}}$ charged field

$$q_{\text{elec}}^{\text{CS}} = -k_{\text{eff}}q_{\mathcal{J}} = \left(k + \frac{1}{2}\sum_i n_i^2\text{sign}(n_i\sigma)\right)q_{\mathcal{J}}. \tag{1.58}$$

This formula can be generalized to the case with non-vanishing real masses $m_{\mathbb{R},i}$, it reads

$$q^{\text{CS}}_{\text{elec}} = \left(k + \frac{1}{2}\sum_i n_i^2 \text{sign}(m_i(\sigma))\right) q_{\mathcal{J}} = \left(k + \frac{1}{2}\sum_i n_i^2 \text{sign}(m_{\mathbb{R},i} + n_i\sigma)\right) q_{\mathcal{J}}. \quad (1.59)$$

The same reasoning is true for the generation of effective abelian charges. One-loop $\text{U}(1)$ charges are generated by mixed Chern-Simons term. Let us see this by weakly gauging a $\text{U}(1)$ symmetry. The weakly gauged $\text{U}(1)$ vector bosons a_i interact with the heavy fermions charged under the same $\text{U}(1)$, with charge $\tilde{n}_i$. Integrating out these massive fermionic fields we encounter the vacuum polarization diagram which generates one-loop mixed Chern-Simons term involving both a_i and A_j. These diagrams induce one-loop mixed CS terms of the form

$$\mathcal{L}^{\text{mixed}}_{\text{CS}} \sim k^{\text{mix}}_{\text{eff}}(\sigma)\text{Tr}(\epsilon^{ijk} a_i F_{jk}), \quad (1.60)$$

with

$$k^{\text{mix}}_{\text{eff}}(\sigma) = \frac{1}{2}\sum_i n_i \tilde{n}_i \text{sign}(n_i\sigma). \quad (1.61)$$

Analogously with the previous case, the non-vanishing mixed CS term will modify Gauss' law for the weakly gauged $\text{U}(1)$ (i.e. the a_0 equation of motion), which in turns will induce a $\text{U}(1)$ charge for fields which have a non-vanishing $q_{\mathcal{J}}$. This reads

$$\tilde{q} = \frac{1}{2}\left(\sum_i n_i \tilde{n}_i \text{sign}(n_i\sigma)\right) q_{\mathcal{J}}. \quad (1.62)$$

In this way the Abelian charges mix in a non trivial way, as a result, the Y's operators, despite the fact that they are not charged classically, acquire Abelian charges quantum mechanically. In this way, as we will review in the next paragraph, they acquire also charge under the gauge group. But, in order to parametrize the CB we need gauge-invariant operator. We construct them by "dressing" the bare monopole operators with massless matter fields.

CHAPTER 2

String Theory and M-Theory

Contents

Quantum Field Theory (QFT) provides the mathematical framework for understanding the subatomic world. Alongside it, the theory of the Standard Model (SM), whose basis is QFT, gave us deep insights into the behavior and interactions of the basic building blocks of our world: particles. Together with the SM, which precisely describes electromagnetic, weak, and strong interactions, the theory of General Relativity (GR) provided an accurate description of the fourth fundamental force: gravity. Both these theories, while incredibly precise within their respective energy ranges, break down in other regimes. GR does not account for the quantum nature of gravity [1], and the predictive power of the SM fails when attempting to describe gravitational interactions. Among various fundamental problems, whether a consistent theory of quantum gravity exists has fascinated physicists for decades.

In this context, String Theory (ST) emerges as an intriguing answer. Initially developed to understand the non-perturbative nature of quantum chromodynamics [186, 298, 305], it soon became evident that there was more to String Theory than met the eye. Particularly, ST naturally incorporates gravity, presenting a unified framework that potentially bridges the gap between quantum mechanics and general relativity [205].

The main idea of ST is that the fundamental constituents are one-dimensional objects called strings, rather than point particles. These strings can vibrate with different modes, each corresponding to a different particle. Remarkably, one of these modes corresponds to the graviton, the hypothetical quantum particle that mediates gravitational interactions. As was later found, ST is not really a single theory but comes in different versions labeled as type-I, type-IIA, type-IIB, and two heterotic string theories. It was later shown that these different versions are actually connected by dualities, leading to the developement to M-theory. M-theory provides a more comprehensive framework that unifies the five distinct string theories [222, 315].

One of the most remarkable developments in String Theory is the AdS/CFT correspondence, proposed by Maldacena in his seminal paper [264], which relates a low-energy description of string theory in anti-de Sitter spacetime in d+1 dimensions (AdS_{d+1}), to a supersymmetric and conformal theory (SCFT), living on the d-dimensional conformal boundary of AdS. Specifically, it suggests that a gravitational theory in (d+1)-dimensional AdS space is equivalent to a d-dimensional CFT without gravity. The best understood realization of this correspondence is the one between type-IIB string theory on an $\text{AdS}_5 \times S^5$ background with fluxes and the maximally supersymmetric Yang-Mills theory in four dimensions (SYM) with gauge group $\text{SU}(N)$.

In the large-N limit, string theory admits a classical description in terms of *supergravity*, a supersymmetric version of Einstein's gravity, on AdS_5. In turn, this theory admits a large family of solutions representing rotating, electrically charged, supersymmetric black holes with an S^3 event horizon of finite area A_{hor}, and therefore a finite number n_{micro} of microstates[1]. The enumeration of the microstates is related to the counting of states in the Hilbert space of the dual SCFT on $S^3 \times S^1$. A fundamental work in which the entropy of a black hole was obtained from a microstate counting was the one by Strominger and Vafa [292], in which they managed to reproduce the Bekenstein–Hawking area law for a class of five-dimensional asymptotically flat extremal black holes in string theory.

We begin this Chapter with Section 2.1, where we review the first example of string theory: the bosonic string. Although illustrative, this theory is incomplete for various reasons. The primary issue is that its critical dimension is 26, significantly higher than the familiar four dimensions of spacetime. Additionally, the spectrum lacks fermions, necessitating the inclusion of supersymmetry on the string worldsheet. Nonetheless, the main features of string theory can be encapsulated in this relatively simple, though incomplete, model.

In Subsection 2.1.2, we discuss how fermions can be added to the worldsheet and how this inclusion influences the bulk spectrum, resulting in a supersymmetric theory when tachyons are projected out through a process known as the GSO projection. The quantization of the superstring and the GSO projection will be the focus of Subsection 2.1.3.

Moving forward, in Section 2.2, we will explore how the boundary conditions for the open string reveal themselves to be dynamical extended objects, known as branes, which are fundamentally important for a consistent treatment of string theory, especially in the context of the AdS/CFT correspondence. This correspondence will be examined in Section 2.4. Finally, in Section 2.3, we will introduce M-theory, discussing how and why it emerges, and its significance in the broader framework of theoretical physics.

The discussions in this Chapter are in view of the contents of Chapters 4 and 5.

2.1 The Bosonic String and the Superstring

2.1.1 Classical Bosonic String

The classical action for bosonic string theory was first introduced by Nambu and Goto as the simplest lagrangian governing the evolution of an extended 1d object, a string, in a d-dimensional space-time. This is given in terms of the pull-back of the space-time

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[1] Since the semi-classical black hole entropy can be obtained via the Bekenstein-Hawking formula $S_{\text{BH}} = A_{\text{hor}}/4G_{\text{Newton}}$ [77, 216], and this should be equal to $\log n_{\text{micro}}$.

matric onto the string worldsheet, i.e. the proper area of the string worldsheet. That is to say that, given two coordinates on the string worldsheet $\sigma^\alpha = (\sigma, \tau) \equiv (\xi_1, \xi_2)$, the action reads

$$S_{\text{NG}} = -\frac{1}{2\pi\alpha'} \int \mathrm{d}^2\sigma \sqrt{-\det\gamma}, \qquad \gamma_{\alpha\beta} = \eta_{\mu\nu} \frac{\partial X^\mu}{\partial\eta_\alpha} \frac{\partial X^\nu}{\partial\eta_\beta}, \tag{2.1}$$

where $T = 1/(2\pi\alpha')$ is the string tension and α' is known as *universal Regge slope*. The fields $X^\mu(\sigma, \tau)$ are the coordinate embeddings of the 2d worldsheet in spacetime. Although the NG action is perfectly valid, a better formulation of it is given in terms of the Polyakov action

$$S_{\text{P}} = -\frac{1}{2\pi\alpha'} \int \mathrm{d}^2\sigma \sqrt{-h} h^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}. \tag{2.2}$$

Varying (2.2) with respect to $h_{\alpha\beta}$ gives back (2.1). One might question if there are any other possible terms which could be added to (2.2). Indeed, if we restrict ourselves to closed strings moving in empty Minkowski space, there are two possible additional renormalizable terms we could add

$$S_1 = \lambda_1 \int_\Sigma \mathrm{d}^2\sigma \sqrt{-h}, \qquad S_2 = \frac{\lambda_2}{4\pi} \int_\Sigma \mathrm{d}^2\sigma \sqrt{-h} R = \lambda_2 \chi(\Sigma), \tag{2.3}$$

where Σ is the string worldsheet and R is the scalar curvature of $h_{\alpha\beta}$. The second term is a topological term, being proportional to the Euler characteristic of the string worldsheet, and therefore does not change the classical equations of motion[2]. The action S_1 is instead set to zero by the equations of motion for $h_{\alpha\beta}$. We will therefore only consider the action (2.2) for what follows.

The Polyakov action is manifestly invariant under the following classical symmetries

- **Global symmetries**:
  - *Poincarè invariance*: In generic dimensions, this is the usual space-time invariance $\text{ISO}(1, d-1) = \text{SO}(1, d-1) \ltimes \mathbb{R}$. On the fields X^μ it acs as expected

$$\delta X^\mu = a^\mu_\nu X^\nu + b^\mu, \qquad \delta h^{\alpha\beta} = 0, \tag{2.4}$$

    where a^μ_ν describes Lorentz transformation and b^μ space-time translations.

- **Local symmetries**:
  - *Diffeomorphism invariance*: This is a gauge symmetry of the worldsheet acting as follows

$$\delta X^\mu = -\xi^\alpha \partial_\alpha X^\mu, \tag{2.5}$$

$$\delta h_{\alpha\beta} = -(\xi^\gamma \partial_\gamma h_{\alpha\beta} + \partial_\alpha \xi^\gamma h_{\gamma\beta} + \partial_\beta \xi^\gamma h_{\alpha\gamma}) = -\nabla_{(\alpha}\xi_{\beta)}, \tag{2.6}$$

$$\delta\sqrt{-h} = -\partial_\alpha(\xi^\alpha \sqrt{-h}), \tag{2.7}$$

---

[2] As we will see later, in the spectrum of the closed string one excitation is the dilaton field Φ. λ_2 turns out to be a constant background value for Φ.

where ξ^α is an arbitrary function parametrizing the infinitesimal transformation on the worldsheet $\sigma^\alpha \to \sigma^\alpha - \xi^\alpha(\sigma)$.

- *Weyl invariance*: This is another gauge symmetry of the worldsheet acting as

$$\delta X^\mu = 0, \qquad \delta h_{\alpha\beta} = 2\Lambda h_{\alpha\beta} \tag{2.8}$$

which is the infinitesimal version of the scaling transformation $h_{\alpha\beta}(\sigma) \to \Omega(\sigma)^2 h_{\alpha\beta}$.

Poincaré invariance and worldsheet diffeomorphism, as well as Weyl invariance, allow one to gauge-fix three independent components of the worldsheet metric and set, in the conformal gauge, $h_{\alpha\beta} = \eta_{\alpha\beta}$, the flat metric. With this choice, the action is greatly simplified

$$S = -\frac{1}{4\pi\alpha'} \int \mathrm{d}^2\sigma \left(\dot{X}^2 - (X')^2\right), \tag{2.9}$$

where the dot denotes differentiation with respect to the worldsheet time τ, while the prime denotes differentiation with respect to the worldsheet space σ. There is still an additional constraint coming from the equations of motion for $h_{\alpha\beta}$ in (2.2)

$$T_{\alpha\beta} := -\frac{2}{T}\frac{1}{\sqrt{-h}}\frac{\delta S}{\delta h_{\alpha\beta}} = 0, \tag{2.10}$$

which implies that the worldsheet energy-momentum tensor must vanish. Altogether, one must solve the following equations of motion:

$$(\partial_\tau^2 - \partial_\sigma^2) X^\mu = 0, \qquad (X' \pm \dot{X})^2 = 0, \tag{2.11}$$

in addition to imposing the vanishing of the boundary term from the variation of the action, which amounts to

$$\delta S \supset -T \int_{\tau_0}^{\tau_1} \mathrm{d}\tau \, X'_\mu \delta X^\mu \Big|_{\sigma=0}^{\sigma=\pi}. \tag{2.12}$$

The possible solutions to (2.12) define open and closed strings. For closed strings, the most general solution is

$$X^\mu(\tau,\sigma) = M^\mu_\nu X^\nu(\tau, \sigma+\pi), \qquad M^\mu_\nu \in \mathrm{O}(1, d-1). \tag{2.13}$$

Whenever M is a non-trivial element of $\mathrm{O}(1, d-1)$, these are called twisted boundary conditions. Hereafter, we will only consider the case where $M^\mu_\nu = \delta^\mu_\nu$. Open strings, instead, can satisfy either Neumann or Dirichlet boundary conditions. In detail

$$\partial_\sigma X^\mu \Big|_{\sigma=0,\pi} = 0 \qquad \text{Neumann (N) boundary conditions,} \tag{2.14}$$

$$\delta X^\mu \Big|_{\sigma=0,\pi} = 0 \qquad \text{Dirichlet (D) boundary conditions.} \tag{2.15}$$

Neumann boundary conditions (2.14) require that the momentum normal to the boundary of the worldsheet vanishes. Thus, when these are chosen for all coordinates $\mu =$

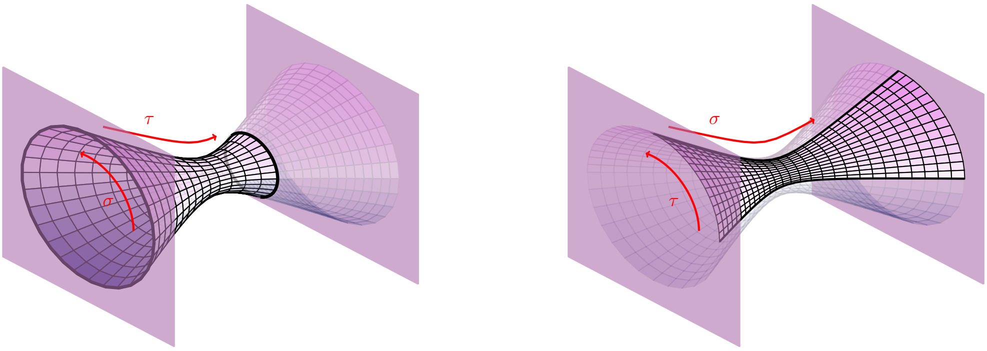

Figure 2.1: Depction of a string worldsheet fixed between two D-branes. This worldsheet can be interpreted as either the one of an open string with its ends on the two branes (right figure), moving in a circle, or as a closed string being created on one brane and absorbed by the other (left figure). The worldsheet coordinates (τ, σ) for the two configurations are also shown.

$0, \dots, \mathrm{d}-1$, Poincaré invariance is conserved. On the other hand, Dirichlet boundary conditions (2.15) break Poincaré invariance. This is due to the fact that when imposing Dirichlet conditions in some directions $\mu = 0, \dots, \mathrm{d}-p-1$, the endpoints of the open string must move along a $(p+1)$-dimensional hypersurface. Although one might generally believe that breaking Poincaré invariance is undesirable, it will turn out that in certain situations Dirichlet conditions are unavoidable. The $(p+1)$-dimensional hypersurface will be interpreted as the world-volume of dynamical objects called Dp-branes. An example of an open string stretching between two branes is given in equation 2.1. These will be essential in the construction of QFTs from string theory setups and will be the topic of discussion in Section 2.2.

From here one proceeds to construct solutions to the equations of motion consistent with given boundary conditions by mode expansion. One further step is to choose a convenient coordinate system on the worldsheet, the light-cone coordinates $\sigma^{\pm} = \tau \pm \sigma$, where the general solution splits into left- and right-moving

$$X^{\mu}(\tau, \sigma) = X^{\mu}_{\mathrm{R}}(\tau - \sigma) + X^{\mu}_{\mathrm{L}}(\tau + \sigma), \tag{2.16}$$

as well as requiring that X^{μ} be a real function. The solution consistent with the closed string boundary conditions is

$$X^{\mu}_{\mathrm{L}}(\sigma^{+}) = \frac{1}{2}x^{\mu} + \frac{1}{2}\alpha' p^{\mu}\sigma^{+} + \mathrm{i}\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\tilde{\alpha}^{\mu}_{n}\mathrm{e}^{-\mathrm{i}n\sigma^{+}}, \tag{2.17}$$

$$X^{\mu}_{\mathrm{R}}(\sigma^{-}) = \frac{1}{2}x^{\mu} + \frac{1}{2}\alpha' p^{\mu}\sigma^{-} + \mathrm{i}\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\alpha^{\mu}_{n}\mathrm{e}^{-\mathrm{i}n\sigma^{-}}, \tag{2.18}$$

where x^{μ} and p^{μ} are position and momentum of the center of mass of the string respectively. The open string is a bit more cumbersome since we have to choose a Neumann or Dirichlet boundary conditions on both ends of the strings. These leaves us with four possible choices: either choose the same boundary condition on both ends or different ones. Here we leave the results without going into the details if not by underlying that, contrary to the closed string, the open string solutions depend only on one set of oscillators α_n^μ. Again we choose the boundaries of the open string to be at $\sigma = 0, \pi$ so that

$$X^\mu(\tau,\sigma) = x^\mu + 2\alpha' p^\mu \tau + \mathrm{i}\sqrt{2\alpha'} \sum_{n\neq 0} \frac{1}{n} \alpha_n^\mu \mathrm{e}^{-\mathrm{i}n\tau} \cos(n\sigma) \qquad \text{(NN)}, \tag{2.19}$$

$$X^\mu(\tau,\sigma) = x_0^\mu + (x_1^\mu - x_0^\mu)\frac{\sigma}{\pi} + \sqrt{2\alpha'} \sum_{n\neq 0} \frac{1}{n} \alpha_n^\mu \mathrm{e}^{-\mathrm{i}n\tau} \sin(n\sigma) \qquad \text{(DD)}, \tag{2.20}$$

$$X^\mu(\tau,\sigma) = x^\mu + \mathrm{i}\sqrt{2\alpha'} \sum_{r\in\mathbb{Z}+\frac{1}{2}} \frac{1}{r} \alpha_r^\mu \mathrm{e}^{-\mathrm{i}r\tau} \cos(r\sigma) \qquad \text{(ND)}, \tag{2.21}$$

$$X^\mu(\tau,\sigma) = x^\mu + \mathrm{i}\sqrt{2\alpha'} \sum_{r\in\mathbb{Z}+\frac{1}{2}} \frac{1}{r} \alpha_r^\mu \mathrm{e}^{-\mathrm{i}r\tau} \sin(r\sigma) \qquad \text{(DN)}. \tag{2.22}$$

In equation (2.20) we labelled the position of the two boundary hypersurfaces as x_0^μ, x_1^μ. As a last comment, jumping a bit, when quantizing the bosonic strings we will encounter the problem of ghosts. This is going to be solved by the fact that the Polyakov action (2.2) has an additional infinite-dimensional symmetry algebra, the Virasoro algebra, which can be used to eliminate such ghosts in the critical dimension d=26.

2.1.2 Adding Fermions: the Superstring

As stated at the start of the Section , the bosonic string comes with many drawbacks. Here we are going to see how one can add fermions to the bosonic string, which turns out to be possible in two equivalent ways[3]

- The Ramond-Neveu-Schwarz (RNS) formalism, where one introduces fermions by adding supersymmetry at the level of the worldsheet.
- The Green-Schwarz (GS) formalism, where supersymmetry is introduced at the level of the target space.

Here we will focus on the RNS formalism, and thus consider a supersymmetric completion of the Polyakov action (2.2) by introducing a d-plet of worldsheet fermions Ψ^μ transforming in the vector representation of the Lorentz group $\mathrm{SO}(1, d-1)$

$$S = -\frac{1}{4\pi\alpha'} \int \mathrm{d}^2\sigma \left(\partial_\alpha X^\mu \partial^\alpha X_\mu + \mathrm{i}\bar{\Psi}^\mu \rho^\alpha \partial_\alpha \Psi_\mu\right), \tag{2.23}$$

where ρ^α are two-dimensional gamma matrices satisfying the usual Clifford algebra $\left\{\rho^\alpha, \rho^\beta\right\} = -2\eta^{\alpha\beta}$, and the conjugate spinor is defined as $\bar{\Psi} := \mathrm{i}\Psi^\dagger \rho^0$.
It might seem counterintuitive to introduce an anticommuting field Ψ^μ which transforms in a bosonic representation of the space-time Lorentz group. However, this is

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[3] At least in 10d Minkowski space.

not in contradiction with the spin-statistics theorem since these are spinors in the two-dimensional field theory defined by (2.23), not in a d-dimensional field theory. From the point of view of the worldsheet, Lorentz symmetry is merely a global symmetry and therefore does not clash with the spin-statistics theorem. A more urgent problem is that of ghosts. For (2.23), the Virasoro constraint is not enough to eliminate both the ghosts coming from worldsheet bosons and fermions. What saves the day is an additional symmetry: supersymmetry, or more precisely superconformal symmetry. This shows that supersymmetry is a fundamental requirement of the theory rather than a mere addition. The action (2.23) is in fact invariant under the transformation

$$\delta X^\mu = \bar{\epsilon}\Psi^\mu, \qquad \delta\Psi^\mu = -\mathrm{i}\rho^\alpha\partial_\alpha X^\mu\epsilon, \tag{2.24}$$

where ϵ is a constant, anti-commuting, Majorana spinor. These are supersymmetric transformations. One expected property of a supersymmetry transformation is that composing two of them leads to a translation, and in fact

$$\begin{aligned}[\delta_1,\delta_2]X^\mu &= \delta_1(\bar{\epsilon}_2\Psi^\mu) - \delta_2(\bar{\epsilon}_1\Psi^\mu)\\ &= -\mathrm{i}\bar{\epsilon}_2\rho^\alpha\partial_\alpha\epsilon_1 X^\mu + \mathrm{i}\bar{\epsilon}_1\rho^\alpha\partial_\alpha\epsilon_2 X^\mu\\ &= 2\mathrm{i}\bar{\epsilon}_1\rho^\alpha\epsilon_2\partial_\alpha X^\mu = a^\alpha\partial_\alpha X^\mu,\end{aligned} \tag{2.25}$$

where $a^\alpha = 2\mathrm{i}\bar{\epsilon}_1\rho^\alpha\epsilon_2$ and we used the condition on 2d Majorana fermions $\bar{\epsilon}_1\rho^\alpha\epsilon_2 = -\bar{\epsilon}_2\rho^\alpha\epsilon_1$. The same goes for Ψ^μ.

Decomposing the worldsheet fermion into Weyl spinors $\Psi^\mu = (\psi_-^\mu, \psi_+^\mu)$, its equations of motion in light-cone coordinates read

$$\partial_-\psi_+^\mu = \partial_+\psi_-^\mu = 0. \tag{2.26}$$

As in the bosonic case, we have left- and right-moving fermions. In these coordinates, supersymmetry becomes apparent from the equations of motion for both $\psi_\pm^\mu$ and X^μ

$$0 = \partial_+\psi_-^\mu = \partial_+(\partial_- X^\mu), \tag{2.27}$$

$$0 = \partial_-\psi_+^\mu = \partial_-(\partial_+ X^\mu). \tag{2.28}$$

Thus, $\partial_- X^\mu$ and ψ_-^μ are both functions of only σ^+ while the others are both functions of only σ^-. Supersymmetry is the symmetry between $\partial_\pm X^\mu$ and $\psi_\pm^\mu$ which obey the same equations.

Concerning the fermionic fields, the open string boundary condition imposes

$$\psi_+^\mu = \pm\psi_-^\mu, \tag{2.29}$$

at each end of the string. On one side, the choice is a matter of convention, and one usually chooses the plus sign. On the other side, we have two possible choices

$$\psi_+(\tau,\pi) = +\psi_-(\tau,\pi) \qquad \text{Ramond (R) boundary condition,} \tag{2.30}$$

$$\psi_+(\tau,\pi) = -\psi_-(\tau,\pi) \qquad \text{Neveu-Schwarz (NS) boundary condition.} \tag{2.31}$$

The Ramond boundary condition gives rise to spacetime fermions and the mode expansion in this sector is

$$\psi^{\mu}_{-}(\tau,\sigma) = \frac{1}{\sqrt{2}}\sum_{n\in\mathbb{Z}} d^{\mu}_{n}\mathrm{e}^{-\mathrm{i}n(\tau-\sigma)}, \quad \psi^{\mu}_{+}(\tau,\sigma) = \frac{1}{\sqrt{2}}\sum_{n\in\mathbb{Z}} d^{\mu}_{n}\mathrm{e}^{-\mathrm{i}n(\tau+\sigma)}. \tag{2.32}$$

The Neveu-Schwarz boundary condition, instead, gives rise to spacetime bosons and the mode expansion in this sector is

$$\psi^{\mu}_{-}(\tau,\sigma) = \frac{1}{\sqrt{2}}\sum_{r\in\mathbb{Z}+\frac{1}{2}} d^{\mu}_{r}\mathrm{e}^{-\mathrm{i}r(\tau-\sigma)}, \quad \psi^{\mu}_{+}(\tau,\sigma) = \frac{1}{\sqrt{2}}\sum_{r\in\mathbb{Z}+\frac{1}{2}} d^{\mu}_{r}\mathrm{e}^{-\mathrm{i}r(\tau+\sigma)}. \tag{2.33}$$

For open string the discussion is much the same, with the addition that we have to impose (anti-)periodic boundary conditions to both components of Φ^{μ} separately, i.e,

$$\psi_{\pm}(\tau,\sigma) = \pm\psi_{\pm}(\tau,\sigma+\pi), \tag{2.34}$$

giving rise to left- and right-moving modes. Again we can impose either Ramond or Neveu-Schwarz to both left- and right-movers. Therefore we have four distinct closed-string sectors: the states in the R-R and NS-NS sector are spacetime bosons, while the states in R-NS and NS-R sectors are spacetime fermions.

2.1.3 The Quantum String

We are now ready to tackle the problem of quantizing the string. In this Section , we will outline the main steps for quantizing the bosonic string and demonstrate that the addition of fermions gives rise to five possible superstring theories.
There are three main equivalent methods to quantize the bosonic string

- **Covariant quantization**: This method is analogous to first quantization in quantum field theory, where one imposes equal-time commutation relations for the coordinates X^{μ}
$$\left[\dot{X}^{\mu}(\tau,\sigma), X^{\nu}(\tau,\sigma')\right] = -\mathrm{i}\pi\delta(\sigma-\sigma')\eta^{\mu\nu}. \tag{2.35}$$
The restrictions on the physical states arise from the constraints of the Virasoro algebra. When adding fermions, this quantization results in two separate spectra, which are truncated and related by means of the GSO projection.

- **Light-cone gauge quantization**: After setting $h_{\alpha\beta} = \eta_{\alpha\beta}$ in the action (2.2), there remains a residual gauge symmetry. One can make a specific non-covariant choice that explicitly solves the Virasoro constraint and describes the theory in a Fock space consisting only of physical states. This formalism is not manifestly covariant but is manifestly ghost-free.

- **BRST quantization**: In this approach, the path integral formulation is employed, and gauge-fixing leads to the introduction of Faddeev-Popov ghosts. This method provides an explicitly covariant way of quantizing the theory.

Table 2.1: Open and closed string spectrum for the lowest levels. The first closed string excited state comes from the decomposition of the $\mathbf{24} \otimes \mathbf{24}$ in $\mathrm{SO}(24)$.

	Mass	Open String	Closed String
Ground state	$M^2 < 0$	Tachyon	Tachyon
First excited state	$M = 0$	A_μ	$g_{\mu\nu}, B_{\mu\nu}, \Phi$

For the purposes of this discussion, we will focus on covariant quantization. The procedure involves promoting the fields to operators and imposing the equal-time commutation relations (2.35). The Fourier modes α_n^μ and $\tilde{\alpha}_n^\mu$ are then promoted to creation and annihilation operators, which act on a Hilbert space and define a vacuum state $|0\rangle$. By acting on the vacuum state with creation operators, the full spectrum of the theory is obtained. Since each mode is labeled by an integer, an infinite set of ladder operators is found, suitably normalized as $a_n^\mu = \alpha_n^\mu/\sqrt{n}$.
The first significant challenge arises when considering the first mass levels for the open string spectrum, which are found to be *tachyonic*, i.e., possessing negative mass squared. This state cannot remain in the spectrum and is projected out in the full superstring theory via the GSO projection, which will be discussed in detail later. The first excited state gives rise to a vector boson transforming in the symmetric traceless rank-two representation of $\mathrm{SO}(24)$. Concerning the closed string spectrum, their modes can be constructed as tensor products of the open string modes. We summarize the first two level spectrum of both the open and closed string in table 2.1. Note that the first excited state of the closed string spectrum transforms in the $\mathbf{24} \otimes \mathbf{24}$ representation of $\mathrm{SO}(24)$ which decomposes into the irreducible representations as

$$\mathbf{24} \otimes \mathbf{24} = \underset{\substack{\uparrow \\ \text{Traceless symmetric} \\ (0,2)\text{-tensor } g_{\mu\nu}}}{\mathbf{299}} \oplus \underset{\substack{\uparrow \\ \text{Totally anti-symmetric} \\ \text{two-form } B_{\mu\nu}}}{\mathbf{276}} \oplus \underset{\substack{\uparrow \\ \text{Singlet } \Phi}}{\mathbf{1}} \tag{2.36}$$

where $g_{\mu\nu}$ is the massless graviton, $B_{\mu\nu}$ is the massless Kalb-Ramond field and Φ is a real massless scalar, the dilaton. The Kalb-Ramond field is especially interesting since it souced by strings, in the same fashion as electrically charged particles are sources for the potential A_μ. Following quantization, we observe that not only does a tachyonic state appear in the spectrum, but there are also states in the Hilbert space of the theory with negative norm. These ghosts can be removed from the bosonic string spectrum, as we hinted before, using the Virasoro constraint by setting the dimension of the ambient spacetime to 26. Tachyons, however, are different beasts, and even after setting d$= 26$, they still remain in the spectrum. To eliminate them, we necessarily need to add fermions and consider the superstring.

Table 2.2: The spectrum of the first few right-moving open string states. The decomposition of the states is done following Wigner's classification, with respect to the stabilizer subgroup of the momentum in the rest frame. For massive representations, the stabilizer subgroup is $\mathrm{SO}(9)$, and for massless representations, it is $\mathrm{SO}(8)$.

Mass	States and their $\mathrm{SO}(8)$ representation	Representation with respect to little group
NS-sector (Bosons)		
$-\frac{1}{2}$	$\|0\rangle$ $\mathbf{1}$	$\mathbf{1}$
0	$b^{\mu}_{-\frac{1}{2}}\|0\rangle$ $\mathbf{8}_v$	$\mathbf{8}_v$
$\frac{1}{2}$	$\alpha^{\mu}_{-1}\|0\rangle \quad b^{\mu}_{-\frac{1}{2}}b^{\nu}_{-\frac{1}{2}}\|0\rangle$ $\mathbf{8}_v \quad \mathbf{28}$	$\mathbf{36}$
1	$b^{\mu}_{-\frac{1}{2}}b^{\nu}_{-\frac{1}{2}}b^{\rho}_{-\frac{1}{2}}\|0\rangle$ $\mathbf{56}_v$ $\alpha^{\mu}_{-1}b^{\nu}_{-\frac{1}{2}}\|0\rangle \quad b^{\mu}_{-\frac{3}{2}}\|0\rangle$ $\mathbf{1}\oplus\mathbf{28}\oplus\mathbf{35}_v \quad \mathbf{8}_v$	$\mathbf{84}\oplus\mathbf{44}$
R-sector (Fermions)		
0	$\|a\rangle$ $\mathbf{8}_s$ $\|\bar{a}\rangle$ $\mathbf{8}_c$	$\mathbf{8}_s$ $\mathbf{8}_c$
1	$\alpha^{\mu}_{-1}\|a\rangle \quad d^{\mu}_{-1}\|\bar{a}\rangle$ $\mathbf{8}_c\oplus\mathbf{56}_c \quad \mathbf{8}_c\oplus\mathbf{56}_s$ $\alpha^{\mu}_{-1}\|\bar{a}\rangle \quad d^{\mu}_{-1}\|a\rangle$ $\mathbf{8}_c\oplus\mathbf{56}_s \quad \mathbf{8}_c\oplus\mathbf{56}_c$	$\mathbf{128}$ $\mathbf{128}$

The canonical quantization of the superstring proceeds in an analogue way to the one just discussed, with the only difference that now the Virasoro constraint requires the space-time dimension to be 10 in order to project out ghosts. A summary of the first few excited states in both NS and R sectors is given in table 2.2.

Nevertheless, the RNS model, as described in Section 2.1.2, is an inconsistent quantum theory and a truncation of the spectrum is required [198]. The argument is two-fold. On one hand, we want to project the tachyonic state out of the theory while retaining the massless particles of interest. On the other hand, we seek to make sense of the anti-commuting vector operators Ψ^{μ}, introduced in (2.23), which, perhaps unnervingly, map bosonic states to bosonic states. Consider a bosonic state $|\phi\rangle$. When we act on $|\phi\rangle$ with an anti-commuting operator Ψ^{μ}, the resulting state $\Psi^{\mu}|\phi\rangle$ is still bosonic. More generally, a state

$$\Psi^{\mu_1}(\sigma_1)\Psi^{\mu_2}(\sigma_2)\cdots\Psi^{\mu_n}(\sigma_n)\,|\phi\rangle \tag{2.37}$$

is bosonic for all n. For even n, this state does not seem strange since the product of an even number of anti-commuting operators is commuting. However, for odd n, the situation is different, and we might be inclined to keep only those states generated by acting with an even number of Ψ^μ and discard those generated by acting with an odd number. This can be done formally by introducing an operator $G = (-1)^F$, known as the *fermion number*[4], under which the bosonic fields X^μ are even and the fermionic ones Ψ^μ are odd. For a general state in (2.37), we then have

$$(-1)^F \left(\Psi^{\mu_1}(\sigma_1)\Psi^{\mu_2}(\sigma_2)\cdots\Psi^{\mu_n}(\sigma_n)\,|\phi\rangle\right) = (-1)^n \left(\Psi^{\mu_1}(\sigma_1)\Psi^{\mu_2}(\sigma_2)\cdots\Psi^{\mu_n}(\sigma_n)\,|\phi\rangle\right). \tag{2.38}$$

The GSO projection states that we must keep only states for which $(-1)^F = +1$. In the fermionic (R) sector one can define a similar projection operation $\bar{\Gamma} = \Gamma_{11}(-1)^F$. Following this procedure, the R and NS spectra split into $\mathrm{R}_\pm$ and $\mathrm{NS}_\pm$, where NS_-, being the sector that contains the tachyon, is projected out. The GSO projection does more than meets the eye. It also yields a supersymmetric theory in the ambient spacetime. Let us rewind to appreciate the beauty of this fact: we added fermions to our bosonic worldsheet theory, which turned out to be superconformal under the requirement that the spacetime spectrum does not contain states of negative norm. To eliminate the tachyonic state from the spectrum, we employ the GSO procedure, which also results in a supersymmetric theory in the ambient spacetime! Everything fits very nicely and one may wonder that this procedure, rather than a formal requirement, must naturally arise from the theory. After the projection, the ground state of the R sector is a massless spinor and the ground state of the NS sector is a massless vector boson, both belonging to the right representation of $\mathrm{SO}(8)$[5]. Therefore, the ground state of the open superstring spectrum is a 16-dimensional multiplet in the $\mathbf{8}_v \oplus \mathbf{8}_s$ of $\mathrm{SO}(8)$, where the subscript labels the vector and spinor representations. This spectrum preserves $\mathcal{N} = 1$ in 10d. On the other hand, the closed superstring spectrum is constructed from two copies of the open-string one by combining a left-moving sector (either NS or $\mathrm{R}_\pm$) with a right-moving one (either NS or $\mathrm{R}_\pm$). The spectrum appearing at the massless level of the closed string is summarized in Table 2.3, together with the eigenvalues of the projection operators.
All in all, we arrive at five possible superstring theories, which we summarize here:

- **Type-II superstring theories**: These are maximally supersymmetric theories in ten dimensions, preserving 32 supercharges, which is $\mathcal{N} = 2$ in 10d. They only contain closed string sectors, and one obtains two different theories depending on the chosen chirality in the R-sector
    - *Type-IIA theory*: where $\bar{\Gamma}_\mathrm{L} = -\bar{\Gamma}_\mathrm{R} = 1$.
    - *Type-IIB theory*: where $\bar{\Gamma}_\mathrm{L} = \bar{\Gamma}_\mathrm{R} = 1$.

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[4] Often called G-parity for historical reasons.

[5] In 10d, massless states are classified by their behavior under the $\mathrm{SO}(8)$ rotation that leave the momentum invariant. In fact, a vector A^μ in 10d has 8 propagating degrees of freedom. A spinor has $2^5 = 32$ components, however after imposing Majorana-Weyl condition these get halved and further halved again when requiring that they satisfy the Dirac equation.

Table 2.3: The closed string spectrum up to the massless level before GSO projection. The operators $G_{\mathrm{L,R}}$ and $\bar{\Gamma}_{\mathrm{L,R}}$ are the projection operators in the NS and R sectors respectively. The red colored rows are sectors of type-IIA, while the blue colored ones are for type-IIB. The yellow ones are sectors of both type-II theories. The white rows are going to be projected out by the GSO procedure.

States and their $\mathrm{SO}(8)$ representation	G_{L} (NS) $\bar{\Gamma}_{\mathrm{L}}$ (R)	G_{R} (NS) $\bar{\Gamma}_{\mathrm{R}}$ (R)	Decomposition	Tensors
(NS,NS)-sector (Bosons)				
$\vert 0\rangle_{\mathrm{L}} \otimes \vert 0\rangle_{\mathrm{R}}$ $\mathbf{1}\otimes\mathbf{1}$	-1	-1	$\mathbf{1}$	$[1]$
$\tilde{b}^i_{-\frac{1}{2}} \vert 0\rangle_{\mathrm{L}} \otimes b^j_{-\frac{1}{2}} \vert 0\rangle_{\mathrm{R}}$ $\mathbf{8}_v\otimes\mathbf{8}_v$	$+1$	$+1$	$\mathbf{1}\oplus\mathbf{28}_v\oplus\mathbf{35}_v$	$[0]+[2]+(2)$
(R,R)-sector (Bosons)				
$\vert a\rangle_{\mathrm{L}} \otimes \vert b\rangle_{\mathrm{R}}$ $\mathbf{8}_s\otimes\mathbf{8}_s$	$+1$	$+1$	$\mathbf{1}\oplus\mathbf{28}_v\oplus\mathbf{35}_v$	$[0]+[2]+[4]_s$
$\vert \bar{a}\rangle_{\mathrm{L}} \otimes \vert \bar{b}\rangle_{\mathrm{R}}$ $\mathbf{8}_c\otimes\mathbf{8}_c$	-1	-1	$\mathbf{1}\oplus\mathbf{28}_v\oplus\mathbf{35}_v$	$[0]+[2]+[4]_c$
$\vert \bar{a}\rangle_{\mathrm{L}} \otimes \vert b\rangle_{\mathrm{R}}$ $\mathbf{8}_c\otimes\mathbf{8}_s$	-1	$+1$	$\mathbf{8}_v\oplus\mathbf{56}_v$	$[1]+[3]$
$\vert a\rangle_{\mathrm{L}} \otimes \vert \bar{b}\rangle_{\mathrm{R}}$ $\mathbf{8}_s\otimes\mathbf{8}_c$	$+1$	-1	$\mathbf{8}_v\oplus\mathbf{56}_v$	$[1]+[3]$
(R,NS)-sector (Fermions)				
$\vert a\rangle_{\mathrm{L}} \otimes b^i_{-\frac{1}{2}} \vert 0\rangle_{\mathrm{R}}$ $\mathbf{8}_s\otimes\mathbf{8}_v$	$+1$	$+1$	$\mathbf{8}_c\oplus\mathbf{56}_s$	$[1]+[3]$
$\vert \bar{a}\rangle_{\mathrm{L}} \otimes b^i_{-\frac{1}{2}} \vert 0\rangle_{\mathrm{R}}$ $\mathbf{8}_c\otimes\mathbf{8}_v$	-1	$+1$	$\mathbf{8}_s\oplus\mathbf{56}_c$	$[1]+[3]$
(NS,R)-sector (Fermions)				
$\tilde{b}^i_{-\frac{1}{2}} \vert 0\rangle_{\mathrm{L}} \otimes \vert a\rangle_{\mathrm{R}}$ $\mathbf{8}_v\otimes\mathbf{8}_s$	$+1$	$+1$	$\mathbf{8}_c\oplus\mathbf{56}_s$	$[1]+[3]$
$\tilde{b}^i_{-\frac{1}{2}} \vert 0\rangle_{\mathrm{L}} \otimes \vert \bar{a}\rangle_{\mathrm{R}}$ $\mathbf{8}_v\otimes\mathbf{8}_c$	$+1$	-1	$\mathbf{8}_s\oplus\mathbf{56}_c$	$[1]+[3]$

In Table 2.3, the full spectrum of the two theories is given. Let us expand on the matter content of the various sectors

- *NS-NS sector*: This belongs to both type-IIA and IIB theories. The matter content is given by

  $$\mathbf{8}_V \otimes \mathbf{8}_v = \mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35} = \Phi \oplus B_{\mu\nu} \oplus g_{\mu\nu}, \tag{2.39}$$

  corresponding to the dilaton Φ, the Kalb-Ramond field $B_{\mu\nu}$, and the graviton $g_{\mu\nu}$.
- *NS-R and R-NS sectors*: The matter content for this sector is

  $$\mathbf{8}_v \otimes \mathbf{8}_s = \mathbf{8}_c \oplus \mathbf{56}_s, \quad \mathbf{8}_v \otimes \mathbf{8}_c = \mathbf{8}_s \oplus \mathbf{56}_c, \tag{2.40}$$

  corresponding to the spin-1/2 dilatino λ and the spin-3/2 gravitino Ψ_μ. In type-IIA, where the two sectors are NS-R$_\pm$ and R$_\mp$-NS, the two gravitinos have opposite chiralities, while in type-IIB, coming from the sectors NS-R$_\pm$ and R$_\pm$-NS, they have the same chirality.
- *R-R sector*: These differ between the two type-II theories. Indeed, one has

  $$\text{Type-IIA} \qquad \mathbf{8}_s \otimes \mathbf{8}_c = \mathbf{8}_v \oplus \mathbf{56}_v, \tag{2.41}$$

  $$\text{Type-IIB} \qquad \mathbf{8}_s \otimes \mathbf{8}_s = \mathbf{1} \oplus \mathbf{28}_v \oplus \mathbf{35}_v. \tag{2.42}$$

  The particle content in type-IIA (R$_\pm$-R$_\mp$) comprises a one-form $C^{(1)}_\mu$ and a three-form $C^{(3)}_{\mu\nu\rho}$, while in type-IIB (R$_\pm$-R$_\pm$), it consists of a zero-form $C^{(0)}$, known as the axion, a two-form $C^{(2)}_{\mu\nu}$, and a four-form $C^{(4)}_{\mu\nu\rho\sigma}$ with a self-dual field strength.

- **Type-I superstring theory**: This is the theory arising from open strings. It can be shown that to be consistent, it must also contain closed strings. The ground state of the spectrum consists of a vector multiplet in 10d $\mathcal{N} = 1$[6]. This vector multiplet contains the graviton, the dilaton, the R-R two-form, together with a gravitino and dilatino.

- **Heterotic string theories**: This is obtained by combining the left-movers of the closed bosonic string with the right-movers of the closed superstring. Since the ambient spacetime of the bosonic string must be 26 from the Virasoro constraint, the 16 additional dimensions must be compactified, giving rise to internal gauge symmetries. These are either SO(32) or E$_8$ × E$_8$. These are $\mathcal{N} = 1$ theories whose massless sector contains the graviton, the dilaton, the Kalb–Ramond two-form, together with a gravitino and a dilatino. Additionally, there are vector fields with their related spin-1/2 fermionic partners (called gaugini) that gauge the internal symmetry group.

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[6] The name Type-I comes from the amount of space-time supersymmetry the theory preserves. Same goes for Type-II.

Type-IIB string theory has a low-energy space-time description in terms of a supersymmetric theory with gravity (a supergravity theory). Due to the presence of the R-R four-form with dual field strength $F_5 = \mathrm{d}C_4$, one cannot write a complete action. Nonetheless, one may write an action containing both dualities for C_4 and then impose the self-duality condition $\star F_5 = F_5$. With this in mind, the bosonic part of the action reads

$$\begin{aligned} S_{\text{Type-IIB}} &= \frac{1}{4\kappa^2}\int \mathrm{d}^{10}x\,\sqrt{-g}\mathrm{e}^{-2\Phi}(2R + 8\partial^\mu\Phi\partial_\mu\Phi - |H_3|^2) \\ &- \frac{1}{4\kappa^2}\int \mathrm{d}^{10}x\left[\sqrt{-g}\left(|F_1|^2 + |\tilde{F}_3|^2 + \frac{1}{2}|\tilde{F}_5|^2\right) + C_4\wedge H_3\wedge F_3\right], \end{aligned} \tag{2.43}$$

where κ is a coupling constant, and the fields are defined by

$$F_1 = \mathrm{d}C_0\,, \qquad F_3 = \mathrm{d}C_2\,, \qquad F_5 = \mathrm{d}C_4\,, \qquad H_3 = \mathrm{d}B\,, \tag{2.44}$$

$$\tilde{F}_3 = F_3 - CH_3, \qquad \tilde{F}_5 = F_5 - \frac{1}{2}C_2\wedge H_3 + \frac{1}{2}B\wedge F_3. \tag{2.45}$$

As stated before, the action must be supplemented with the self-duality condition $\star\tilde{F}_5 = \tilde{F}_5$. This action manifests a very interesting symmetry under the non-compact group $\mathrm{SU}(1,1)\sim\mathrm{SL}(2,\mathbb{R})$. By the following change of frame (known as the Einstein frame)

$$g_{\mathrm{E}\,\mu\nu} = \mathrm{e}^{-\frac{\Phi}{2}}g_{\mu\nu}, \qquad \tau = C + \mathrm{i}\mathrm{e}^{-\Phi}, \qquad G_3 = \frac{F_3 - \tau H_3}{\mathrm{Im}\,\tau}, \tag{2.46}$$

the action may be rewritten as

$$\begin{aligned} S^{\text{Einstein}}_{\text{Type-IIB}} &= \frac{1}{4\kappa^2}\int \mathrm{d}^{10}x\,\sqrt{-g_{\mathrm{E}}}\left(2R_{\mathrm{E}} - \frac{\partial_\mu\tau\partial^\mu\bar{\tau}}{(\mathrm{Im}\,\tau)^2} - \frac{1}{2}|F_1|^2 + |G_3|^2 - \frac{1}{2}|\tilde{F}_5|^2\right) \\ &- \frac{1}{4\mathrm{i}\kappa^2}\int \mathrm{d}^{10}x\,C_4\wedge\bar{G}_3\wedge G_3. \end{aligned} \tag{2.47}$$

This action is still invariant under $\mathrm{SL}(2,\mathbb{R})$, which acts on the axion-dilaton field as

$$\tau \to \frac{a\tau + b}{c\tau + d}, \qquad ad - bc = 1, \quad a,b,c,d\in\mathbb{R}, \tag{2.48}$$

while simultaneously mixing the C_2 and B fields under the linear transformation associated with this Möbius transformation. The latter may be recast as an action on the field G_3 as follows

$$G_3 \to \frac{c\bar{\tau} + d}{|c\tau + d|}G_3. \tag{2.49}$$

In the quantum theory, the quantization condition on the axion-dilaton field $\tau\sim\tau+1$ breaks the $\mathrm{SL}(2,\mathbb{R})$ to its subgroup $\mathrm{SL}(2,\mathbb{Z})$, which will be fundamental for the discussion in Chapter 4.

2.2 Extended Objects: The Branes

As briefly stated in the previous Section, D-branes are non-perturbative extended objects defined as the loci where open strings end with Dirichlet boundary conditions [282]. However, from this interpretation, it is not evident that these objects are actually dynamical. To understand this, we need another interpretation, which is going to be of fundamental importance when discussing the AdS/CFT correspondence.

From usual Maxwell theory, one knows that given a theory of some $(p+1)$-form field, one should be able to find objects that are charged under these fields. Indeed, such a field naturally couples to a $(p+1)$-dimensional surface $\Sigma^{(p+1)}$ by the following action

$$S_{p+1} = T_{p+1} \int\limits_{\Sigma^{(p+1)}} A_{p+1} = \frac{T_{p+1}}{(p+1)!} \int\limits_{\Sigma^{(p+1)}} A_{\mu_1,\dots,\mu_{p+1}}\, \mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_{p+1}} \,. \tag{2.50}$$

Solutions to supergravity which carry a non-trivial charge under A_{p+1} are known as p-branes. The possible brane solutions in a given supergravity theory are thus limited by the p-forms present in the field content. Moreover, by Hodge duality, for each gauge field A_{p+1}, there is an associated magnetic field $\tilde{A}_{d-p-3}$ whose field strength is related to that of A_{p+1} by Poincaré duality

$$\mathrm{d}\tilde{A}_{d-p-3} = \star\, \mathrm{d}A_{p+1} \,. \tag{2.51}$$

Therefore, each p-brane comes with a dual $(d-p-4)$-brane carrying charge under $\tilde{A}_{d-p-3}$.

Consider now the case of type-II theories. The R-R spectrum includes higher-form gauge fields whose dimensions depend on the type of theory. Each Dp-brane is electrically charged under a C_{p+1} R-R potential and magnetically charged under a C_{7-p} potential. We can derive the brane content for each type-II theory from (2.41) and (2.42).

- **Type-IIA**: In type-IIA theory, only Dp-branes with p even are present:

  $$\mathrm{D}0, \quad \mathrm{D}2, \quad \mathrm{D}4, \quad \mathrm{D}6, \quad \mathrm{D}8. \tag{2.52}$$

  D0-branes, also known as D-particles, are magnetic duals to D6-branes, while D2-branes are magnetic duals to D4-branes. The D8-brane couples to the R-R potential whose field strength is F_{10}, which has no propagating degrees of freedom.

- **Type-IIB**: In type-IIB theory, only Dp-branes with p odd are present:

  $$\mathrm{D}(-1), \quad \mathrm{D}1, \quad \mathrm{D}3, \quad \mathrm{D}5, \quad \mathrm{D}7, \quad \mathrm{D}9. \tag{2.53}$$

  The peculiar case of the D(-1)-brane describes a particle localized in time, known as a D-instanton, whose magnetic dual is the D7-brane. The D1-brane is a D-string, and its magnetic dual is the D5-brane. D9-branes are space-filling branes with no coupling to any R-R field and lead to Neumann boundary conditions in every direction. The D3-brane is a self-dual brane, which will be fundamental in the forthcoming discussion.

We summarize the various branes, the fields they are charged under, and their magnetic duals in Table 2.4. The object dual to the fundamental string is called an NS5-brane, named because it couples to the degrees of freedom arising from quantizing the NS-NS sector of the type II strings [130].

Table 2.4: Branes in type-II theories

Brane	Type-IIA	Type-IIB	Magnetic dual
D(-1) Instanton	–	$\tau = C + \mathrm{i}e^{-\Phi}$	D7
D0 particle	$C^{(1)}_{\mu}$	–	D6
F1 string	$B_{\mu\nu}$	$B_{\mu\nu}$	NS5
D1 string	–	$C^{(2)}_{\mu\nu}$	D5
D2 brane	$C^{(3)}_{\mu\nu\sigma}$	–	D4
D3 brane	–	$C^{(4)}_{\mu\nu\rho\sigma}$	D3

In the remaining part of this Section , we aim to understand the worldvolume action for D-branes. A complete treatment of this subject is beyond the scope of this thesis, as it requires K-theory as a framework to properly discuss it [61, 269, 312].

The first requirement for the worldvolume theory is the presence of a U(1) gauge field on the brane, since the boundaries of open strings lie on the D-brane and should couple to such a field. Starting from the Nambu-Goto action, one can infer that a possible D-brane action is

$$S_{\text{D-brane}} = -T_p \int \mathrm{d}^{p+1}\sigma \; \mathrm{e}^{-\Phi}\sqrt{\det g_{ab}}, \tag{2.54}$$

where g_{ab} is the pull-back of the metric on the brane worldvolume.

Next, considering the brane in a background generated by the massless NS-NS modes of the closed string sector, namely $B_{\mu\nu}$, additional terms arise. The only gauge-invariant combination that can appear in the D-brane action is

$$\mathcal{F} = B + 2\pi\alpha' F, \tag{2.55}$$

where F is the field strength of the U(1) gauge field A.

This leads to the well-known Dirac-Born-Infeld (DBI) action for the D-brane

$$S_{\text{Dp}} = -T_p \int \mathrm{d}^{p+1}\sigma \; \sqrt{\det(g_{ab} + \mathcal{F}_{ab})}, \tag{2.56}$$

where B_{ab} is the pull-back of $B_{\mu\nu}$ to the worldvolume.

However, this is not the complete picture. The coupling to the higher-dimensional R-R fields must also be included. Since D-branes act as sources for both gravitational and p-form fields, one must require the cancellation of mixed anomalies coming from the zero modes of open strings ending on the intersection of two branes. This leads to the following action [204, 269]

$$S_{\text{R-R}} = \mu_p \int C \wedge \text{ch}(\mathcal{F}) \wedge \frac{\sqrt{\hat{A}(TW)}}{\sqrt{\hat{A}(NW)}}, \tag{2.57}$$

where $C = \sum_p C_{p+1}$ and $\hat{A}$ is the Dirac $\hat{A}$-genus[7] of the tangent and normal bundle to the worldvolume. Interestingly, a p-brane acts not only as a source for C_{p+1} but also for all lower-dimensional forms.

Finally, the tension of a D-brane, T_p, is related to the string coupling g_s and the Regge slope α' as follows:

$$T_p = \frac{1}{g_s(2\pi)^p(\alpha')^{(p-1)/2}}, \tag{2.58}$$

from which we can deduce that in the strongly-coupled string regime ($g_s \to \infty$ or $\alpha' \to \infty$), the branes are very light. Conversely, in the weak-coupling regime, they are very heavy and decouple.

Both the DBI action and the Chern-Simons term can be generalized from a single brane to a stack of N coincident branes [272, 303]. The result is more complex since the $\mathrm{U}(1)$ field A and the transverse coordinates X^i to the brane are now $N \times N$ matrices, and it is not immediately clear how to pull back the bulk fields in this geometry. Nonetheless, this can be accomplished, and we provide here the leading α' term to the DBI action for this system

$$-T_p \int \mathrm{d}^{p+1}\sigma \ \mathrm{e}^{-\Phi} \ \mathrm{Tr}\left(F_{ab}F^{ab} + 2D_a X^i D^a X_i + \left[X^i, X^j\right]^2\right). \tag{2.59}$$

From this, we conclude that the dynamics now corresponds to a $\mathrm{U}(N)$ worldvolume gauge theory, rather than $\mathrm{U}(1)^N$, with gauge coupling

$$g^2_{\text{YM}} \sim g_s T_p^{-1} \alpha'^{-2}. \tag{2.60}$$

This theory turns out to be 4-dimensional maximally supersymmetric $\mathcal{N} = 4$ Yang-Mills. We will briefly discuss how this result forms the basis of the AdS/CFT correspondence, though we will delve into more details in the proceeding sections. The basic idea of the correspondence is that there are two equivalent descriptions of the same physics: one in terms of supergravity in a space of the form $\text{AdS}_{D+1} \times M_{9-D}$ and the other as a D-dimensional field theory living on the boundary of AdS. The AdS space is the maximally symmetric solution to Einstein's equations with a negative cosmological constant, while M_{9-D} is a $(9-D)$-dimensional compact manifold that encodes the global symmetries in the dual field theory.

The first example of this correspondence was presented in [264], which established a relationship between type-IIB supergravity in $\text{AdS}_5 \times S^5$ and 4-dimensional $\mathcal{N} = 4$ super Yang-Mills theory with gauge group $\mathrm{SU}(N)$. Notably, as we showed before, this is the gauge theory living on the worldvolume of a stack of N D3-branes. On the supergrav-

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[7] This is a generalization of the index of the Dirac operator on general spin manifolds.

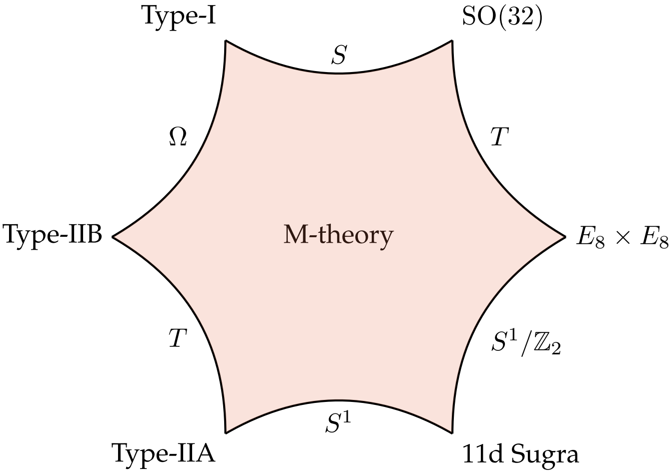


Figure 2.2: Map of various string dualities. From 11d supergravity one can go to either Type-IIA or $E_8 \times E_8$ Heterotic by compactifying on a circle and on an interval respectively. The action of Ω is by orientifolding.

ity side, in the *near-horizon limit*, this brane configuration gives rise to the $\mathrm{AdS}_5 \times S^5$ geometry.
Numerous non-trivial checks of this correspondence have been found, both for the cited example and for various generalizations.
The result for $\mathcal{N} = 4$ SYM is going to be the basis to understand the ideas of [38], which we are going to be discussing in Chapter 4.

2.3 M-Theory

Despite many improvements in the understanding of string theory, several open problems persisted by the end of the 1980s. In the last section, we discovered that, quite surprisingly, there exist five different consistent superstring theories: type-I, type-II A and B and the two heterotic string theories with gauge groups $E_8 \times E_8$ and $\mathrm{SO}(32)$. This issue of having multiple self-consistent string theories found a solution with the discovery of fundamental dualities between them, a picture of which can be found in figure 2.2, culminating in the mid-nineties with the work of Witten [315]. Here, Witten conjectured the existence of M-theory, a joint strong coupling limit for all five superstring theories. The low energy effective field theory description of M-theory is 11-dimensional supergravity, which reduces to the various string theories by Kaluza-Klein compactification, followed by various string dualities.

Concretely, consider performing a circle reduction of a $(\mathrm{d}+1)$-dimensional theory. Here the Kaluza-Klein modes arising from the modes around this S^1 have a mass related to the radius of the circle R and goes like

$$m_{\mathrm{KK}} = \frac{n}{R}, \quad n \in \mathbb{Z}, \tag{2.61}$$

where n labels the n-th excited state. Thus, by considering the small circle limit $R \to 0$, only the zero modes $n = 0$, which are massless, survive, while the other modes are integrated out in the effective d-dimensional IR theory.
If we now consider D0-branes in type-IIA, we know that their mass is given by

$$M_{\text{D0}} = T_{\text{D0}} = g_s \ell_s. \tag{2.62}$$

The mass of a stack of D0-branes in nM_{D0}. By comparison with (2.61), we can interprate this configuration as the n-th excited state of a KK-tower coming from circle reduction of an 11-dimensional theory with radius $R_{11} = g_s \ell_s$. The aformentioned 11-dimensional theory is indeed M-theory. From this simple formulation we can also see that the weak coupling limit of type-IIA string theory arises from the circle reduction since $g_s \to 0$ implies $R_{11} \to 0$. On the contrary, in the strong coupling limit $g_s \to \infty$, the circle direction is not compactified and M-theory arises.
M-theory comes with two known BPS branes, called the M2-brane and the M5-brane. These are related to non-perturbative objects in type-IIA by circle compactification. M5-branes are of interest to us for the discussion of [35], where we are going to consider compactification of M-theory coming from M5-branes wrapping certain geometries. These branes host on their worldvolume a non-lagrangian supersymmetric 6d theory with supersymmetries $\mathcal{N} = (2, 0)$. These theories, altough being non-lagrangian, are labelled by simply laced Lie algebras $\mathfrak{g}$ because both their compactification on a circle and 5d $\mathcal{N} = 2$ SYM on its Coulomb branch, are described by 5d abelian vector multiplets in the Cartan of $\mathfrak{g}$.
Using the tools provided by geometric engineering, people have been able to construct a vast landscape of both lagrangian, and non-lagrangian, theories in various dimensions. The ones we are going to be interested in are theories of class-S type [188] arising from wrapping a stack of M5-branes on a Riemann surface without punctures [67]. In Chapter 5 we are going to delve into the details of such theories.

2.4 The AdS/CFT Correspondence

The foundational argument directly motivating the AdS/CFT correspondence was first stated by Maldacena [264] and is known as the *decoupling argument*. Although this argument does not constitute a proof, it provides a compelling rationale that can be examined through various checks. As we will observe, the argument is fundamentally based on the dual interpretation of D-branes discussed in the previous sections: on one hand as solutions to the supergravity field equations, and the other as boundary condition of open strings. By considering the branes from each perspective, we find that, under certain conditions, two decoupled theories emerge from each viewpoint, leading to a correspondence between, in the simplest and original example, 4d $\mathcal{N} = 4$ SYM in Minkowski space and type-IIB string theory in the space AdS$_5 \times S^5$. This is the result that we alluded at the end of section 2.2. This conjecture represents a fundamental advancement in theoretical physics, providing a profound link between quantum gravity and gauge theories. The Maldacena limit is a particular low-energy limit that isolates

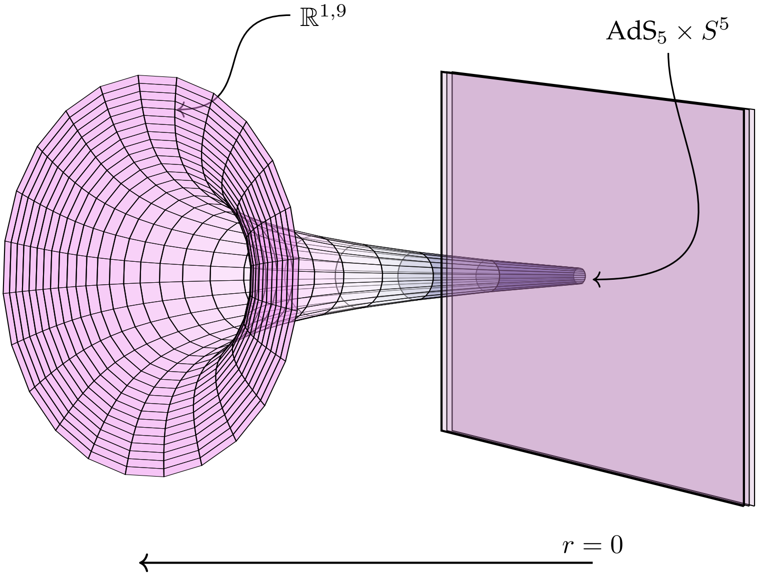


Figure 2.3: A graphical representation of how a stack of D3-branes backreacts on the target spacetime, resulting in a "deep throat" geometry. Inside the throat, the horizon radius is kept constant by the finite size of the S^5 (represented here by an S^1), leading to the $\mathrm{AdS}_5 \times S^5$ geometry. Away from the throat at asymptotic infinity the space is flat, and so we have a Minkowski $\mathbb{R}^{1,9}$ geometry.

the gauge theory dynamics from the bulk gravity effects. This limit involves keeping the string coupling g_s and the number of D3-branes N fixed while taking the string length scale α' to zero. In this limit, the effective 't Hooft coupling $\lambda = g_{YM}^2 N$ remains constant. The decoupling argument, altough abstract, is similar to the situation where an electron is interacting with a proton. Here we also have two distrinct descriptions: one by summing all perturbative contributions coming from the interaction between the two particles, and the other where the heavy proton is decoupled and contributes as a correction to the Coulomb potential in which the electron is moving.

As solution to the supergravity equations of motion, a stack of coincident N D3-branes is described by the following metric

$$\mathrm{d}s^2 = H(r)^{-\frac{1}{2}} \eta_{\mu\nu}\, \mathrm{d}x^\mu\, \mathrm{d}x^\nu + H(r)\big(\mathrm{d}r^2 + r^2\, \mathrm{d}\Omega_5^2\big), \tag{2.63}$$

where

$$H(r) = 1 + \frac{L^4}{r^4}, \qquad L^4 = 4\pi g_s N (\alpha')^2. \tag{2.64}$$

The coordinate r is the distance from the stack of branes and, far from them $r \gg L$, one can see that the geometry is the one of Minkowski space in ten dimensions. The other relevant limit, referred to as near-horizon, $r \ll L$, the geometry is described by the following metric

$$\mathrm{d}s^2 = \frac{r^2}{L^2} \eta_{\mu\nu}\, \mathrm{d}x^\mu\, \mathrm{d}x^\nu + \frac{L^2}{r^2}\, \mathrm{d}r^2 + L^2\, \mathrm{d}\Omega_5^2\,, \tag{2.65}$$

which corresponds the metric of a five-dimensional Anti-de Sitter space times a five sphere. This geometry is depicted in figure 2.3.
The AdS/CFT correspondence strongly depends on the geometric properties of AdS spaces. In the Minkowskian signature, AdS_{d+1} is represented as a hyperboloid in $\mathbb{R}^{2,d}$

$$-X_0^2 - X_{d+1}^2 + \sum_{i=1}^{d} X_i^2 = -L^2, \tag{2.66}$$

with the metric

$$\mathrm{d}s^2 = L^2 \left(-\cosh^2\rho\ \mathrm{d}\tau^2 + \mathrm{d}\rho^2 + \sinh^2\rho\ \mathrm{d}\Omega_{d-1}^2\right), \tag{2.67}$$

where τ is the time coordinate and ρ the radial coordinate. This geometry corresponds to the maximally symmetric solution of Einstein's field equations with negative cosmological constant.
In the Euclidean signature, the AdS space, also known as hyperbolic space H^{d+1}, is described by the equation

$$-X_0^2 + \sum_{i=1}^{d+1} X_i^2 = -L^2, \tag{2.68}$$

with the metric in Poincaré coordinates

$$\mathrm{d}s^2 = L^2 \left(\frac{\mathrm{d}z^2 + \mathrm{d}\vec{x}^2}{z^2}\right). \tag{2.69}$$

Here, z is the radial coordinate and $\mathrm{d}\vec{x}^2$ represents the Euclidean $\mathbb{R}^d$. The isometry group of Euclidean AdS is $\mathrm{SO}(1, \mathrm{d}+1)$, which is the conformal group in d-dimensions, providing a useful framework for defining boundary CFTs.
As we stated before, the core idea of the AdS/CFT correspondence lies in the duality between

- Type IIB Superstring Theory on $\mathrm{AdS}_5 \times S_5$: Including background 5-form flux

$$N = \frac{1}{2\pi} \int_{S^5} F_5^+. \tag{2.70}$$

- $\mathcal{N} = 4$ Super-Yang-Mills Theory: A four-dimensional SCFT with gauge group $\mathrm{SU}(N)$, where N is the same as the units of flux in (2.70).

The parameters in these theories are related by

$$2\pi g_s = g_{YM}^2, \qquad L^4 = 4\pi g_s N (\alpha')^2. \tag{2.71}$$

This duality suggests that the gravitational theory in the bulk $\mathrm{AdS}_5 \times S^5$ space can be mapped to a conformal field theory on its boundary, with precise correspondences between bulk fields and boundary operators. The AdS/CFT correspondence extends beyond the original $\mathrm{AdS}_5/\mathrm{CFT}_4$ framework to other dimensionalities and less symmetric setups. These generalizations broaden the correspondence's applicability, providing insights into various physical systems.

A key aspect of the AdS/CFT correspondence is the computation of correlation functions. The generating functional of CFT correlators is equivalent to the partition function of some quantum gravity theory with specified boundary conditions. Schematically, in string theory

$$\left\langle \exp\left(\int \mathrm{d}^4x\ \mathcal{O}(x)\phi_0(x)\right)\right\rangle_{\text{CFT}} = \mathcal{Z}_{\text{string}}[\phi(x) \to \phi_0(x)], \tag{2.72}$$

where $\mathcal{O}(x)$ is an operator in the CFT, and $\phi_0(x)$ is the boundary value of the bulk field ϕ. This equivalence allows the calculation of gauge theory observables via classical gravity computations. This is an instance of the holographic principle where bulk gravitational dynamics encode boundary quantum field theory information. Some observable quantities can be extracted from the dual supergravity by compactification of 10- or 11-dimensional supergravity. This is achieved geometrically by considering our $D = 10, 11$ dimensional supergravity on a product manifold of the form $\mathcal{M}^{(D)} = \mathcal{M}^{(d)} \times X^{(D-d)}$, where $X^{(D-d)}$ is a compact space. In the limit where the compact space is very small, this procedure defines a lower-dimensional supergravity. By doing so, it may happen that some degrees of freedom associated to the compact direction decouple. In this case, the dimensional reduction defines a *consistent truncation*, meaning that the physics of the higher-dimensional supergravity theory is described by a finite set of fields in the lower-dimensional supergravity. In Chapter 5 we are going to use one such consistent truncation of 11-dimensional supergravity to study the dual solutions to models compactified on a manifold of the form $\Sigma_1^{(2)} \times \Sigma_2^{(2)}$.

Another profound application of the AdS/CFT correspondence is in understanding the thermodynamics of Black Holes. The correspondence allows for the study of Hawking radiation and black hole entropy in a controlled manner. In particular, the Bekenstein-Hawking entropy of a BH, given by

$$S_{\text{BH}} = \frac{\text{Area}}{4G_N}, \tag{2.73}$$

corresponds to the entropy of the dual gauge theory at finite temperature. The counting of the microsciopical degrees of freedom of the Black Hole, contributing to the entropy, is achieved in the dual field theory by enumerating the states in the Hilbert space of the theory on $S^3 \times S^1$. As we will discuss in Chapter 6, this counting in achieved by considering RG-protected quantities known as indices, where the Black Hole behaviour can be extracted by considering certain high temperature limits. Additionally, the AdS/CFT correspondence has been instrumental in understanding the phase transitions of strongly coupled gauge theories. The dual description in terms of BH thermodynamics offers a geometric perspective on such phenomena, given in terms of the Hawking-Page phase transition from thermal AdS to a large BH.

Part II

CHAPTER 3

Supersymmetric Dualities with Four Supercharges

Contents

The low energy dynamics of UV free strongly coupled supersymmetric gauge theories can be often be simplified by the existence of infrared dualities. The dual descriptions are in general associated to (more) weakly coupled QFTs, described by a different set of fields and interactions that share in the IR the same correlation functions for the physically observable conserved currents of the original description. The prototypical example of such dualities is the electromagnetic duality and for this reason the two dual models are usually referred to as the electric and the magnetic phase.

Restricting to cases with four supercharges the basic example of these dualities was discovered by Seiberg in [289] for SU(N_c) 4d SQCD with $N_f > N_c+1$ flavors and vanishing superpotential, which was discussed in details in section 1.2.1. This duality has also a limiting case, where the magnetic description does not correspond to any gauge theory but to a WZ model consisting in a collection of mesons and baryons of the electric description, in addition to a (classical) constraint among them. In this case, corresponding to the choice $N_f = N_c + 1$, the electric gauge theory confines without breaking the chiral symmetry (i.e. s-confines [168]), and the magnetic theory describes the dynamics of the confined degrees of freedom, with a superpotential imposing the classical constraint on the moduli space. There is also another confining case, corresponding to SU(N_c) 4d SQCD with $N_f = N_c$ flavors, where the low energy dynamics described by the mesons and the baryons requires a quantum constraint on the moduli space. Such constraint breaks the chiral symmetry and for this reason this case is referred to as confinement

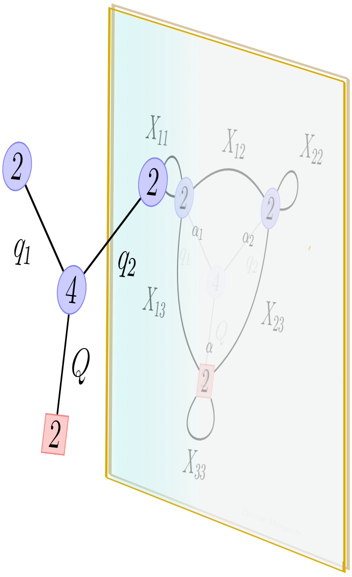


Figure 3.1: Artistic depiction of a ”Duality”.

with chiral symmetry breaking. In [36, 37] we conjectured new type of dualities in 3d $\mathcal{N} = 2$ theories, and in this Chapter the results of these works are given.
In [37] we generalized these constructions to the case of SQCD with two adjoints interacting through a D_{n+2}-type superpotential. The 4d duality was found in [118], and the 3d analogous constructions have been discussed in [275] for the case with non-vanishing CS and more recently in [230] for the case with vanishing CS level. This last case, obtained by dimensional reduction of the 4d duality of [118], through the reduction scheme of [15], reveals a novelty in the structure of the Coulomb branch of the 3d model, because of the presence of charge-two monopole operators in the superpotential of the magnetic phase.

The Chapter is organized as follows. We begin with a brief review of the main materials needed to understand the results of the papers, in particular we give an explanation of deconfinement as a limiting case of Seiber-like confining duality and then proceed with the review of known non-chiral dualities with two adjoint matter fields. Then, we proceed by giving the results of [37] and of [36].

3.1 Relevant Background

3.1.1 A Primer on Tensor Deconfinement

This idea of confinement as a limiting case of a supersymmetric duality was then extended to various generalizations of Seiberg duality. Furthermore a full classification of s-confining gauge theories with vanishing superpotential was worked out in [169] for theories with a single gauge group. In this classification there are many models that do not correspond to any limiting case of any known duality. Such models are characterized usually by the presence of matter fields in a rank-two tensor representation of the gauge group. Despite the fact that gauge theories of this type do not have in general a Seiberg-like dual description, it has been shown in [72] that the s-confining dualities can be derived using only Seiberg-(like) dualities thanks to the rank-2 tensor deconfining technique originally proposed in [93] and subsequently generalized in [262].

The technique consists of substituting a rank-2 tensor matter field with a bifundamental field charged also under another (auxiliary) confining gauge group, such to recover the original description once this new gauge group confines. After deconfining the rank-2 tensors it has been possible to apply sequences of Seiberg dualities (see [116] for a general construction) and than to recover the confined phase proposed in [169], using only the s-confining dualities of $\mathrm{SU}(N_c)$ with N_c+1 flavors of [289] and $\mathrm{USp}(2N_c)$ SQCD with $2N_c+4$ fundamentals of [237]. This construction may require further refinements for models with a superpotential deformation, due to the possible presence of an Higgsing that breaks partially or completely the gauge group (see [158] for a general discussion).

Recently new confining gauge theories have been obtained for 4d models with rank-2 tensors and non-vanishing superpotential [73]. Furthermore the deconfinement techniques have been applied to 3d $\mathcal{N}=2$ gauge theories [91, 279], where the zoo of confining gauge theories is richer, because of the presence of a dual photon and of a Coulomb branch. New confining dualities in this direction have been obtained in [43, 92].

3.1.2 The 3d Partition Functions

In this section we give some basic results on the three-sphere partition function useful for our analysis. The partition function of a 3d $\mathcal{N}=2$ SQFT, computed through localization techniques on the squashed three-sphere S_b^3 [212], corresponds to a matrix integral over a variable associated to the real scalar of the vector multiplet in the Cartan of the gauge group.

The general structure of the partition function consists of classical contributions from the Fayet-Iliopulos (FI) and the CS terms in the action, and contributions coming from the one-loop determinants for the chiral and vector multiplets. If we consider a 3d $\mathcal{N}=2$ supersymmetric gauge theory with gauge group G at CS level k, the S_b^3 partition function takes the following form

$$
\begin{aligned}
\mathcal{Z}_{G_k}(\mu_a;\lambda) = \frac{1}{|W|}\int \prod_{i=1}^{\operatorname{rank} G} \frac{\mathrm{d}\sigma_i}{\sqrt{-\omega_1\omega_2}} \exp\big(-\mathrm{i}\pi k\sigma_i^2 - \mathrm{i}\pi\lambda\sigma_i\big) \\
\times \prod_I \Gamma_h(\omega\Delta_I + \rho_I(\sigma) + \tilde{\rho}_I(\mu)) \cdot \prod_{\alpha\in G_+} \Gamma_h^{-1}(\pm\alpha(\sigma))
\end{aligned}
\tag{3.1}
$$

where μ_a are real parameters associated to the flavour symmetry, while $\tilde{\rho}(\mu)$ and $\rho(\sigma)$ are the weights of the flavour and gauge symmetry respectively. The α are the positive roots of the gauge symmetry and they parametrize the contributions from the one-loop determinant of the vector multiplet. The contribution of the FI, corresponding to the real mass for the topological symmetry $\mathrm{U}(1)_J$, is parameterized by λ. The R-charges of the chiral fields are parameterized by Δ_I. The gaussian factor corresponds to the CS level k. The normalization $|W|$ is the order of the Weyl group of G.

In our notation, the one-loop dererminants are given in terms of hyperbolic Gamma functions which can be written as the following infinite product

$$
\Gamma_h(x) = \mathrm{e}^{\frac{\mathrm{i}\pi}{2\omega_1\omega_2}\left((x-\omega)^2-\frac{\omega_1^2+\omega_2^2}{12}\right)} \prod_{j=0}^{\infty} \frac{1-\mathrm{e}^{\frac{2\pi\mathrm{i}}{\omega_1}(\omega_2-x)}\mathrm{e}^{\frac{2\pi\mathrm{i}\omega_2 j}{\omega_1}}}{1-\mathrm{e}^{-\frac{2\pi\mathrm{i}}{\omega_2}}\mathrm{e}^{-\frac{2\pi\mathrm{i}\omega_2 j}{\omega_2}}}
\tag{3.2}
$$

where $\omega_1 = ib$, $\omega_2 = i/b$ and b is the squashing parameter of the three-sphere S_b^3 which is defined by $b^2(x_1^2+x_2^2)+b^{-2}(x_3^2+x_4^2) = 1$; the ω parameter is defined as $2\omega = \omega_1+\omega_2$. We will often use the compound notation where

$$
\Gamma_h(x;y) \equiv \Gamma_h(x)\Gamma_h(y), \qquad \Gamma_h(\pm x) = \Gamma_h(x)\Gamma_h(-x).
\tag{3.3}
$$

The hyperbolic Gamma function obeys useful identities that are going to play an essential role in our analysis. The first is the inversion formula

$$
\Gamma_h(2\omega - x)\Gamma_h(x) = 1
\tag{3.4}
$$

which in field theory corresponds to integrating out fields appearing in the superpotential through holomorphic mass terms. The second one gives the asymptotic behavior of the hyperbolic Gamma function

$$
\lim_{|x|\to\infty} \Gamma_h(x) = \mathrm{e}^{-\frac{\mathrm{i}\pi}{2}\operatorname{sign}(x)(x-\omega)^2}
\tag{3.5}
$$

and it corresponds in field theory to integrating out a massive field with a large real mass term.

Focusing on the chiral models of our interest, the partition function of a $\mathrm{U}(N_c)_k$ theory with N_f fundamentals, N_a anti-fundamentals and two adjoints X, Y, at CS level k is

$$\begin{aligned}\mathcal{Z}^{(N_f,N_a)}_{\mathrm{U}(N_c)_k}(\vec{\mu};\vec{\nu};\tau_X;\tau_Y;\lambda) = &\frac{\Gamma_h(\tau_X)^{N_c}\Gamma_h(\tau_Y)^{N_c}}{N_c!\sqrt{-\omega_1\omega_2}^{N_c}}\int\prod_{i=1}^{N_c}\mathrm{d}\sigma_i\exp\left(-\mathrm{i}\pi\lambda\sigma_i-\mathrm{i}\pi k\sigma_i^2\right)\\ &\times\prod_{1\le i<j\le N_c}\prod_{\beta=X,Y}\frac{\Gamma_h\big(\tau_\beta\pm(\sigma_i-\sigma_j)\big)}{\Gamma_h\big(\pm(\sigma_i-\sigma_j)\big)}\\ &\times\prod_{i=1}^{N_c}\left(\prod_{a=1}^{N_f}\Gamma_h(\mu_a+\sigma_i)\cdot\prod_{b=1}^{N_a}\Gamma_h(\nu_b-\sigma_i)\right).\end{aligned}\tag{3.6}$$

The parameters μ_a, ν_a refer to the real masses of the fundamentals and anti-fundamentals while the $\tau_{X,Y}$ are the real masses of the adjoints.
The chiral $\mathrm{SU}(N_c)_k$ case can be recovered by the $\mathrm{U}(N_c)_k$ one by gauging the topological $\mathrm{U}(1)_J$ symmetry which at the level of the partition function amounts to adding a factor $\frac{1}{2}\mathrm{e}^{\mathrm{i}\pi\lambda N_c m_B}$ and integrating over λ, imposing the tracelessness condition on the adjoint fields [24, 91]

$$\begin{aligned}\mathcal{Z}^{(N_f,N_a)}_{\mathrm{SU}(N_c)_k}(\vec{\mu};\vec{\nu};\tau_X;\tau_Y) = &\frac{\Gamma_h(\tau_X)^{N_c-1}\Gamma_h(\tau_Y)^{N_c-1}}{N_c!\sqrt{-\omega_1\omega_2}^{N_c}}\int\prod_{i=1}^{N_c}\mathrm{d}\sigma_i\,\delta\left(\sum_{i=1}^{N_c}\sigma_i\right)\exp\left(-\mathrm{i}\pi k\sigma_i^2\right)\\ &\times\prod_{1\le i<j\le N_c}\prod_{\beta=X,Y}\frac{\Gamma_h\big(\tau_\beta\pm(\sigma_i-\sigma_j)\big)}{\Gamma_h\big(\pm(\sigma_i-\sigma_j)\big)}\\ &\times\prod_{i=1}^{N_c}\left(\prod_{a=1}^{N_f}\Gamma_h(\mu_a+m_B+\sigma_i)\cdot\prod_{b=1}^{N_a}\Gamma_h(\nu_b-m_B-\sigma_i)\right).\end{aligned}\tag{3.7}$$

3.1.3 Non-chiral 3d Dualities with Adjoint Matter

We start our analysis from the 3d $\mathcal{N}=2$ duality for $\mathrm{U}(N_c)_0$ SQCD with N_f pairs of fundamentals and anti–fundamentals and two adjoints interacting through a D_{n+2}-type superpotential. The duality has been obtained in [230], from the circle reduction of the 4d duality of [118], by following the prescription of [15]. In the 3d limit the duality is characterized by the unusual presence of superpotential interactions involving (dressed) monopole operators of charge two. The duality relates

- 3d $\mathcal{N}=2$ $\mathrm{U}(N_c)_0$ SQCD with N_f flavours $Q, \tilde{Q}$ with two adjoints fields X, Y and superpotential
$$\mathcal{W}_{\text{ele}} = \mathrm{Tr}\,X^{n+1} + \mathrm{Tr}\,XY^2 \tag{3.8}$$
with n odd.
- 3d $\mathcal{N}=2$ $\mathrm{U}(\tilde{N}_c)_0$ SQCD with $\tilde{N}_c = 3nN_f - N_c$, N_f dual flavours $q, \tilde{q}$ and two

Table 3.1: Matter content of electric (upper) and magnetic (lower) $\mathrm{U}(N_c)_0$ theories.

	Gauge		Global				
Field	$\mathrm{U}(N_c)$	$\mathrm{U}(\tilde{N}_c)$	$\mathrm{SU}(N_f)_L$	$\mathrm{SU}(N_f)_R$	$\mathrm{U}(1)_A$	$\mathrm{U}(1)_J$	$\mathrm{U}(1)_R$
Q	$\square$	1	$\square$	1	1	0	r_Q
$\tilde{Q}$	$\overline{\square}$	1	1	$\overline{\square}$	1	0	r_Q
X	Adj	1	1	1	0	0	$\frac{2}{n+1}$
Y	Adj	1	1	1	0	0	$\frac{n}{n+1}$
$V^{\pm}_{j\ell}$	1	1	1	1	$-N_f$	±1	$(1-r_Q)N_f + \frac{2j+n\ell-(N_c-1)}{n+1}$
$W^{\pm}_{q}$	1	1	1	1	$-2N_f$	±2	$2(1-r_Q)N_f + \frac{2+4q-2(N_c-1)}{n+1}$
q	1	$\overline{\square}$	1	$\overline{\square}$	-1	0	$\frac{2-n}{n+1} - r_Q$
$\tilde{q}$	1	$\square$	$\square$	1	-1	0	$\frac{2-n}{n+1} - r_Q$
x	1	Adj	1	1	0	0	$\frac{2}{n+1}$
y	1	Adj	1	1	0	0	$\frac{n}{n+1}$
$\mathcal{M}_{j\ell}$	1	1	$\square$	$\overline{\square}$	2	0	$2r_Q + \frac{2j+n\ell}{n+1}$
$\tilde{V}^{\pm}_{j\ell}$	1	1	1	1	N_f	∓1	$(r_Q-1)N_f + \frac{2j+n\ell+(N_c+1)}{n+1}$
$\tilde{W}^{\pm}_{q}$	1	1	1	1	$2N_f$	∓2	$2(r_Q-1)N_f + \frac{2+4q+2(N_c+1)}{n+1}$

adjoint fields x, y interacting through the superpotential

$$\begin{aligned} \mathcal{W}_{\text{mag}} &= \operatorname{Tr} x^{n+1} + \operatorname{Tr} xy^2 + \sum_{j=0}^{n-1}\sum_{\ell=0}^{2} \operatorname{Tr}\left(\mathcal{M}^{j,\ell} q x^{n-1-j} y^{2-\ell} \tilde{q}\right) \\ &+ \sum_{\substack{j=0,\dots,n-1 \\ \ell=0,1,2 \\ j\ell=0}} V^{\pm}_{j,\ell}\widetilde{V}^{\pm}_{n-j,2-\ell} + \sum_{q=0}^{\frac{n-3}{2}} W^{\pm}_{q}\widetilde{W}^{\pm}_{\frac{n-3}{2}-q}\,, \end{aligned} \tag{3.9}$$

where singlets $\mathcal{M}^{j,\ell}$ are dual to the dressed mesons of the electric theory $QX^jY^\ell\tilde{Q}$, the $V^{\pm}_{j,\ell}$ and $W^{\pm}_{q}$ are the monopole operators of the electric theory with topological charges ±1 and ±2 respectively, acting as singlets in the magnetic phase. Notice that the monopole operators $V^{\pm}_{j,\ell}$ enter in the superpotential with the condition $j\ell = 0$. This is going to be the case also for some of our dualities, where the condition is going to be understood and explicitly given only in the superpotential.

Observe that such monopole operators, defined through radial quantization from states on S^2 that carry a non–trivial magnetic flux background, are mapped to the states $\operatorname{Tr} X^jY^\ell|\pm 1, 0\dots, 0\rangle$ and $\operatorname{Tr} X^{2q}|\pm 1, \pm 1\dots, 0\rangle$ respectively.

The global symmetry is $\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R \times \mathrm{U}(1)_A \times \mathrm{U}(1)_J \times \mathrm{U}(1)_R$, where $\mathrm{U}(1)_A$ is the axial symmetry, $\mathrm{U}(1)_J$ is the topological symmetry and $\mathrm{U}(1)_R$ is the R-symmetry. The various fields transform as in table 3.1. The mapping of the chiral ring between the two phases is

$$\begin{array}{llcll}
\mathrm{Tr}\, X^j Y^\ell & & & \mathrm{Tr}\, x^j y^\ell & j=0,\dots,N_c-1,\ \ell=0,1,2 \\
Q X^j Y^\ell \tilde{Q} & & \Longleftrightarrow & \mathcal{M}^{j,\ell} & j=0,\dots,N_c-1,\ \ell=0,1,2 \\
V^\pm_{j,\ell} & & & V^\pm_{j,\ell} & j=0,\dots,N_c-1,\ \ell=0,1,2,\ j\cdot\ell=0 \\
W^\pm_q & & & W^\pm_q & q=0,\dots,\frac{N_c-3}{2}
\end{array} \tag{3.10}$$

where, with a slight abuse of notation, $V^\pm_{j,\ell}$ and $W^\pm_q$ are monopole operators in the elecrtic theory and singlets in the magnetic theory. On the other hand, the monopoles of the magnetic theory are set to zero on the chiral ring by the electric monopoles acting as singlets in this phase.

The 4d/3d reduction of the duality of [118] has been recently studied also in [44] by circle reduction of the conjectured identity between the 4d superconformal indices. The final result, once the divergent contributions between the electric and the magnetic side of the identity have been matched and canceled, corresponds to the identity that reproduces the 3d $\mathcal{N}=2$ duality of [230] on the squashed three sphere partition function. The identity is

$$\begin{aligned}
\mathcal{Z}^{N_f}_{\mathrm{U}(N_c)}(\mu_a;\nu_a;\tau_X;\tau_Y;\lambda) = {} & \mathcal{Z}^{N_f}_{\mathrm{U}(\tilde{N}_c)}(\tau_X-\tau_Y-\nu_a;\tau_X-\tau_Y-\mu_a;\tau_X;\tau_Y;-\lambda) \\
& \times \prod_{j=0}^{n-1}\prod_{\ell=0}^{2}\prod_{a,b=1}^{N_f}\Gamma_h(j\tau_X+\ell\tau_Y+\mu_a+\nu_b) \\
& \times \prod_{\substack{j=0,\dots,n-1\\ \ell=0,1,2\\ j\ell=0}}\Gamma_h\left(\pm\frac{\lambda}{2}+N_f\omega-\frac{N_c-1}{2}\tau_X-\frac{1}{2}\sum_{a=1}^{N_f}(\mu_a+\nu_a)+j\tau_X+\ell\tau_Y\right) \\
& \times \prod_{q=0}^{\frac{n-3}{2}}\Gamma_h\left(\pm\lambda+2N_f\omega+(N_c-1)\tau_X-\sum_{a=1}^{N_f}(\mu_a+\nu_a)+(2q+1)\tau_X\right).
\end{aligned} \tag{3.11}$$

The parameters associated to the N_f fundamentals and anti-fundamentals satisfy the constraint $\sum_{a=1}^{N_f}\mu_a=\sum_{a=1}^{N_f}\nu_a=N_f m_A$. The parameters associated to the adjoint are fixed as

$$\tau_X=\frac{2\omega}{n+1}, \qquad \tau_Y=\frac{n\omega}{n+1} \tag{3.12}$$

reflecting the constraints imposed by the superpotential (3.8).

3.1.4 Classification of Chiral Dualities

We conclude our review by surveying the case of chiral dualities for ordinary 3d $\mathcal{N}=2$ SQCD worked out in [85]. These dualities are characterized by a different number of fundamentals N_f and anti-fundamentals N_a, and by a CS level k. By comparing $\Delta F \equiv |N_f-N_a|$ and $2k$ three different cases have been identified.

For historical reasons the classification proposed in [85] for such dualities reflects the one worked out in the mathematical literature for the hyperbolic integral identities, corresponding to the matching of the three sphere partition functions (see for example [123]). In this case the chiral SQCD models are labelled by two non-negative integers $[\mathbf{p}, \mathbf{q}]$. These integral identities are related to the physics of CS theories with chiral matter.
The relation between the integers $[\mathbf{p}, \mathbf{q}]$ and the physical quantities can be made explicit by defining these integers in terms of the effective CS level of a $\mathrm{U}(N_c)_k$ theory

$$k_\pm = k \pm \frac{1}{2}(N_f - N_a). \tag{3.13}$$

According to the signs of $k_\pm$, there are four possible definitions

$$\begin{aligned} &[\mathbf{p}, \mathbf{q}]_a \equiv [-k_+, -k_-]_a, \qquad &&[\mathbf{p}, \mathbf{q}]_a^* \equiv [-k_+, k_-]_a^*, \\ &[\mathbf{p}, \mathbf{q}]_b \equiv [k_+, k_-]_b, \qquad &&[\mathbf{p}, \mathbf{q}]_b^* \equiv [k_+, -k_-]_b^* \end{aligned} \tag{3.14}$$

where the theory type a, b is chosen such that $\mathbf{p}, \mathbf{q} > 0$.
This means that for any choice of k, N_f and N_a one has to compute $k_\pm$ using (3.13) and then one has to select in (3.14) the one with both $\mathbf{p}$ and $\mathbf{q}$ positive. The flip of the sign of the CS term under duality imposes also that the dual of an a-theory is a b-theory and viceversa [85].
We survey the classification the dualities of [85] following this notation and based on the difference between ΔF and $2k$ (with $k > 0$, the case of $k < 0$ can be studied analogously). In each case the electric theories are $\mathrm{U}(N_c)_k$ SQCD with N_f fundamentals and N_a anti-fundamentals and vanishing superpotential. Depending on the value of $[\mathbf{p}, \mathbf{q}]$ one has

$$\begin{array}{lll} [\mathbf{p}, \mathbf{0}] & \Delta F = 2k & N_a < N_f, \\ [\mathbf{p}, \mathbf{q}] & \Delta F < 2k & N_a \neq N_f, \\ [\mathbf{p}, \mathbf{q}]^* & \Delta F > 2k & N_a \neq N_f. \end{array} \tag{3.15}$$

The gauge group of the dual magnetic chiral SQCD is

$$\begin{array}{ll} [\mathbf{p}, \mathbf{0}] & \mathrm{U}(N_f - N_c)_{-k} , \\ [\mathbf{p}, \mathbf{q}] & \mathrm{U}\Big(\frac{1}{2}(N_f + N_a) + |k| - N_c\Big)_{-k} , \\ [\mathbf{p}, \mathbf{q}]^* & \mathrm{U}\big(\max(N_a, N_f) - N_c\big)_{-k} \end{array} \tag{3.16}$$

with N_a fundamentals and N_f anti-fundamentals. In the last two cases the dual superpotential is given by

$$\mathcal{W}_{\text{mag}} = M q \tilde{q} \tag{3.17}$$

while in the $[\mathbf{p}, \mathbf{0}]$ case only there is an additional singlet T_+ in the magnetic phase and the superpotential is given by

$$\mathcal{W}_{\text{mag}} = M q \tilde{q} + T_+ t_- \tag{3.18}$$

where T_+ is dual to the electric monopole.
An analogous description holds for the SU(N_c) cases (see [11, 24]). Furthermore the discussion has been extended to the case of adjoint SQCD with A_n-type superpotential and unitary gauge group [24, 232]. In the following we will discuss the generalization to the case of two adjoint SQCD with D_{n+2}-type superpotential and unitary gauge group.

3.2 Chiral Dualities for SQCD$_3$ with D-type Superpotential

The classification program of 3d $\mathcal{N} = 2$ dualities is a fruitful field of research that boosted once localization techniques made powerful tools available [281]. Indeed after the discovery of mirror symmetry [110, 111, 238] and of Aharony duality [6] (see also [248]), it took the community a decade to have another class of examples of 3d $\mathcal{N} = 2$ dualities [197, 275]. These examples were derived from the type-IIB Hanany Witten (HW) setup [214], and they were motivated by the ABJ(M) results [8, 9].
The matrix integral for the 3d $\mathcal{N} = 2$ three sphere partition function, derived in full generality [211, 212, 240, 247], allowed to check the validity of these dualities and to define new ones, thanks to the possibility to engineer real mass and Higgs flows. Such flows are ubiquitous in the analysis of 3d SUSY gauge theories, and they usually give rise to a chiral like field content, for the case of SQCD with unitary gauge group, i.e. there is a different number of fundamentals N_f and on anti-fundamentals N_a. This difference requires a non zero Chern-Simons (CS) coupling for the invariance under large gauge transformations. Surprisingly the integral identities relating the three sphere partition functions of the new dualities [85] were already known to the mathematical community for U(N_c) SQCD. The classification of chiral dualities for 3d $\mathcal{N} = 2$ SQCD was then extended to the SU(N_c) case in [11].
A further goal has been to formulate new 3d $\mathcal{N} = 2$ dualities analogous to the the various generalization of 4d $\mathcal{N} = 1$ Seiberg-like dualities. The simplest extension, due to [258], regarded the case of adjoint SQCD with A_n type superpotential. Two dualities have been proposed, with [275] or without [253] CS action and with $N_f = N_a$ [1]. The chiral case was then partially studied in [232, 276] for the U(N_c) case, while a uniform treatment was then provided in [24] for both the U(N_c) and SU(N_c) case, generalizing the case without adjoints (corresponding to the case of A_1 type superpotential.)
Through a series of real mass and Higgs flows we generalize the web found in [85] and in [24] for the cases of SQCD and A_n adjoint SQCD respectively.
Furthermore we gauge the topological symmetry generalizing the construction to the SU(N_c) case as well. We corroborate the various steps of our derivation by reproducing the flow on the three sphere partition function, showing the cancellation of the divergent contributions, matching of the CS contact terms and proposing the new integral identities for the chiral dualities. We conclude our analysis by studying the case of two antisymmetric USp$(2N_c)_{2k}$ SQCD with D_{n+2}-type superpotential, previously uncovered in the literature.

---

[1] Actually more dualities have been obtained by adding monopole superpotentials [22], but here we will not discuss such possibility.

3.2.1 Dualities for U(N_c) Chiral SQCD with Two Adjoints

In this section, we study the chiral limit of the U$(N_c)_0$ duality studied in [230]. We will use the above-mentioned notation to differentiate the various theories with the addition of the subscript $[\mathbf{p},\mathbf{q}]_{X,Y}$ to underline the presence of two adjoints, similar to the notation of [24].

We introduce real mass flows on the electric side by turning on background fields for the flavour symmetry and giving large vacuum expectation values to the scalars in the vector multiplet of the flavour symmetry. This flow will lead in the IR to $[\mathbf{p},\mathbf{q}]_{X,Y}$ theories. Then we turn on background fields for the gauge symmetry on the magnetic side and consider large vacuum expectation values to the scalars.

This procedure is rephrased on the partition function (3.11) by considering consistent assignments of shifts on the parameters associated to the flavour and to the gauge symmetry. For large shifts the asymptotic of the integral identities gives new finite identities for the partition functions after factoring and canceling out the divergent contributions between the electric and magnetic phases.

3.2.1.1 The $[\mathbf{p},\mathbf{p}]_{X,Y}$ Case

The $[\mathbf{p},\mathbf{p}]_{X,Y}$ duality (corresponding to the one studied in [275]) is obtained from the $[\mathbf{0},\mathbf{0}]_{X,Y}$ duality with N_f+k flavours by assigning a positive large real mass to k of them. In the magnetic phase k dual quarks and anti-quarks are shifted with opposite signs while $2n(k^2+2N_fk)$ components of the (dressed) mesons acquire a large mass accordingly. In the IR, this will lead to the following duality:

- U$(N_c)_k$ SQCD with N_f fundamentals and anti-fundamentals $Q,\tilde{Q}$ and two adjoints X,Y interacting through the superpotential

$$\mathcal{W}_{\text{ele}} = \operatorname{Tr} X^{n+1} + \operatorname{Tr} XY^2. \tag{3.19}$$

- U$(\tilde{N}_c)_{-k}$ SQCD, with $\tilde{N}_c = 3n(N_f+|k|)-N_c$, N_f fundamentals and anti-fundamentals $q,\tilde{q}$ and two adjoints x,y, interacting through the superpotential

$$\mathcal{W}_{\text{mag}} = \operatorname{Tr} x^{n+1} + \operatorname{Tr} xy^2 + \sum_{j=0}^{n-1}\sum_{\ell=0}^{2} \operatorname{Tr}\left(\mathcal{M}^{j,\ell} q x^{n-1-\ell} y^{2-l} \tilde{q}\right). \tag{3.20}$$

The two theories acquire a CS level k and $-k$ respectively. The CS term lifts the Coulomb branch of the U(N_c) model. It reflects in the dual side to integrate out the singlets corresponding to the monopole of the electric phase.

To reproduce the duality on the partition function, we start from the equality (3.11) and consider the following shifts of the real masses

$$\begin{cases} m_A \to m_A + \frac{k}{N_f+k}s & \\ m_a \to m_a - \frac{k}{N_f+k}s & a = 1,\dots,N_f \\ m_a \to m_a + \frac{N_f}{N_f+k}s & a = 1,\dots,k \\ n_a \to n_a - \frac{k}{N_f+k}s & a = 1,\dots,N_f \\ n_a \to n_a + \frac{N_f}{N_f+k}s & a = 1,\dots,k \end{cases} \tag{3.21}$$

where we split the abelian axial part, m_A, of the real masses for the flavour symmetry from its non-abelian part m_a, n_a.

At this level, when the shift is finite, the equality (3.11) still holds. To reproduce the flow, we need study the large s limit on the partition functions by making use of the asymptotic behavior of the hyperbolic Gamma function (3.5). One needs to be careful when taking this limit since an infinite shift in the variables makes the integrals divergent. Therefore we need check that in the limit the leading saddle point contributions cancel between the electric and magnetic partition functions [32]. We are left then with the equality between

$$\mathcal{Z}_{\text{ele}} = \mathcal{Z}^{(N_f,N_f)}_{\text{U}(N_c)_k}(\mu_a;\nu_a;\tau_X;\tau_Y;\lambda) \tag{3.22}$$

and

$$\begin{aligned} \mathcal{Z}_{\text{mag}} &= \mathrm{e}^{\mathrm{i}\pi\phi}\mathrm{e}^{-\frac{3i\pi}{4}n\lambda^2}\mathcal{Z}^{(N_f,N_f)}_{\text{U}(\tilde{N}_c)_{-k}}(\tau_X-\tau_Y-\nu_a;\tau_X-\tau_Y-\mu_a;\tau_X;\tau_Y;-\lambda) \\ &\times \prod_{j=0}^{n-1}\prod_{\ell=0}^{2}\prod_{a,b=1}^{N_f}\Gamma_h(j\tau_X+\ell\tau_Y+\mu_a+\nu_b), \end{aligned} \tag{3.23}$$

where $\mu_a = m_a + m_A$ and $\nu_b = n_b + m_A$, satisfying the constraint $\sum_{a=1}^{N_f}\mu_a = \sum_{b=1}^{N_f}\nu_b = N_f m_A$.

There is a non-trivial complex exponential phase in the identity between $\mathcal{Z}_{\text{ele}}$ and $\mathcal{Z}_{\text{mag}}$. This phase is essential for matching the partition functions of the two dual theories. It was shown [150, 151] that the exponents are related to CS contact terms in two-point functions of global symmetry currents. The complex phase ϕ in this case has the following form

$$\begin{aligned} \phi &= 2N_f m_A\tau_Y(\tilde{N}_c - 2N_c + 3N_f + 3k(n-1)) - \frac{\tau_X^2}{8} \\ &- \frac{1}{4}\tau_X\tau_Y\Big((1+n+n^2) + 6N_f^2 + k^2 + 6N_c^2 - 4N_f(k+3N_c) + 4(N_c+\tilde{N}_c)(N_f+k) \\ &- (12k^2 + 12N_fN_c + 18kN_c)n + 6(N_f^2 + 4N_fk + 4k^2)n^2\Big) + 3nN_fm_A^2(k-N_f) \\ &+ \frac{3}{2}kn\sum_{a=1}^{N_f}(m_a^2+n_a^2). \end{aligned} \tag{3.24}$$

This phase can be reproduced from the computation of the contact terms by a linear combination of $\Delta k_{ij} \equiv k_{ij}^{\text{ele}} - k_{ij}^{\text{mag}}$, where the indices run over the abelian symmetries. Explicitly, for this flow, the non-zero contact terms from the field theory [33, 85] are given by

$$
\begin{aligned}
k_{AA}^{\text{ele}} &= 2 \times \frac{1}{2} N_c N_f , \\
k_{AA}^{\text{mag}} &= -2 \times \frac{1}{2} \tilde{N}_c N_f + 6nN_f^2 \equiv k_{AA}^{\text{ele}} - 3nN_f(k - N_f), \\
k_{rA}^{\text{ele}} &= 2 \times \frac{1}{2} N_c N_f (\Delta - 1), \\
k_{rA}^{\text{mag}} &= -2 \times \frac{1}{2} \tilde{N}_c N_f \left(\frac{2-n}{n+1} - \Delta - 1 \right) + \frac{1}{2} N_f^2 \sum_{\substack{j=0,\ldots,n-1 \\ \ell=0,1,2}} \left(2\Delta + \frac{2j + n\ell}{n+1} - 1 \right) \\
k_{rr}^{\text{ele}} &= 2 \times \frac{1}{2} N_c N_f (\Delta - 1)^2 + \frac{1}{2} N_c^2 \left(\frac{2}{n+1} - 1 \right)^2 + \frac{1}{2} N_c^2 \left(\frac{n}{n+1} - 1 \right)^2 - \frac{1}{2} N_c^2 \\
k_{rr}^{\text{mag}} &= -2 \times \frac{1}{2} \tilde{N}_c N_f \left(\frac{2-n}{n+1} - \Delta - 1 \right)^2 - \frac{1}{2} N_c^2 \left(\frac{2}{n+1} - 1 \right)^2 - \frac{1}{2} N_c^2 \left(\frac{n}{n+1} - 1 \right)^2 \\
&\quad + \frac{1}{2} N_f^2 \sum_{\substack{j=0,\ldots n-1 \\ \ell=0,1,2}} \left(2\Delta + \frac{2j + n\ell}{n+1} - 1 \right)^2 + \frac{1}{2} N_c^2
\end{aligned}
$$

from which we find

$$
\begin{aligned}
\Delta k_{AA} &= 3nN_f(k - N_f) m_A^2 \\
\Delta k_{rA} &= \frac{6nN_f}{1+n} (N_f - k - N_c + nN_f + 2nk) m_A \omega \\
\Delta k_{rr} &= -\frac{1}{2(n+1)^2} \Big((4n-1) \left(9k^2n^2 - 6knN_c + 2N_c^2 \right) \\
&\quad - 6nN_f \left(-k \left(8n^2 + n - 1 \right) + 3nN_c + N_c \right) + n(3n(6n+1) + 5) N_f^2 \Big)
\end{aligned}
\tag{3.25}
$$

where Δ is the R-charge. To reproduce (3.24) from (3.25) we can set $\Delta = 0$ and make explicit the real masses for the symmetries. This is done since in the partition function, the mass parameter is a combination of the various abelian and non-abelian charges. For the other dualities, the calculation is similar and we won't carry it out explicitly. Observe that in this case we have only a discrepancy in Δk_{rr} with respect to the result red from the exponent (3.24) in the partition function. This is nevertheless unphysical because it only acts as a pure phase in the identity between $\mathcal{Z}_{\text{ele}}$ and $\mathcal{Z}_{\text{mag}}$.

3.2.1.2 The $[\mathbf{p}, \mathbf{q}]_{X,Y}$ Case

The flow to the $[\mathbf{p}, \mathbf{q}]_{X,Y}$ duality is obtained starting from the $[\mathbf{0}, \mathbf{0}]_{X,Y}$ $\mathrm{U}(N_c)_0$ duality with N_f flavours by assigning a positive large real mass to $N_f - N_{f_1}$ fundamentals and $N_f - N_{f_2}$ anti-fundamentals. In the IR, this will lead the following duality:

- $\mathrm{U}(N_c)_k$ SQCD with N_{f_1} fundamentals and N_{f_2} anti-fundamentals $Q, \tilde{Q}$, two adjoints X and Y interacting through the superpotential

$$\mathcal{W}_{\text{ele}} = \operatorname{Tr} X^{n+1} + \operatorname{Tr} XY^2. \tag{3.26}$$

- $\mathrm{U}(\tilde{N}_c)_{-k}$ SQCD, with $\tilde{N}_c = 3nN_f - N_c$, N_{f_2} fundamentals q, N_{f_1} anti-fundamentals $\tilde{q}$ and two adjoint fields x, y, interacting through the superpotential

$$\mathcal{W}_{\text{mag}} = \operatorname{Tr} x^{n+1} + \operatorname{Tr} xy^2 + \sum_{j=0}^{n-1}\sum_{\ell=0}^{2} \operatorname{Tr}\left(\mathcal{M}^{j,\ell} q x^{n-1-j} y^{2-\ell} \tilde{q}\right). \tag{3.27}$$

The CS levels of the two phases are given by $k = N_f - \frac{1}{2}(N_{f_1} + N_{f_2})$ and $-k$ respectively. The Coulomb branch of the electric phase is lifted and in the dual phase the dressed electric monopoles acting as singlets are now massive and we integrated them out.
To reproduce the duality on the partition function, we start from the equality (3.11) and consider the following shifts in the real masses

$$\begin{cases} m_A \to m_A + \frac{2N_f - N_{f_1} - N_{f_2}}{2N_f} s & \\ m_a \to m_a - \frac{N_f - N_{f_1}}{N_f} s & a = 1, \dots, N_{f_1} \\ m_a \to m_a + \frac{N_{f_1}}{N_f} s & a = 1, \dots, N_f - N_{f_1} \\ n_a \to n_a - \frac{N_f - N_{f_2}}{N_f} s & a = 1, \dots, N_{f_2} \\ n_a \to n_a + \frac{N_{f_2}}{N_f} s & a = 1, \dots, N_f - N_{f_2} \\ \sigma_i \to \sigma_i - \frac{N_{f_1} - N_{f_2}}{2N_f} s & i = 1, \dots, N_c \\ \tilde{\sigma}_i \to \tilde{\sigma}_i - \frac{N_{f_1} - N_{f_2}}{2N_f} s & i = 1, \dots, 3nN_f - N_c \\ \lambda \to \lambda + (N_{f_2} - N_{f_1}) s & \end{cases} \tag{3.28}$$

We study the limit of large s as stated before, checking that the divergent contributions cancel between the electric and magnetic phases. We are left with the equality between

$$\mathcal{Z}_{\text{ele}} = \mathcal{Z}_{\mathrm{U}(N_c)_k}^{(N_{f_1}, N_{f_2})}\left(\mu_a; \nu_b; \tau_X; \tau_Y; \hat{\lambda}\right) \tag{3.29}$$

where

$$\hat{\lambda} = \lambda + (N_{f_1} - N_{f_2})(m_A - \omega), \tag{3.30}$$

and

$$\begin{aligned} \mathcal{Z}_{\text{mag}} = {} & e^{i\pi\phi} e^{-\frac{3i\pi}{4} n\lambda^2} \mathcal{Z}_{\mathrm{U}(\tilde{N}_c)_{-k}}^{(N_{f_2}, N_{f_1})}(\tau_X - \tau_Y - \nu_a; \tau_X - \tau_Y - \mu_a; \tau_X; \tau_Y; \tilde{\lambda}) \\ & \times \prod_{j=0}^{n-1}\prod_{\ell=0}^{2}\prod_{a=1}^{N_{f_1}}\prod_{b=1}^{N_{f_2}} \Gamma_h(j\tau_X + \ell\tau_Y + \mu_a + \nu_b) \end{aligned} \tag{3.31}$$

where

$$\tilde{\lambda} = -\lambda - (N_{f_1} - N_{f_2})(m_A - \tau_X + \tau_Y + \omega). \tag{3.32}$$

We set $\mu_a = m_a + m_A$ and $\nu_b = n_b + m_A$, satisfying the constraint $\sum_{a=1}^{N_f} \mu_a = \sum_{b=1}^{N_f} \nu_b = N_f m_A$.
The complex exponent ϕ, necessary for the equality between the partition functions, has the following form

$$\begin{aligned}
\phi = & -\frac{3}{2} m_A \tau_Y \Big((1+n)(N_{f_1}+N_{f_2})(2k+N_{f_1}+N_{f_2}) - 2\tilde{N}_c(N_{f_1}+N_{f_2}) + 4N_{f_1}N_{f_2}(n-2) \Big) \\
& - \frac{\tau_X^2}{8} - \frac{\tau_X \tau_Y}{4} \Big((1+n+n^2) + 11N_{f_1}N_{f_2} + 6n(n-2)N_{f_1}N_{f_2} - 6N_c\tilde{N}_c \\
& + \frac{1}{4}(1+24n^2)(2k+N_{f_1}+N_{f_2})^2 - 3(N_c - \tilde{N}_c(1-n))(N_{f_1}+N_{f_2}) \\
& - \frac{3}{2}(2k+N_{f_1}+N_{f_2})(N_{f_1}+N_{f_2})(1-n+n^2) \Big) \\
& + \frac{3}{4} n m_A^2 \big((N_{f_1}+N_{f_2})(2k+N_{f_1}+N_{f_2}) - 8N_{f_1}N_{f_2} \big) \\
& + \frac{3}{4} n \left((2k+N_{f_1}-N_{f_2}) \sum_{a=1}^{N_{f_1}} m_a^2 + (2k-N_{f_1}+N_{f_2}) \sum_{a=1}^{N_{f_2}} n_a^2 \right).
\end{aligned} \tag{3.33}$$

Again the phase in (3.33) can be reproduced from the difference between the contact terms for the global abelian symmetries of the electric and the magnetic theories. We observe here the same unphysical mismatch in Δk_{rr} discussed in the $[\mathbf{p},\mathbf{p}]_{X,Y}$ case.

3.2.1.3 The $[\mathbf{p},\mathbf{0}]_{X,Y}$ Case

The flow to the $[\mathbf{p},\mathbf{0}]_{X,Y}$ theory, we start from the $[\mathbf{0},\mathbf{0}]_{X,Y}$ $\mathrm{U}(N_c)_0$ duality with N_f flavours, and give a positive large real mass to $N_f - N_{f_1}$ fundamentals. In the IR, this will lead the following duality:

- $\mathrm{U}(N_c)_k$ theory with N_{f_1} fundamentals and N_f anti-fundamentals $Q, \tilde{Q}$, two adjoint X and Y interacting through the superpotential

$$\mathcal{W}_{\text{ele}} = \mathrm{Tr}\, X^{n+1} + \mathrm{Tr}\, XY^2. \tag{3.34}$$

- $\mathrm{U}(\tilde{N}_c)_{-k}$, where $\tilde{N}_c = 3nN_f - N_c$, with N_f fundamentals and N_{f_1} anti-fundamentals $q, \tilde{q}$, two adjoint fields x, y interacting through the superpotential

$$\begin{aligned}
\mathcal{W}_{\text{mag}} = & \mathrm{Tr}\, x^{n+1} + \mathrm{Tr}\, xy^2 + \sum_{j=0}^{n-1} \sum_{\ell=0}^{2} \mathrm{Tr}\left(\mathcal{M}^{j,\ell} q x^{n-1-j} y^{2-\ell} \tilde{q} \right) \\
& + \sum_{\substack{j=0,\dots,n-1 \\ \ell=0,1,2 \\ j\ell=0}} V_{j,\ell}^{+} \widetilde{V}_{n-j,2-\ell}^{+} + \sum_{q=0}^{\frac{n-3}{2}} W_q^{+} \widetilde{W}_{\frac{n-3}{2}-q}^{+}.
\end{aligned} \tag{3.35}$$

The CS levels of the two phases are given by $k = \frac{1}{2}(N_f - N_{f_1})$ and $-k$ respectively. Half of the Coulomb branch is left in this case and it reflects in the presence of the singlets $V_{j,\ell}^{+}$

and W_q^+ in the spectrum of the dual model.

To reproduce this duality on the partition function, we start from the equality (3.11) and consider the following shifts for the real masses

$$\begin{cases} m_A \to m_A + \frac{N_f - N_{f_1}}{2N_f} s & \\ m_a \to m_a - \frac{N_f - N_{f_1}}{N_f} s & a = 1, \dots, N_{f_1} \\ m_a \to m_a + \frac{N_{f_1}}{N_f} s & a = 1, \dots, N_f - N_{f_1} \\ \sigma_i \to \sigma_i + \frac{N_f - N_{f_1}}{2N_f} s & i = 1, \dots, N_c \\ \tilde{\sigma}_i \to \tilde{\sigma}_i + \frac{N_f - N_{f_1}}{2N_f} s & i = 1, \dots, 3nN_f - N_c \\ \lambda \to \lambda + (N_f - N_{f_1}) s & \end{cases} \tag{3.36}$$

where we split the axial abelian part, m_A, for the flavour symmetry from its non-abelian part m_a, n_a.

We study the large s limit as stated before, checking that the leading saddle point contributions cancel between the electric and magnetic partition functions, and we are left with the equality between

$$\mathcal{Z}_{\text{ele}} = \mathcal{Z}_{\text{U}(N_c)_k}^{(N_{f_1}, N_f)}(\mu_a; \nu_a; \tau_X; \tau_Y; \hat{\lambda}) \tag{3.37}$$

where

$$\hat{\lambda} = \lambda + (N_f - N_{f_1})(m_A - \omega) \tag{3.38}$$

and

$$\begin{aligned} \mathcal{Z}_{\text{mag}} = {} & \mathrm{e}^{\mathrm{i}\pi\phi} \mathrm{e}^{-\frac{1}{4}\lambda(\frac{3}{2}n\lambda - \eta)} \mathcal{Z}_{\text{U}(\tilde{N}_c)_{-k}}^{(N_f, N_{f_1})}(\tau_X - \tau_Y - \nu_a; \tau_X - \tau_Y - \mu_a; \tau_X; \tau_Y; \tilde{\lambda}) \\ & \times \prod_{j=0}^{n-1} \prod_{\ell=0}^{2} \prod_{a=1}^{N_{f_1}} \prod_{b=1}^{N_f} \Gamma_h(j\tau_X + \ell\tau_Y + \mu_a + \nu_b) \\ & \times \prod_{\substack{j=0,\dots n-1 \\ \ell=0,1,2 \\ j\ell=0}} \Gamma_h\left(\frac{\lambda}{2} + N_f\omega - \frac{N_c - 1}{2}\tau_X - \frac{1}{2}\sum_{a=1}^{N_f}(m_a + n_a) + j\tau_X + \ell\tau_Y\right) \\ & \times \prod_{q=0}^{\frac{n-3}{2}} \Gamma_h\left(\lambda + 2N_f\omega - (N_c - 1)\tau_X - \sum_{a=1}^{N_f}(m_a + n_a) + (2q+1)\tau_A\right) \end{aligned} \tag{3.39}$$

where $\mu_a = m_a + m_A$ and $\nu_b = n_b + m_A$, which solve the constrain $\sum_{a=1}^{N_f} \mu_a = \sum_{b=1}^{N_f} \nu_b = N_f m_A$, and

$$\tilde{\lambda} = -\lambda + (N_f - N_{f_1})(m_A - \tau_X + \tau_Y + \omega), \tag{3.40}$$

$$\eta = \tau_X + 6\tau_Y - 2\omega + n\Big(6(2k + N_{f_1})(\omega - m_A) + \tau_X(2n - 3N_c) - 4\omega\Big). \tag{3.41}$$

The complex exponent ϕ necessary for the equality between the partition functions to hold has the following form

$$\begin{aligned}\phi =\,& 3m_A\tau_Y((2k+N_{f_1})^2(n-2)+3N_{f_1}(2k+N_{f_1})-N_cN_{f_1})\\ &-\frac{\tau_X^2}{16}-\frac{\tau_X\tau_Y}{8}\Big((1+n+n^2)+6N_c^2+2(\tilde{N}_c+N_c)^2-20k(2k+N_{f_1})\\ &+6N_cN_{f_1}(n-2)+6N_{f_1}(2k+N_{f_1})(1-2n^2)+6n(2k+N_{f_1})(4k+2N_{f_1}-3N_c)\Big)\\ &+\frac{3}{2}m_A^2n(8k^2+2kN_{f_1}-N_{f_1}^2)+3kn\sum_{a=1}^{N_f}n_a^s\,.\end{aligned} \tag{3.42}$$

Again the phase in (3.33) can be reproduced from the difference between the contact terms for the global abelian symmetries of the electric and the magnetic theories. We observe here the same unphysical mismatch in Δk_{rr} discussed in the $[\mathbf{p},\mathbf{p}]_{X,Y}$ and in the $[\mathbf{p},\mathbf{q}]_{X,Y}$ case.

3.2.1.4 The $[\mathbf{p},\mathbf{q}]^*_{X,Y}$ Case

The flow to the $[\mathbf{p},\mathbf{q}]^*_{X,Y}$ theory, we start from the $[\mathbf{0},\mathbf{0}]_{X,Y}$ $\mathrm{U}(N_c)_0$ duality with N_f flavours and give a positive large real mass to N_{f_1} anti-fundamentals and a negative large real mass to N_{f_2} anti-fundamentals. In the IR, this will lead the following duality:

- $\mathrm{U}(N_c)_k$ theory with N_f fundamentals and $N_a=N_f-N_{f_1}-N_{f_2}$ anti-fundamentals $Q,\tilde{Q}$, two adjoint X and Y interacting through the superpotential

$$\mathcal{W}_{\text{ele}}=\mathrm{Tr}\,X^{n+1}+\mathrm{Tr}\,XY^2. \tag{3.43}$$

- $\mathrm{U}(\tilde{N}_c)_{-k}$, where $\tilde{N}_c=3nN_f-N_c$, with N_a fundamentals and N_f anti-fundamentals $q,\tilde{q}$, two adjoint fields x,y interacting through the superpotential

$$\mathcal{W}_{\text{mag}}=\mathrm{Tr}\,x^{n+1}+\mathrm{Tr}\,xy^2+\sum_{j=0}^{n-1}\sum_{\ell=0}^{2}\mathrm{Tr}\left(\mathcal{M}^{j,\ell}qx^{n-1-j}y^{2-\ell}\tilde{q}\right). \tag{3.44}$$

The CS levels of the two phases are given by $k=\frac{1}{2}(N_{f_1}-N_{f_2})$ and $-k$ respectively. The CB is lifted and the monopoles acting as singlets in the magnetic theory are integrated out.

To reproduce the duality on the partition function, we start from equality (3.11) and consider the following shifts in the real masses

$$\begin{cases} m_A \to m_A + \frac{N_{f_1}-N_{f_1}}{2N_f} s & \\ n_a \to n_a - \frac{N_{f_1}-N_{f_2}}{N_f} s & a = 1, \dots, N_f - N_{f_1} - N_{f_2} \\ n_a \to n_a + \frac{N_f - N_{f_1}+N_{f_2}}{N_f} s & a = 1, \dots, N_{f_1} \\ n_a \to n_a - \frac{N_f + N_{f_1}-N_{f_2}}{N_f} s & a = 1, \dots, N_{f_2} \\ \sigma_i \to \sigma_i + \frac{N_{f_2}-N_{f_1}}{2N_f} s & i = 1, \dots, N_c \\ \tilde{\sigma}_i \to \tilde{\sigma}_i + \frac{N_{f_2}-N_{f_1}}{2N_f} s & i = 1, \dots, 3nN_f - N_c \\ \lambda \to \lambda - (N_{f_1} + N_{f_2}) s & \end{cases} \tag{3.45}$$

where we split the abelian axial part m_A of the real masses for the flavour symmetry from its non abelian part m_a, n_a.
We study the large s limit by making use of the asymptotic behavior of the hyperbolic Gamma function (3.5). We check that the leading saddle point contributions cancel between the electric and magnetic partition functions, and we are left with the equality between

$$\mathcal{Z}_{\text{ele}} = \mathcal{Z}_{\text{U}(N_c)_k}^{(N_f, N_a)}(\mu_a; \nu_a; \tau_X; \tau_Y; \hat{\lambda}) \tag{3.46}$$

where

$$\hat{\lambda} = \lambda + (N_{f_1} - N_{f_2})(m_A - \omega), \tag{3.47}$$

and

$$\mathcal{Z}_{\text{mag}} = \mathrm{e}^{\mathrm{i}\pi\phi} \mathrm{e}^{\mathrm{i}\pi\lambda\left(m_A(N_c + \tilde{N}_c) - 3\tau_Y(N_f(n+1) - N_c)\right)} \mathcal{Z}_{\text{U}(\tilde{N}_c)_{-k}}^{(N_f, N_{f_1})}(\tau_X - \tau_Y - \nu_a; \tau_X - \tau_Y - \mu_a; \tilde{\lambda}) \\ \times \prod_{j=0}^{n-1} \prod_{\ell=0}^{2} \prod_{a=1}^{N_f} \prod_{b=1}^{N_a} \Gamma_h(j\tau_X + \ell\tau_Y + \mu_a + \nu_b) \tag{3.48}$$

where

$$\tilde{\lambda} = -\lambda - (N_{f_1} - N_{f_2})(m_A - \tau_X + \tau_Y + \omega), \tag{3.49}$$

with $\mu_a = m_a + m_A$ and $\nu_b = n_b + m_A$ solving the constrain $\sum_{a=1}^{N_f} \mu_a = \sum_{b=1}^{N_f} \nu_b = N_f m_A$.
The complex exponent ϕ necessary for the equality between the partition functions to hold has the following form

$$\phi = (N_{f_1} - N_{f_2})\Big(3m_A\tau_Y(N_c - 3N_f) + \frac{\tau_X\tau_Y}{4}(3N_c(n-2) + N_f(8 - 6n^2)) \\ + \frac{9}{2}m_A^2 n N_f + \frac{3}{2}n\sum_{a=1}^{N_f} m_a^2\Big). \tag{3.50}$$

In this case the CS contact terms can be computed using the reduction of the $[\mathbf{p}, \mathbf{q}]^*_{X,Y}$ duality to the $[\mathbf{0}, \mathbf{0}]_{X,Y}$ one. This requires a Higgsing in the magnetic phase and we obtain the same results discussed above. We match all the contributions except the one of the Δk_{rr}, but this mismatch is unphysical because it involves a pure phase.

3.2.2 Dualities for SU(N_c) Chiral SQCD with Two Adjoints

As discussed in section 3.1.2, one can start from the $[\mathbf{0},\mathbf{0}]$ $\mathrm{U}(N_c)_0$ duality and obtain a duality for $\mathrm{SU}(N_c)_0$ by gauging the topological $\mathrm{U}(1)_J$ symmetry [15]. This is achieved by introducing a dynamical background multiplet for the topological symmetry. This procedure introduces a mixed CS term in the action

$$\mathcal{L} \supset A^{\mathrm{U}(1)} \wedge \mathrm{dTr}\, A^{\mathrm{U}(N_c)} \tag{3.51}$$

at level -1 between the new $\mathrm{U}(1)$ symmetry coming from the topological $\mathrm{U}(1)_J$ and the abelian subgroup of the gauge symmetry $\mathrm{U}(1) \subset \mathrm{U}(N_c)$. In addition a new topological $\mathrm{U}(1)_{J'}$ is generated from the hodge dual of the gauged $\mathrm{U}(1)_J$ field strength which is conserved by virtue of the Bianchi identity. In the absence of monopoles, which are charged under the gauged topological symmetry, the mixed CS term makes the two $\mathrm{U}(1)$ photons massive and can be integrated out in the IR. In this case, the gauge group becomes $\mathrm{SU}(N_c)_0$ and the topological $\mathrm{U}(1)_{J'}$ can be considered as the baryonic symmetry $\mathrm{U}(1)_B$ under which the flavour has canonically normalized charge $1/N_c$.
In presence of fields charged under $\mathrm{U}(1)_J$ the analysis has to be modified. It is the case for example of many of the dual phases, where the electric monopoles are singlets in the dual description. It follows that we cannot decouple the dynamics of the gauged $\mathrm{U}(1)_J$ and of the $\mathrm{U}(1) \subset \mathrm{U}(\tilde{N}_c)$ symmetries. In this case, the magnetic side will be a $\mathrm{U}(\tilde{N}_c)_0 \times \mathrm{U}(1)$ gauge theory with a level -1 mixed CS term. In some case some further local duality simplifies such sectors. For example when considering Aharony duality one can use the SQED/XYZ duality.
Another common feature of the gauging of the topological symmetry leading from $\mathrm{U}(N_c)$ to $\mathrm{SU}(N_c)$ in presence of adjoint matter, consists of imposing the tracelessness condition [24, 91] on the adjoints. This can be achieved in two ways: one can either add a flipping term in the superpotential on the magnetic side

$$\mathcal{W}_{\text{flip}} = \alpha_0 \,\mathrm{Tr}\, x + \beta_0 \,\mathrm{Tr}\, y, \tag{3.52}$$

which imposes then the tracelessness of the adjoints by the F-term equations for the singlets α_0 and β_0, or consider the traceless adjoint representation of $\mathrm{U}(\tilde{N}_c)$.
The procedure just described gives the following duality

- 3d $\mathcal{N}=2$ $\mathrm{SU}(N_c)_0$ SQCD with N_f flavours $Q, \tilde{Q}$ and two adjoint fields X, Y, with superpotential
$$\mathcal{W}_{\text{ele}} = \mathrm{Tr}\, X^{n+1} + \mathrm{Tr}\, XY^2. \tag{3.53}$$
- 3d $\mathcal{N}=2$ $\mathrm{U}(\tilde{N}_c)_0 \times \mathrm{U}(1)$ SQCD with $\tilde{N}_c = 3nN_f - N_c$, N_f dual flavours $q, \tilde{q}$ in the non-abelian sector and two adjoints X, Y, $n+2$ pairs of fields $V_{j,\ell}^{\pm}$ and $\frac{1}{2}(n-1)$ pairs of fields $W_q^{\pm}$ in the abelian gauge sector with opposite gauge charge. These fields interact with the dressed monopoles and anti-monopoles of the $\mathrm{U}(\tilde{N}_c)$ sector, that

Table 3.2: Matter content of the electric (upper) and magnetic (lower) theories after gauging the topological symmetry. The subscript shows the charge of the fields under the gauged $\mathrm{U}(1)_J$.

	Gauge	Global				
Field	$\mathrm{SU}(N_c)_0$	$\mathrm{SU}(N_f)_L$	$\mathrm{SU}(N_f)_R$	$\mathrm{U}(1)_A$	$\mathrm{U}(1)_B$	$\mathrm{U}(1)_R$
Q	$\square$	$\square$	1	1	1	r_Q
$\tilde{Q}$	$\overline{\square}$	1	$\overline{\square}$	1	-1	r_Q
X	Adj	1	1	0	0	$\frac{2}{n+1}$
Y	Adj	1	1	0	0	$\frac{n}{n+1}$

Field	$\mathrm{U}(\tilde{N}_c)_0\times\mathrm{U}(1)$	$\mathrm{SU}(N_f)_L$	$\mathrm{SU}(N_f)_R$	$\mathrm{U}(1)_A$	$\mathrm{U}(1)_B$	$\mathrm{U}(1)_R$
q	$\overline{\square}_0$	1	$\overline{\square}$	-1	0	$\frac{2-n}{n+1} - r_Q$
$\tilde{q}$	$\square_0$	$\square$	1	-1	0	$\frac{2-n}{n+1} - r_Q$
x	Adj_0	1	1	0	0	$\frac{2}{n+1}$
y	Adj_0	1	1	0	0	$\frac{n}{n+1}$
$\mathcal{M}_{j\ell}$	1_0	$\square$	$\overline{\square}$	2	0	$2r_Q + \frac{2j+n\ell}{n+1}$
$V^{\pm}_{j\ell}$	$1_{\pm1}$	1	1	$-N_f$	$\pm N_c$	$(1-r_Q)N_f + \frac{2j+n\ell-(N_c-1)}{n+1}$
$W^{\pm}_q$	$1_{\pm2}$	1	1	$-2N_f$	$\pm N_c$	$2(1-r_Q)N_f + \frac{2+4q-2(N_c-1)}{n+1}$

in this case carry ±1 charge under the new $\mathrm{U}(1)$ gauged factor as well. There is also a level -1 CS level between the abelian $\mathrm{U}(1)$ subgroup in the $\mathrm{U}(\tilde{N}_c)$ and the other $\mathrm{U}(1)$ gauge factor. The superpotential is given by

$$\begin{aligned}\mathcal{W}_{\text{mag}} &= \mathrm{Tr}\,x^{n+1} + \mathrm{Tr}\,xy^2 + \alpha_0\,\mathrm{Tr}\,x + \beta_0\,\mathrm{Tr}\,y + \mathrm{Tr}\sum_{j=0}^{n-1}\sum_{\ell=0}^{2}\mathrm{Tr}\left(\mathcal{M}^{j,\ell}qx^{n-1-j}y^{2-\ell}\tilde{q}\right)\\ &\quad + \sum_{\substack{j=0,\dots,n-1\\ \ell=0,1,2\\ j\ell=0}} V^{\pm}_{j,\ell}\widetilde{V}^{\pm}_{n-j,2-\ell} + \sum_{q=0}^{\frac{n-3}{2}} W^{\pm}_q\widetilde{W}^{\pm}_{\frac{n-3}{2}-q}\,. \end{aligned} \tag{3.54}$$

The non-anomalous global symmetry of the theories is $\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R \times \mathrm{U}(1)_A \times \mathrm{U}(1)_B \times \mathrm{U}(1)_R$ under which the fields are charged as in table 3.2.

At the level of the partition function, the gauging procedure, is implemented by adding a factor $\frac{1}{2}\mathrm{e}^{\mathrm{i}\pi\lambda N_c m_B}$ to both side of the identity (3.11) and then integrating over λ. The integral over the FI corresponds to the gauging of the $\mathrm{U}(1)_J$ while the added exponential factor carries the additional baryonic symmetry, whose real mass we label as m_B. The numerical factor $1/2$ is added to ensure the proper normalization of the $V^{\pm}_{j,\ell}$ and $W^{\pm}_q$ under the gauged $\mathrm{U}(1)_J$. On the electric side, the only dependence on the FI is in the exponential term that can be integrated upon shifting the Cartan variables by m_B. This will lead to flavour matter fields carrying baryonic charge. On the magnetic side, the fields $V^{\pm}_{j,\ell}$ and $W^{\pm}_q$ are charged under the topological $\mathrm{U}(1)_J$ and therefore the integration is not straightforward, i.e. a local mirror duality should be necessary to get rid of this sector. For our purpose we will leave the integration on λ explicit on both sides whilst shifting the Cartan on the electric side, making explicit the baryonic symmetry of the matter. This will lead us to the identity between the electric partition function

$$
\begin{aligned}
\mathcal{Z}_{\text{ele}} = {} & \frac{\Gamma_h(\tau_X)^{N_c-1}\Gamma_h(\tau_Y)^{N_c-1}}{N_c!\sqrt{-\omega_1\omega_2}^{N_c}} \int \mathrm{d}\xi \int \prod_{i=1}^{N_c} \mathrm{d}\sigma_i \exp(-2\mathrm{i}\pi\xi\sigma_i) \\
& \times \prod_{i=1}^{N_c}\prod_{a=1}^{N_f} \Gamma_h(\mu_a+m_B+\sigma_i;\nu_a-m_B-\sigma_i) \prod_{1\le i<j\le N_c}\prod_{\beta=X,Y} \frac{\Gamma_h(\tau_\beta\pm(\sigma_i-\sigma_j))}{\Gamma_h(\pm(\sigma_i-\sigma_j))}
\end{aligned}
\tag{3.55}
$$

and the magnetic partition function

$$
\begin{aligned}
\mathcal{Z}_{\text{mag}} = {} & \frac{\Gamma_h(\tau_X)^{\tilde{N}_c-1}\Gamma_h(\tau_Y)^{\tilde{N}_c-1}}{\tilde{N}_c!\sqrt{-\omega_1\omega_2}^{\tilde{N}_c}} \prod_{j=0}^{n-1}\prod_{\ell=0}^{2}\prod_{a,b=1}^{N_f} \Gamma_h(j\tau_X+\ell\tau_Y+\mu_a+\nu_b) \\
& \times \int \mathrm{d}\xi \int \prod_{i=1}^{\tilde{N}_c} \mathrm{d}\sigma_i \exp\left(2\mathrm{i}\pi\xi\left(\sigma_i+\frac{N_c}{\tilde{N}_c}m_B\right)\right) \\
& \times \prod_{a=1}^{N_f} \Gamma_h(\tau_X-\tau_Y-\nu_a+\sigma_i;\tau_X-\tau_Y-\mu_a-\sigma_i) \prod_{1\le i<j\le \tilde{N}_c}\prod_{\beta=X,Y} \frac{\Gamma_h(\tau_\beta\pm(\sigma_i-\sigma_j))}{\Gamma_h(\pm(\sigma_i-\sigma_j))} \\
& \times \prod_{\substack{j=0,\dots,n-1\\ \ell=0,1,2\\ j\ell=0}} \Gamma_h\left(\pm\xi+N_f\omega-\frac{N_c-1}{2}\tau_X-\frac{1}{2}\sum_{a=1}^{N_f}(\mu_a+\nu_a)+j\tau_X+\ell\tau_Y\right) \\
& \times \prod_{q=0}^{\frac{n-3}{2}} \Gamma_h\left(\pm 2\xi+2N_f\omega+(N_c-1)\tau_X-\sum_{a=1}^{N_f}(\mu_a+\nu_a)+(2q+1)\tau_X\right)
\end{aligned}
\tag{3.56}
$$

where we rescaled the FI as $\lambda = 2\xi$ [2]. The addition of the flipping terms in the magnetic superpotential is reflected on the partition function by an additional $\Gamma_h(2\omega-\tau_X;2\omega-\tau_Y)$ factor. Using the inversion formula (3.4), these factors cancel out with a factor of $\Gamma_h(\tau_X)$ and $\Gamma_h(\tau_Y)$.

3.2.2.1 The $[\mathbf{p},\mathbf{p}]_{X,Y}$ Case

To flow to the $[\mathbf{p},\mathbf{p}]_{X,Y}$ theory, we start from the $[\mathbf{0},\mathbf{0}]_{X,Y}$ $\mathrm{SU}(N_c)_0$ duality with N_f+k fundamentals and anti-fundamentals and give a positive large, but finite, real mass to k flavours. In the IR, this will lead to the following duality

- $\mathrm{SU}(N_c)_k$ SQCD with N_f fundamentals and anti-fundamentals $Q,\tilde{Q}$, two adjoints X,Y interacting through the superpotential

$$
\mathcal{W}_{\text{ele}} = \mathrm{Tr}\,X^{n+1} + \mathrm{Tr}\,XY^2. \tag{3.57}
$$

---

[2] With this normalization, the exponential term for the FI has an added factor 2 which will be implicit.

- $\mathrm{U}(\tilde{N}_c)_{-k} \times \mathrm{U}(1)_{3n}$ SQCD with $\tilde{N}_c = 3n(N_f + |k|) - N_c$, a level -1 mixed CS, N_f dual fundamentals and anti-fundamentals $q, \tilde{q}$, two traceless adjoint fields x, y interacting through the superpotential

$$\mathcal{W}_{\text{mag}} = \mathrm{Tr}\, x^{n+1} + \mathrm{Tr}\, xy^2 + \sum_{j=0}^{n-1} \sum_{\ell=0}^{2} \mathrm{Tr}\left(\mathcal{M}^{j,\ell} q x^{n-1-j} y^{2-\ell} \tilde{q}\right). \tag{3.58}$$

The fields $V^{\pm}_{j,\ell}$ and $W^{\pm}_{q}$ in the dual phase are massive and they are integrated out. The electric and the magnetic theories acquire a CS level k and $-k$ respectively. From here on we omit in the magnetic phase the singlets needed to make the adjoint traceless since we can integrate them out in the IR.
At this stage, one could also decouple the dynamics of the massive photons in the two gauge U(1) in the magnetic side, because we do not have matter charged under $\mathrm{U}(1)_J$.
On the partition function, the real mass flow that produces the above-mentioned duality, is given by the assignment of real masses of (3.21). The real mass associated to the baryonic $\mathrm{U}(1)_B$ symmetry does not get shifted.
We study the limit of large s on both the electric (3.55) and magnetic (3.56) partition functions by making use of the asymptotic behavior of the hyperbolic Gamma function (3.5). We check that the leading saddle point contributions cancel between the electric and magnetic phases, and we are left with the equality between

$$\mathcal{Z}_{\text{ele}} = \mathcal{Z}^{(N_f,N_f)}_{\mathrm{SU}(N_c)_k}(\mu_a; \nu_a; \tau_X; \tau_Y) \tag{3.59}$$

in which we have no FI since after the flow we integrate on ξ, and

$$\begin{aligned}\mathcal{Z}_{\text{mag}} = &\prod_{j=0}^{n-1} \prod_{\ell=0}^{2} \prod_{a,b=1}^{N_f} \Gamma_h(j\tau_X + \ell\tau_Y + \mu_a + \nu_b) \\ &\times e^{i\pi\phi} \int d\xi\, e^{i\pi(2m_B N_c \xi - 3n\xi^2)} \mathcal{Z}^{(N_f,N_f)}_{\mathrm{U}(\tilde{N}_c)_{-k}}(\tau_X - \tau_Y - \nu_a; \tau_X - \tau_Y - \mu_a; \tau_X; \tau_Y; -\xi).\end{aligned} \tag{3.60}$$

From (3.60) we can see the level $3n$ CS of the U(1) gauge factor and the -1 mixed CS, in the term $m_B\xi$, between the abelian subgroup of the $\mathrm{U}(\tilde{N}_c)$ and the abelian U(1) gauge group. Again the real masses satisfy the constrain $\sum_{a=1}^{N_f} \mu_a = \sum_{a=1}^{N_f} \nu_a = N_f m_A$ where $\mu_a = m_a + m_A$ and $\nu_a = n_a + m_A$.

The complex exponent ϕ needed for the matching is given by

$$\begin{aligned}\phi = {} & 3m_A^2 n N_f (k - N_f) - k m_B^2 N_c + \frac{\tau_X}{2}\omega m_A \Big(3n(n+1)(2N_f^2 + k^2 - (2n^2+1)(N_f+k) \\ & - 3N_c n(N_f - k) + kN_f(6n(n-1) + 2N_c)\Big) - \frac{\tau_X^2}{8}\Big(1 + N_c(2k+1) - 2(N_f+k)\Big) \\ & - \frac{\tau_X \tau_Y}{8}\Big((2n^2 + 8n - 1) - 4(N_f+k)(1 + 2n + 2n^2) + 12(1 + 2n + n^2)N_f^2 \\ & + (54n^2 + 33n - 1)k^2 + 2(30n^2 + 33n - 1)N_f k - 12(n+1)N_f N_c - 4(6n+1)kN_c \\ & + N_c(3N_c + 4n)\Big) + \frac{3}{2}kn\sum_{a=1}^{N_f}\big(m_a^2 + n_a^2\big).\end{aligned} \tag{3.61}$$

3.2.2.2 The $[\mathbf{p},\mathbf{q}]_{X,Y}$ Case

To flow to the $[\mathbf{p},\mathbf{q}]_{X,Y}$ theory, we start from the $[\mathbf{0},\mathbf{0}]_{X,Y}$ SU$(N_c)_0$ duality with N_f fundamentals and anti-fundamentals and give a positive large real mass to $N_f - N_{f_1}$ fundamentals and $N_f - N_{f_2}$ anti-fundamentals. This will lead to the following duality

- SU$(N_c)_k$ SQCD with N_{f_1} fundamentals and N_{f_2} anti-fundamentals $Q, \tilde{Q}$, two adjoints X, Y interacting through the superpotential

$$\mathcal{W}_{\text{ele}} = \operatorname{Tr} X^{n+1} + \operatorname{Tr} XY^2. \tag{3.62}$$

- U$(\tilde{N}_c)_{-k} \times$ U$(1)_{3n}$ SQCD with $\tilde{N}_c = 3nN_f - N_c$, a level -1 mixed CS between the two U(1)s, N_{f_2} dual fundamentals and N_{f_1} dual anti-fundamentals $q, \tilde{q}$, two traceless adjoint fields x, y interacting through the superpotential

$$\mathcal{W}_{\text{mag}} = \operatorname{Tr} x^{n+1} + \operatorname{Tr} xy^2 + \sum_{j=0}^{n-1}\sum_{\ell=0}^{2} \operatorname{Tr}\big(\mathcal{M}^{j,\ell} q x^{n-1-j} y^{2-\ell}\tilde{q}\big). \tag{3.63}$$

The fields $V_{j,\ell}^{\pm}$ and $W_q^{\pm}$ in the dual phase are massive and they are integrated out. The electric and the magnetic theories acquire a CS level $k = N_f - \frac{1}{2}(N_{f_1} + N_{f_2})$ and $-k$ respectively.

To reproduce this duality at the level of the partition function, we start from the equality between (3.55) and (3.56) and consider the following shifts in the real masses

$$\begin{cases} m_A \to m_A + \frac{2N_f - N_{f_1} - N_{f_2}}{2N_f} s & \\ m_B \to m_B - \frac{N_{f_1} - N_{f_2}}{2N_f} s & \\ m_a \to m_a - \frac{N_f - N_{f_1}}{N_f} s & a = 1, \dots, N_{f_1} \\ m_a \to m_a + \frac{N_{f_1}}{N_f} s & a = 1, \dots, N_f - N_{f_1} \\ n_a \to n_a - \frac{N_f - N_{f_2}}{N_f} s & a = 1, \dots, N_{f_2} \\ n_a \to n_a + \frac{N_{f_2}}{N_f} s & a = 1, \dots, N_f - N_{f_2} \\ \tilde{\sigma}_i \to \tilde{\sigma}_i - \frac{N_{f_1} - N_{f_2}}{2N_f} s & i = 1, \dots, 3nN_f - N_c \\ \xi \to \xi - \frac{N_{f_1} - N_{f_2}}{2} s & \end{cases} \tag{3.64}$$

We study the limit of large s on both the electric (3.55) and magnetic (3.56) partition functions by making use of the asymptotic behavior of the hyperbolic Gamma function (3.5). We check that the leading saddle point contributions cancel between the electric and magnetic phases, and we are left with the equality between

$$\mathcal{Z}_{\text{ele}} = \mathcal{Z}^{(N_{f_1}, N_{f_2})}_{\text{SU}(N_c)_k}(\mu_a; \nu_a; \tau_X; \tau_Y) \tag{3.65}$$

in which we have no FI since after the flow we integrate on ξ, and

$$\mathcal{Z}_{\text{mag}} = \prod_{j=0}^{n-1} \prod_{\ell=0}^{2} \prod_{a=1}^{N_{f_1}} \prod_{b=1}^{N_{f_2}} \Gamma_h(j\tau_X + \ell\tau_Y + \mu_a + \nu_b) \\ \times e^{i\pi\phi} \int d\xi \, e^{i\pi(2m_B N_c \xi - 3n\xi^2)} \mathcal{Z}^{(N_{f_1}, N_{f_2})}_{\text{U}(\tilde{N}_c)_{-k}}(\tau_X - \tau_Y - \nu_a; \tau_X - \tau_Y - \mu_a; \tau_X; \tau_Y; \hat{\xi}) \tag{3.66}$$

where

$$\hat{\xi} = \xi - \frac{N_{f_1} - N_{f_2}}{2}(m_A - \tau_X + \tau_Y + \omega). \tag{3.67}$$

From (3.66) we can see the level $3n$ CS of the $\text{U}(1)$ gauge factor and the -1 mixed CS, in the term $m_B \xi$, between the abelian subgroup of the $\text{U}(\tilde{N}_c)$ and the abelian $\text{U}(1)$ gauge group. Again the real masses satisfy the constrain $\sum_{a=1}^{N_f} \mu_a = \sum_{a=1}^{N_f} \nu_a = N_f m_A$ where $\mu_a = m_a + m_A$ and $\nu_a = n_a + m_A$.

The complex exponent ϕ needed for the matching is given by

$$\begin{aligned}\phi &= 3m_A^2 n\Bigl(N_f^2 - N_f(4N_{f_2}+k) + 2N_{f_2}(N_{f_2}+2k))\Bigr) - m_B^2 k N_c \\ &+ \frac{\tau_X}{2} m_A\Bigl(2nN_{f_2}^2(n-3) + 3nN_f^2(3n-1) + 2k(N_c(1+3n) + 3nN_f(n-3)) \\ &+ N_f(2+2n^2+12kn^2+3nN_c+6nN_{f_2}(n-3))\Bigr) + 2m_B\omega N_c(N_{f_2}-N_f+k) \\ &- \frac{\tau_X^2}{8}\Bigl(1+2N_f+N_c(1+2k)\Bigr) - \frac{\tau_X\tau_Y}{8}\Bigl(-1+8n+2n^2+N_f^2(-1+33n+6n^2) \\ &+ 2N_{f_2}k(-13+9n-6n^2) + N_{f_2}(-13+9n-6n^2) + 2N_f\bigl((-2+13N_{f_2}-6N_c) \\ &- n(4+9N_{f_2}+6N_c) + n^2(-4+6N_{f_2}+24k)\bigr) + N_c(4n+k(8-12n)) \\ &+ 3N_c^2\Bigr) + \frac{3}{2}n\left((N_f-N_{f_2})\sum_{a=1}^{N_{f_1}}\mu_a^2 + (N_f-N_{f_1})\sum_{a=1}^{N_{f_2}}\nu_a^2\right).\end{aligned} \tag{3.68}$$

3.2.2.3 The $[\mathbf{p},\mathbf{0}]_{X,Y}$ Case

To flow to the $[\mathbf{p},\mathbf{0}]_{X,Y}$ theory, we start from the $[\mathbf{0},\mathbf{0}]_{X,Y}$ $\mathrm{SU}(N_c)_0$ duality with N_f flavours and give a positive large real mass to $N_f - N_{f_1}$ fundamentals. This will lead to the following duality

- $\mathrm{SU}(N_c)_k$ SQCD with N_{f_1} fundamentals and N_f anti-fundamentals $Q, \tilde{Q}$, two adjoints X and Y interacting through the superpotential

$$\mathcal{W}_{\text{ele}} = \operatorname{Tr} X^{n+1} + \operatorname{Tr} XY^2. \tag{3.69}$$

- $\mathrm{U}(\tilde{N}_c)_{-k} \times \mathrm{U}(1)_{\frac{3}{2}n}$ SQCD with $\tilde{N}_c = 3nN_f - N_c$, a level -1 mixed CS between the two $\mathrm{U}(1)s$, N_f dual fundamentals and N_{f_1} dual anti-fundamentals $q, \tilde{q}$, two traceless adjoints x, y interacting through the superpotential

$$\begin{aligned}\mathcal{W}_{\text{mag}} = \operatorname{Tr} x^{n+1} + \operatorname{Tr} xy^2 + \sum_{j=0}^{n-1}\sum_{\ell=0}^{2} \operatorname{Tr}\left(\mathcal{M}^{j,\ell} q x^{n-1-j} y^{2-\ell} \tilde{q}\right) \\ + \sum_{\substack{j=0,\dots,n-1 \\ \ell=0,1,2 \\ j\ell=0}} V_{j,\ell}^{+} \widetilde{V}_{n-j,2-\ell}^{+} + \sum_{q=0}^{\frac{n-3}{2}} W_q^{+} \widetilde{W}_{\frac{n-3}{2}-q}^{+}\,.\end{aligned} \tag{3.70}$$

The fields $V_{j,\ell}^{\pm}$ and $W_q^{\pm}$ in the dual phase are massive and they are integrated out. The electric and the magnetic theories acquire a CS level $k = \frac{1}{2}(N_f - N_{f_1})$ and $-k$ respectively. To reproduce this duality on the partition function, we start from the equality between

(3.55) and (3.56) and consider the following shifts of the real masses

$$\begin{cases} m_A \to m_A + \frac{N_f - N_{f_1}}{2N_f} s & \\ m_B \to m_B + \frac{N_f - N_{f_1}}{2N_f} s & \\ m_a \to m_a - \frac{N_f - N_{f_1}}{N_f} s & a = 1, \dots, N_{f_1} \\ m_a \to m_a + \frac{N_{f_1}}{N_f} s & a = 1, \dots, N_f - N_{f_1} \\ \tilde{\sigma}_i \to \tilde{\sigma}_i + \frac{N_f - N_{f_1}}{2N_f} s & i = 1, \dots, 3nN_f - N_c \\ \xi \to \xi + \frac{N_f - N_{f_1}}{2} s & \end{cases} \tag{3.71}$$

We study the limit of large s on both the electric (3.55) and magnetic (3.56) partition functions by making use of the asymptotic behavior of the hyperbolic Gamma function (3.5). We check that the leading saddle point contributions cancel between the electric and magnetic phases, and we are left with the equality between

$$\mathcal{Z}_{\text{ele}} = \mathcal{Z}^{(N_{f_1}, N_f)}_{\text{SU}(N_c)_k}(\mu_a; \nu_a; \tau_X; \tau_Y) \tag{3.72}$$

in which we have no FI since after the flow we integrate on ξ, and

$$\begin{aligned} \mathcal{Z}_{\text{mag}} = & \prod_{j=0}^{n-1} \prod_{\ell=0}^{2} \prod_{a=1}^{N_{f_1}} \prod_{b=1}^{N_f} \Gamma_h(j\tau_X + \ell\tau_Y + \mu_a + \nu_b) \\ & \times e^{\mathrm{i}\pi\phi} \int d\xi \, e^{i\frac{\pi}{2}\xi(\eta - 3n\xi)} \mathcal{Z}^{(N_f, N_{f_1})}_{\text{U}(\tilde{N}_c)_{-k}}(\tau_X - \tau_Y - \nu_a; \tau_X - \tau_Y - \mu_a; \tau_X; \tau_Y; \hat{\xi}) \\ & \times \prod_{q=0}^{\frac{n-3}{2}} \Gamma_h\left(2\xi + 2N_f\omega - (N_c - 1)\tau_X - \sum_{a=1}^{N_f}(m_a + n_a) + (2q+1)\tau_A\right) \\ & \times \prod_{\substack{j=0,\dots n-1 \\ \ell=0,1,2 \\ j\ell=0}} \Gamma_h\left(\xi + N_f\omega - \frac{N_c - 1}{2}\tau_X - \frac{1}{2}\sum_{a=1}^{N_f}(m_a + n_a) + j\tau_X + \ell\tau_Y\right) \end{aligned} \tag{3.73}$$

where

$$\begin{aligned} \hat{\xi} &= -\xi + \frac{N_f - N_{f_1}}{2}(m_A - \tau_X + \tau_Y + \omega), \\ \eta &= 4m_B N_c - 6nm_A N_f + (1 - 3nN_c + 2n^2)\tau_X + 6\tau_Y - 2\omega(1 + 2n - 3nN_f). \end{aligned} \tag{3.74}$$

From (3.73) we can see the level $\frac{3}{2}n$ CS of the $\text{U}(1)$ gauge factor and the -1 mixed CS, in the term $m_B\xi$, between the abelian subgroup of the $\text{U}(\tilde{N}_c)$ and the abelian $\text{U}(1)$ gauge group. Again the real masses satisfy the constrain $\sum_{a=1}^{N_f} \mu_a = \sum_{a=1}^{N_f} \nu_a = N_f m_A$ where $\mu_a = m_a + m_A$ and $\nu_a = n_a + m_A$.

The complex exponent ϕ needed for the matching is given by

$$\begin{aligned}\phi = &-\frac{3}{2}m_A^2 n N_f(N_f-6k) - kN_c m_B^2 + \frac{\tau_X}{4}m_A\Big(6nN_f^2(n+1)+4kN_c(1+3n)\\ &+N_f\left(1+2n^2+3nN_c+12kn(3+n)\right)\Big) + 2kN_c m_B\omega - \frac{\tau_X^2}{16}\Big(1-2N_f+N_c(1+4k)\Big)\\ &-\frac{\tau_X\tau_Y}{16}\Big((-1+8n+2n^2)+3N_c^2+12N_f^2(1+n)^2+4N_f\big(-(1+2n+2n^2)\\ &-3N_c(1+n)+(-13+9n+18n^2)k\big)+N_c\big(4n+k(16-24n)\big)\Big)+3kn\sum_{a=1}^{N_f}n_a^2\,.\end{aligned} \tag{3.75}$$

3.2.2.4 The $[\mathbf{p},\mathbf{q}]^*_{X,Y}$ Case

To flow to the $[\mathbf{p},\mathbf{q}]^*_{X,Y}$ theory, we start from the $[\mathbf{0},\mathbf{0}]_{X,Y}$ SU$(N_c)_0$ duality with N_f flavours and give a positive large real mass to N_{f_1} anti-fundamentals and a negative large real mass to N_{f_2} anti-fundamentals. This will lead to the following duality

- SU$(N_c)_k$ SQCD with N_f fundamentals and $N_a = N_f - N_{f_1} - N_{f_2}$ anti-fundamentals $Q, \tilde{Q}$, two adjoints X and Y interacting through the superpotential

$$\mathcal{W}_{\text{ele}} = \text{Tr}\,X^{n+1} + \text{Tr}\,XY^2. \tag{3.76}$$

- U$(\tilde{N}_c)_{-k}\times$U(1) SQCD with $\tilde{N}_c = 3nN_f - N_c$, a level -1 mixed CS between the two U(1)s, N_a dual fundamentals and N_f dual anti-fundamentals $q, \tilde{q}$, two traceless adjoints x, y interacting through the superpotential

$$\mathcal{W}_{\text{mag}} = \text{Tr}\,x^{n+1} + \text{Tr}\,xy^2 + \sum_{j=0}^{n-1}\sum_{\ell=0}^{2}\text{Tr}\left(\mathcal{M}^{j,\ell}qx^{n-1-j}y^{2-\ell}\tilde{q}\right). \tag{3.77}$$

The fields $V^{\pm}_{j,\ell}$ and $W^{\pm}_q$ in the dual phase are massive and they are integrated out. The electric and the magnetic theories acquire a CS level $k = \frac{1}{2}(N_{f_1} - N_{f_2})$ and $-k$ respectively.

To reproduce this duality on the partition function we start from the equality between (3.55) and (3.56) and consider the following shifts of the real masses

$$\begin{cases} m_A \to m_A + \frac{N_{f_1}-N_{f_2}}{2N_f}s & \\ m_B \to m_B - \frac{N_{f_1}-N_{f_2}}{2N_f}s & \\ n_a \to n_a - \frac{N_{f_1}-N_{f_2}}{N_f}s & a = 1,\ldots,N_f - N_{f_1} - N_{f_2} \\ n_a \to n_a + \frac{N_f-N_{f_1}+N_{f_2}}{N_f}s & a = 1,\ldots,N_{f_1} \\ n_a \to n_a - \frac{N_f-N_{f_1}+N_{f_2}}{N_f}s & a = 1,\ldots,N_{f_2} \\ \tilde{\sigma}_i \to \tilde{\sigma}_i + \frac{N_{f_1}-N_{f_2}}{2N_f}s & i = 1,\ldots,3nN_f - N_c \\ \xi \to \xi - \frac{N_{f_1}+N_{f_2}}{2}s & \end{cases} \tag{3.78}$$

We study the limit of large s on both the electric (3.55) and magnetic (3.56) partition functions by making use of the asymptotic behavior of the hyperbolic Gamma function (3.5). We check that the leading saddle point contributions cancel between the electric and magnetic phases, and we are left with the equality between

$$\mathcal{Z}_{\text{ele}} = \mathcal{Z}^{(N_f,N_a)}_{\text{SU}(N_c)_k}(\mu_a;\nu_a;\tau_X;\tau_Y) \tag{3.79}$$

in which we have no FI since after the flow we integrate on ξ, and

$$\begin{aligned}\mathcal{Z}_{\text{mag}} = &\prod_{j=0}^{n-1}\prod_{\ell=0}^{2}\prod_{a=1}^{N_f}\prod_{b=1}^{N_a}\Gamma_h(j\tau_X+\ell\tau_Y+\mu_a+\nu_b)\\ &\times \mathrm{e}^{\mathrm{i}\pi\phi}\int \mathrm{d}\xi\, \mathrm{e}^{\mathrm{i}\pi\eta\xi}\mathcal{Z}^{(N_a,N_f)}_{\text{U}(\tilde{N}_c)_{-k}}(\tau_X-\tau_Y-\nu_a;\tau_X-\tau_Y-\mu_a;\tau_X;\tau_Y;-\hat{\xi})\end{aligned} \tag{3.80}$$

where

$$\begin{aligned}\hat{\xi} &= \xi+\frac{N_{f_1}-N_{f_2}}{2}(m_A-\tau_X+\tau_Y+\omega),\\ \eta &= 2m_BN_c+6nm_AN_f-(1-3nN_c+2n^2)\tau_X-6\tau_Y+2\omega(1+2n-3nN_f).\end{aligned} \tag{3.81}$$

From (3.73) we can see the level -1 mixed CS, in the term $m_B\xi$, between the abelian subgroup of the $\text{U}(\tilde{N}_c)$ and the abelian $\text{U}(1)$ gauge group. Observe that in this case, differently from the $[\mathbf{p},\mathbf{q}]$ cases studied above, the CS level associated to the gauged topological symmetry is vanishing. This is because the competition between the shift that gives rise to the non-trivial vacuum of the $\text{U}(1)_J$ sector and the shift associated to the axial symmetry give opposite signs to the divergent masses of the charged fields, i.e. the fields $V^{\pm}_{j,\ell}$ and $W^{\pm}_q$. This implies that in this case the duality is preserved with a flat direction in the Coulomb branch of the gauged $\text{U}(1)_J$ symmetry. Again the real masses satisfy the constrain $\sum_{a=1}^{N_f}\mu_a = \sum_{a=1}^{N_f}\nu_a = N_fm_A$ where $\mu_a = m_a+m_A$ and $\nu_a = n_a+m_A$.

The complex exponent ϕ needed for the matching is given by

$$\begin{aligned}\phi = &\,9nN_fkm_A^2-kN_cm_B^2+\frac{\tau_X}{2}(N_{f_1}-N_{f_2})\Big(N_c(1+3n)-3nN_f(n+3)\Big)m_A\\ &+N_c(N_{f_2}-N_{f_1})m_B\omega+\frac{\tau_X^2}{2}\Big(nN_f(13-9n-18n^2)+N_c(-1-4n+6n^2)\Big)\\ &+\frac{3}{2}n(N_{f_1}-N_{f_2})\sum_{a=1}^{N_f}m_a^2.\end{aligned} \tag{3.82}$$

3.2.3 Dualities for USp$(2N_c)$ Chiral SQCD with Two Anti-Symmetric

In this section we start by setting up the notation for $\text{USp}(2N_c)$ theories and their dualities. The partition function of a CS-theory with $\text{USp}(2N_c)_k$ gauge group can be found starting from (3.1). In particular, for the case of our interest, with $2N_f$ fundamentals and two anti-symmetric rank two tensors A, B, the partition function is given by

$$\mathcal{Z}^{2N_f}_{\mathrm{USp}(2N_c)_{2k}}(\vec{\mu};\tau_A;\tau_B) = \frac{\Gamma_h(\tau_A)^{N_c}\Gamma_h(\tau_B)^{N_c}}{2^{N_c}N_c!\sqrt{-\omega_1\omega_2}^{N_c}} \int \prod_{i=1}^{N_c} \mathrm{d}\sigma_i \exp\left(-\mathrm{i}\pi k \sigma_i^2\right) \frac{\prod_{a=1}^{2N_f}\Gamma_h(\mu_a \pm \sigma_i)}{\Gamma_h(\pm 2\sigma_i)} \\ \times \prod_{1\le i<j\le N_c} \prod_{\alpha=A,B} \frac{\Gamma_h(\tau_\alpha \pm \sigma_i \pm \sigma_j)}{\Gamma_h(\pm\sigma_i \pm \sigma_j)}. \tag{3.83}$$

The 3d duality for the USp$(2N_c)_0$ case was worked out in [44] and relates

- 3d $\mathcal{N}=2$ USp$(2N_c)_0$ SCQD with $2N_f$ flavours Q and two anti-symmetric rank-two tensors A, B interacting through the superpotential

$$\mathcal{W}_{\text{ele}} = \operatorname{Tr} A^{n+1} + \operatorname{Tr} AB^2. \tag{3.84}$$

- 3d $\mathcal{N}=2$ USp$(2\tilde{N}_c)_0$ SQCD where $\tilde{N}_c = 3nN_f - N_c - 2n - 1$, with $2N_f$ dual flavours q, two anti-symmetric rank-two tensors a, b interacting through the superpotential

$$\mathcal{W}_{\text{mag}} = \operatorname{Tr} a^{n+1} + \operatorname{Tr} ab^2 + \sum_{j=0}^{n-1}\sum_{\ell=0}^{2} \mathcal{M}_{j,\ell}\, q a^j b^\ell q \\ + \sum_{\substack{j=0,\dots,n-1\\ \ell=0,1,2\\ j\ell=0}} Y_{j,\ell}\widetilde{Y}_{n-j,2-\ell} + \sum_{q=0}^{\frac{n-3}{2}} Z_q \widetilde{Z}_{\frac{n-3}{2}-q}\,. \tag{3.85}$$

The non-anomalous global symmetry of the teories is SU$(2N_f)$ × U$(1)_A$ × U$(1)_R$ under which the fields transform as in table 3.3.
At the level of the partition function, this duality corresponds to the identity

$$\mathcal{Z}^{2N_f}_{\mathrm{USp}(2N_c)}(\mu_a;\tau_A;\tau_B) = \mathcal{Z}^{2N_f}_{\mathrm{USp}(2\tilde{N}_c)}(\tau_A - \tau_B - \mu_a;\tau_A;\tau_B) \\ \times \prod_{j=0}^{n-1}\prod_{\ell=0}^{2}\prod_{1\le a<b\le 2N_f} \Gamma_h(j\tau_A + \ell\tau_B + \mu_a + \mu_b) \\ \times \prod_{q=0}^{\frac{n-3}{2}}\prod_{a=1}^{2N_f} \Gamma_h((2q+1)\tau_A + \tau_B + 2\mu_a) \\ \times \prod_{\substack{j=0,\dots,n-1\\ \ell=0,1,2\\ j\ell=0}} \Gamma_h\left(j\tau_A + \ell\tau_B + 2N_f\omega - (N_c+n)\tau_A - \sum_{a=1}^{2N_f}\mu_a\right) \\ \times \prod_{q=0}^{\frac{n-3}{2}} \Gamma_h\left((2q+1)\tau_A + 4N_f\omega - 2(N_c+n)\tau_A - 2\sum_{a=1}^{2N_f}\mu_a\right) \tag{3.86}$$

Table 3.3: Matter content of $\mathrm{USp}(2N_c)_0$ and $\mathrm{USp}(2\tilde{N}_c)_0$ dual theories.

	Gauge		Global		
Field	$\mathrm{USp}(2N_c)$	$\mathrm{USp}(2\tilde{N}_c)$	$\mathrm{SU}(2N_f)$	$\mathrm{U}(1)_A$	$\mathrm{U}(1)_R$
Q	$\square$	1	$\square$	1	r_Q
A	$\begin{smallmatrix}\square\\\square\end{smallmatrix}$	1	1	0	$\frac{2}{n+1}$
B	$\begin{smallmatrix}\square\\\square\end{smallmatrix}$	1	1	0	$\frac{n}{n+1}$
$Y_{j\ell}^{\pm}$	1	1	1	$-2N_f$	$2(1-r_Q)N_f+\frac{2j+n\ell-2(N_c+n)}{n+1}$
$Z_q^{\pm}$	1	1	1	$-4N_f$	$4(1-r_Q)N_f+\frac{2+4q-4(N_c+n)}{n+1}$
q	1	$\square$	$\overline{\square}$	-1	$\frac{2-n}{n+1}-r_Q$
a	1	$\begin{smallmatrix}\square\\\square\end{smallmatrix}$	1	0	$\frac{2}{n+1}$
b	1	$\begin{smallmatrix}\square\\\square\end{smallmatrix}$	1	0	$\frac{n}{n+1}$
$\mathcal{M}_{j,0}^{j=0,\dots,n-1}$	1	1	$\begin{smallmatrix}\square\\\square\end{smallmatrix}$	2	$2r_Q+\frac{2j}{n+1}$
$\mathcal{M}_{2j,1}^{j=0,\dots,\frac{n-1}{2}}$	1	1	$\begin{smallmatrix}\square\\\square\end{smallmatrix}$	2	$2r_Q+\frac{4j+n}{n+1}$
$\mathcal{M}_{2j+1,1}^{j=0,\dots,\frac{n-3}{2}}$	1	1	$\begin{smallmatrix}\square\\\square\end{smallmatrix}$	2	$2r_Q+\frac{4j+n+2}{n+1}$
$\mathcal{M}_{j,2}^{j=0,\dots,n-1}$	1	1	$\begin{smallmatrix}\square\\\square\end{smallmatrix}$	2	$2r_Q+\frac{2j+2n}{n+1}$
$\tilde{Y}_{j\ell}^{\pm}$	1	1	1	$2N_f$	$2(r_Q-1)N_f+\frac{n\ell+2(j+N_c+n+1)}{n+1}$
$\tilde{Z}_q^{\pm}$	1	1	1	$4N_f$	$4(r_Q-1)N_f+\frac{4(q+N_c+n)+6}{n+1}$

The superpotential (3.84) fixes the values of the real masses for the adjoint fields

$$\tau_A=\frac{2\omega}{n+1},\qquad \tau_B=\frac{n\omega}{n+1}. \tag{3.87}$$

This duality is going to be our starting point.
In the following we want to construct the duality for non-vanishing CS level. We consider the duality without CS terms and with $2(N_f+k)$ fundamental flavours and assign a positive real mass to $2k$. By integrating out the massive fields we arrive at the duality between

- $\mathrm{USp}(2N_c)_{2k}$ SQCD with $2N_f$ fundamentals Q and two anti-symmetric rank-two tensors A, B interacting through the superpotential

$$\mathcal{W}_{\text{ele}}=\operatorname{Tr}A^{n+1}+\operatorname{Tr}AB^2. \tag{3.88}$$

- $\mathrm{USp}(2\tilde{N}_c)$ SQCD with $\tilde{N}_c=3n(N_f+|k|)-N_c-2n-1$, $2N_f$ fundamentals q and two anti-symmetric rank-two tensors a, b interacting through the superpotential

$$\mathcal{W}_{\text{mag}}=\operatorname{Tr}a^{n+1}+\operatorname{Tr}ab^2+\sum_{j=0}^{n-1}\sum_{\ell=0}^{2}\mathcal{M}_{j,\ell}qa^jb^\ell q. \tag{3.89}$$

The electric and the magnetic theories acquire a CS level $2k$ and $-2k$ respectively

The dressed monopole operators of the electric theory acting as singlets in the dual phase become massive and are integrated out.

To reproduce this flow at the level of the partition function, we start from the equality (3.86) and consider the following shifts in the real masses

$$\begin{cases} m_A \to m_A + \frac{k}{N_f+k} s & \\ m_a \to m_a - \frac{k}{N_f+k} s & a = 1, \dots, 2N_f \\ m_a \to m_a + \frac{N_f}{N_f+k} s & a = 1, \dots, 2k \end{cases} \tag{3.90}$$

We study the limit of large s on both sides of the identity (3.86) by making use of the asymptotic behavior of the hyperbolic Gamma function (3.5). We check that the leading saddle point contributions cancel between the electric and magnetic phases, and we are left with the equality between

$$\mathcal{Z}_{\text{ele}} = \mathcal{Z}^{2N_f}_{\text{USp}(2N_c)_{2k}}(\mu_a; \tau_A; \tau_B), \tag{3.91}$$

and

$$\begin{aligned} \mathcal{Z}_{\text{mag}} = {} & \mathrm{e}^{\mathrm{i}\pi\phi} \mathcal{Z}^{2N_f}_{\text{USp}(2\tilde{N}_c)_{-2k}}(\tau_A - \tau_B - \mu_a; \tau_A; \tau_B) \\ & \times \prod_{j=0}^{n-1} \prod_{\ell=0}^{2} \prod_{1 \le a < b \le 2N_f} \Gamma_h(j\tau_A + \ell\tau_B + \mu_a + \mu_b). \end{aligned} \tag{3.92}$$

To evaluate the asymptotic behavior of the factor coming from the singlets, or the electric mesons under the duality map, in the magnetic theory, we make use of the following decomposition formula

$$\begin{aligned} \sum_{\substack{a,b=1,\dots,2k \\ a<b}} (j\tau_A + \ell\tau_B + \mu_a + \mu_b - \omega) = {} & 2(k-1) \sum_{a=1}^{2k} \mu_a^2 + \left(\sum_{a=1}^{2k} \mu_a \right)^2 \\ & + (4k-2)(j\tau_A + \ell\tau_B - \omega) \sum_{a=1}^{2k} \mu_a + k(2k-1)(j\tau_A + \ell\tau_B - \omega)^2. \end{aligned} \tag{3.93}$$

The complex exponent ϕ necessary for the equality between the partition functions to hold has the following form

$$\begin{aligned} \phi = {} & 12nN_f(k - N_f)m_A^2 + 6\tau_A\Big(-1 - 2n - 2N_c + 2N_f(1+n) + 2k(-1+2n)\Big) m_A \\ & + \frac{\tau_A^2}{24}(6k-3) + \frac{\tau_A\tau_B}{12}\Big(4k - 12k^2(1+24n^2) + 3\big(-7 - 25n(1+n) - 24N_f^2(1+n)^2 \\ & - 24N_c(1+2n) - 24N_c^2 + 24N_f(1+n)(1+2n+2N_c) \\ & + 4k(12N_f - 72nN_f(1+n) + n(39 + 77n + 54N_c))\big)\Big) - 3n\left(\left(\sum_{a=1}^{2N_f} m_a \right)^2 - 2k \sum_{a=1}^{2N_f} m_a^2 \right). \end{aligned} \tag{3.94}$$

3.2.4 Discussion and Conclusions

In this section we studied 3d $\mathcal{N} = 2$ dualities for two adjoint U(N_c) and SU(N_c) SQCD with D_{n+2}-type superpotential and odd n. This generalizes the constructions of [11, 85] for SQCD and of [24, 232, 276] for adjoint SQCD. The dualities are obtained starting from the one obtained in [230] from the 4d/3d reduction. The classification is constructed through real mass flows, Higgs flows and the gauging of the topological symmetry. We corroborated these construction by checking the various steps with the help of the three sphere partition function. Furthermore we matched the CS contact terms across the dual phases with the complex phases that can be read in the integral identities on the three sphere. We concluded by proposing a duality for USp$(2N)_{2k}$ SQCD with two antisymmetric and D_{n+2} type superpotential that was overlooked in the literature.

There are interesting aspects of such dualities and possible generalizations that we did not investigate and that we leave for future projects. For example we did not match the superconformal index across the new dual phases. This should provide a stronger check of the dualities obtained here. Another aspect that we did not investigate corresponds to find mirror dualities for the U(1) sectors in the duals of the SU(N_c) dualities in presence of charged matter fields. Such mirrors could simplify the structure of the dual models, that so far are given in terms of product groups. A further generalization of the construction is related to chiral models with monopole superpotentials. In the SQCD case such possibility has been discussed in [81] for the case of linear monopole superpotential and in [30] for SQCD with quadratic monopole superpotential. In the A_n case a similar extension (with quadratic monopoles) has been proposed in [23]. The analysis could be extended also to the dualities with monopole superpotential for the D_{n+2} case studied in [231]. We conclude observing that a full list of 4d dualities for SU(N_c) SQCD with two tensors has been provided in [119]. It should be possible to reduce such dualities to 3d and that to study the chiral limit of these cases as well. This is an interesting possibility because some of the models in the classification have also an interpretation in terms of the HW setup [121]. This may allow to study the 4d/3d reduction from a T-duality in the HW setup along the lines of [27, 41] and the chiral dualities as discussed in [24]

3.3 Sporadic Dualities from Tensor Deconfinement

In this section we study a 3d confining duality recently obtained in [278], corresponding to $\mathrm{USp}(4)$ with two rank-2 anti-symmetric tensors and two fundamentals. The existence of such a duality has been claimed by extending to the 3d bulk a boundary duality constructed from $\mathcal{N} = (0,2)$ half-BPS boundary conditions in 3d $\mathcal{N} = 2$. Again we deconfine the two rank-2 anti-symmetric tensors and then provide the sequential dualities leading to the final description in terms of the gauge singlets of the original model.

3.3.1 The Okazaki-Smith 3d duality

In this section we study a 3d $\mathcal{N} = 2$ confining duality recently proposed in [278]. The electric model is $\mathrm{USp}(4)$ SQCD with two fundamentals and two rank-2 anti-symmetric tensors. The model has an $\mathrm{U}(2)^2 \times \mathrm{U}(1)_R$ global symmetry and the charges of the fields under these symmetries are summarized in (3.95).

	$\mathrm{USp}(4)$	$\mathrm{SU}(2)_A$	$\mathrm{SU}(2)_a$	$\mathrm{U}(1)_A$	$\mathrm{U}(1)_a$	$\mathrm{U}(1)_R$
A	6	2	1	1	0	0
Q	4	1	2	0	1	0

(3.95)

The model has vanishing superpotential and its low energy dynamics is described by the gauge invariant combinations $M = Q_1 Q_2$, $\phi_I = \mathrm{Tr}\, A_I$, $\phi_{IJ} = \mathrm{Tr}(A_I A_J)$, $B_{\alpha\beta} = Q_\alpha A_1 A_2 Q_\beta$ and $B_I = Q_1 A_I Q_2$. These fields interact through a superpotential with a singlet $\mathcal{T}_4$ that corresponds to the minimal monopole of $\mathrm{USp}(4)$. The charges of the fields with respect to the global $\mathrm{U}(2)^2 \times \mathrm{U}(1)_R$ symmetry are:

	$\mathrm{SU}(2)_A$	$\mathrm{SU}(2)_a$	$\mathrm{U}(1)_A$	$\mathrm{U}(1)_a$	$\mathrm{U}(1)_R$
M	1	1	0	2	0
$B_{\alpha\beta}$	1	3	2	2	0
ϕ_{IJ}	3	1	2	0	0
ϕ_I	2	1	1	0	0
B_I	2	1	1	2	0
$\mathcal{T}_4$	1	1	-4	-4	2

(3.96)

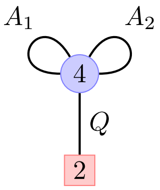
In the following we will derive this confining duality by deconfining the anti-symmetric tensors and then by sequentially dualizing the gauge groups. We found that in order to proceed it is very useful to make the rank-2 tensors traceless by adding the flippers $\beta_{1,2}$ interacting through the superpotential

$$W = \sum_{I=1,2} \beta_I \,\mathrm{Tr}\, A_I \tag{3.97}$$

where the USp-invariant trace is defined as $\mathrm{Tr}\, A_I = A_I^{ij} J_{ij}$ with J_{ij} the totally anti-symmetric USp tensor.

3.3.1.1 Field theory analysis

In the following we will derive the duality using the field theory approach. We proceed by representing the model in terms of a quiver gauge theory, using the same conventions of the previous section: the blue circles refer to symplectic gauge groups while the red squares identify the special unitary flavor groups.
We start by considering the model with the flip in formula (3.97)

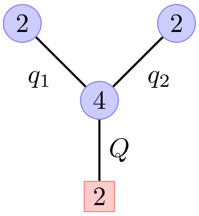


$$W = \beta_1 \,\mathrm{Tr}\, A_1 + \beta_2 \,\mathrm{Tr}\, A_2 \,. \tag{3.98}$$

We then deconfine the two fields A_I with two auxiliary $\mathrm{USp}(2)$ nodes with the assignment

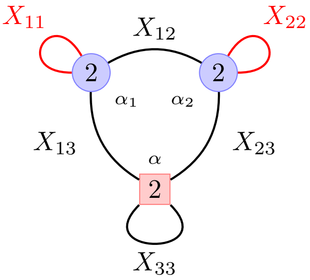
$$A_1^{ij} = q_1^{\alpha_1 i} q_1^{\beta_1 j} \epsilon_{\alpha_1 \beta_1} \,, \qquad A_2^{ij} = q_2^{\alpha_2\, i} q_2^{\beta_2\, j} \epsilon_{\alpha_2 \beta_2} \,, \tag{3.99}$$

where the i-index refers to the $\mathrm{USp}(4)$ node and $(\alpha_{1,2}, \beta_{1,2})$ are indices of the two $\mathrm{USp}(2)_{1,2}$ gauge groups. Therefore the deconfined theory is



$$W = \sum_{I=1,2} \Big(\beta_I \,\mathrm{Tr}\, q_I^2 + \sigma_I Y_2^{(I)} \Big), \tag{3.100}$$

where $Y_2^{(I)}$ are monopoles for the two $\mathrm{USp}(2)$ nodes. The central $\mathrm{USp}(4)$ node in this theory has 6 fundamentals and therefore it confines [6]. The IR description has two $\mathrm{USp}(2)_1$ and $\mathrm{USp}(2)_2$ gauge groups connected by a bifundamental field. There is still a manifest $\mathrm{SU}(2)$ flavor symmetry associated to a node in the quiver and there are further fundamental fields for both the $\mathrm{USp}(2)_{1,2}$ gauge factors. There is also a singlet Y_4 identified with the monopole of the $\mathrm{USp}(4)$ gauge group for the model in (3.100), that interacts through a superpotential with the generalized meson of $\mathrm{USp}(4)$ itself. The quiver and the superpotential for this dual theory are



$$\begin{aligned} W = {} & Y_4 \,\mathrm{Pf}\, X + \beta_1 X_{11} + \beta_2 X_{22} \\ & + \sigma_1 Y_2^{(1)} X_{23}^2 + \sigma_2 Y_2^{(2)} X_{13}^2 \,, \end{aligned} \tag{3.101}$$

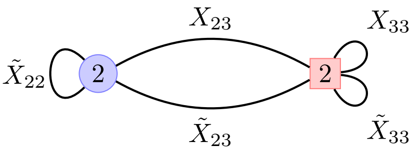
where the explicit superpotential is given by

$$\begin{aligned}W =\; & Y_4\epsilon_{\alpha_1\beta_1}\epsilon_{\alpha_2\beta_2}\epsilon_{\alpha\beta}\left(-X_{12}^{\alpha_1\alpha_2}X_{23}^{\beta_2\alpha}X_{13}^{\beta_1\beta}+\frac{1}{8}X_{11}^{\alpha_1\beta_1}X_{22}^{\alpha_2\beta_2}X_{33}^{\alpha\beta}-\frac{1}{4}X_{13}^{\alpha_1\alpha}X_{13}^{\beta_1\beta}X_{22}^{\alpha_2\beta_2}\right.\\ & \left.-\frac{1}{4}X_{12}^{\alpha_1\alpha_2}X_{12}^{\beta_1\beta_2}X_{33}^{\alpha\beta}-\frac{1}{4}X_{23}^{\alpha_2\alpha}X_{23}^{\beta_2\beta}X_{11}^{\alpha_1\beta_1}\right)\\ & +\sigma_1 Y_2^{(1)}\epsilon_{\alpha_2\alpha}\epsilon_{\beta_2\beta}X_{23}^{\alpha_2\alpha}X_{23}^{\beta_2\beta}+\sigma_2 Y_2^{(2)}\epsilon_{\alpha_1\alpha}\epsilon_{\beta_1\beta}X_{13}^{\alpha_1\alpha}X_{13}^{\beta_1\beta}+\beta_1X_{11}+\beta_2X_{22}.\end{aligned} \tag{3.102}$$

The fields X_{11}, X_{22}, β_1 and β_2 are massive and can be integrated out. The final superpotential in the IR is therefore

$$\begin{aligned}W =\; & -\frac{Y_4}{4}\epsilon_{\alpha_1\beta_1}\epsilon_{\alpha_2\beta_2}\epsilon_{\alpha\beta}\left(4X_{12}^{\alpha_1\alpha_2}X_{23}^{\beta_2\alpha}X_{13}^{\beta_1\beta}+X_{12}^{\alpha_1\alpha_2}X_{12}^{\beta_1\beta_2}X_{33}^{\alpha\beta}\right)\\ & +\sigma_1 Y_2^{(1)}\epsilon_{\alpha_2\beta_2}\epsilon_{\alpha\beta}X_{23}^{\alpha_2\alpha}X_{23}^{\beta_2\beta}+\sigma_2 Y_2^{(2)}\epsilon_{\alpha_1\beta_1}\epsilon_{\alpha\beta}X_{13}^{\alpha_1\alpha}X_{13}^{\beta_1\beta}.\end{aligned} \tag{3.103}$$

The two $\mathrm{USp}(2)$ nodes have each 4 fundamentals and are therefore confining [6]. Here we choose to confine the $\mathrm{USp}(2)_1$ group. The other choice is completely equivalent because of the $\mathrm{SU}(2)_A$ global symmetry that rotates the two anti-symmetric in the original description of the model. After confining the $\mathrm{USp}(2)_1$ gauge group we are left with a $\mathrm{USp}(2)_2$ SQCD with four fundamentals and a non-trivial superpotential. The quiver and the operator mapping are reported below



$$\begin{aligned}\tilde{X}_{23}^{\alpha_2\alpha} &= \epsilon_{\alpha_1\beta_1}X_{12}^{\alpha_1\alpha_2}X_{13}^{\beta_1\alpha}\\ \tilde{X}_{22}^{\alpha_2\beta_2} &= \epsilon_{\alpha_1\beta_1}X_{12}^{\alpha_1\alpha_2}X_{12}^{\beta_1\beta_2}\\ \tilde{X}_{33}^{\alpha\beta} &= \epsilon_{\alpha_1\beta_1}X_{13}^{\alpha_1\alpha}X_{13}^{\beta_1\beta}\end{aligned} \tag{3.104}$$

while the superpotential is

$$\begin{aligned}W =\; & -\frac{1}{4}\epsilon_{\alpha_2\beta_2}\epsilon_{\alpha\beta}\left[Y_4\left(4\tilde{X}_{23}^{\alpha_2\beta}X_{23}^{\beta_2\alpha}+\tilde{X}_{22}^{\alpha_2\beta_2}X_{33}^{\alpha\beta}\right)-Y_2^{(1)}(\tilde{X}_{22}^{\alpha_2\beta_2}\tilde{X}_{33}^{\alpha\beta}-2\tilde{X}_{23}^{\alpha_1\alpha}\tilde{X}_{23}^{\beta_1\beta})\right]\\ & +\sigma_1 Y_2^{(1)}\epsilon_{\alpha\beta}\epsilon_{\alpha_2\beta_2}X_{23}^{\alpha_2\alpha}X_{23}^{\beta_2\beta}+\sigma_2 Y_2^{(2)}\epsilon_{\alpha\alpha'}\epsilon_{\beta\beta'}\tilde{X}_{33}^{\alpha\beta}\tilde{X}_{33}^{\alpha'\beta'}\,.\end{aligned} \tag{3.105}$$

The second block in the first line comes from the confining superpotential $Y_2^{(1)}\,\mathrm{Pf}\,\tilde{X}$, where the field $Y_2^{(1)}$ is the monopole of the $\mathrm{USp}(2)_1$ gauge group acting as a singlet in the confined phase.

We conclude the sequence by confining the $\mathrm{USp}(2)_2$ gauge group, that has indeed four fundamentals. This leads to the final, confined, theory where the new mesons are mapped to the fundamentals of $\mathrm{USp}(2)_2$ as

$$V^{\alpha\beta}=\epsilon_{\alpha_2\beta_2}\tilde{X}_{23}^{\alpha_2\beta}X_{23}^{\beta_2\alpha},\quad U^{\alpha\beta}=\epsilon_{\alpha_2\beta_2}X_{23}^{\alpha_2\alpha}X_{23}^{\beta_2\beta},\quad T^{\alpha\beta}=\epsilon_{\alpha_2\beta_2}\tilde{X}_{23}^{\alpha_2\alpha}\tilde{X}_{23}^{\beta_2\beta}. \tag{3.106}$$

Furthermore there is a singlet of $\mathrm{USp}(2)_2$ that we redefine as $R = \epsilon_{\alpha_2\beta_2}\tilde{X}_{22}^{\alpha_2\beta_2}$. The superpotential of this final WZ model is

$$\begin{aligned} W = -\frac{\epsilon_{\alpha\beta}}{4}\Big[& Y_4\left(4V^{\alpha\beta} + RX_{33}^{\alpha\beta}\right) - Y_2^{(1)}(R\tilde{X}_{33}^{\alpha\beta} - 2T^{\alpha\beta} + 4\sigma_1 U^{\alpha\beta}) \\ & -Y_2^{(2)}\epsilon_{\ell m}(U^{\alpha\ell}T^{\beta m} - V^{\alpha\ell}V^{\beta m})\Big] + \sigma_2 Y_2^{(2)}\epsilon_{\alpha\alpha'}\epsilon_{\beta\beta'}\tilde{X}_{33}^{\alpha\beta}\tilde{X}_{33}^{\alpha'\beta'} . \end{aligned} \tag{3.107}$$

The expression (3.107) needs some massage in order to simplify its interpretation. For example some fields appear quadratically in the superpotential and they can be integrated out. By writing

$$V_{\alpha\beta} = \sigma^{\mu}_{\alpha\beta}v_{\mu}, \qquad \sigma^{\mu} = (\mathbf{1}, \sigma^i) \tag{3.108}$$

we see that the v_3 field is massive. The singlet field $\epsilon_{\alpha\beta}T^{\alpha\beta}$ also acquires a mass and it can be integrated out in the IR. By considering the various F-term conditions and rescaling the fields appropriately, we get the final superpotential

$$W = Y_2^{(2)}\left[RU\tilde{X}_{33} + \sigma_1 U^2 + \sigma_2\tilde{X}_{33}^2 + R^2 X_{33}^2 - \det V_{\alpha\beta}\right], \tag{3.109}$$

where the squares are understood with the right contractions. We can then identify the fields here with the ones in formula (3.96). Looking at the global symmetry structure the explicit mapping is then

$$\begin{aligned} Y_2^{(2)} &\leftrightarrow \mathcal{T}_4 \\ X_{33} &\leftrightarrow M \\ (R, \sigma_1, \sigma_2) &\leftrightarrow (\phi_{12}, \phi_{11}, \phi_{22}) \\ V_{\alpha\beta} &\leftrightarrow B_{\alpha\beta} \\ (\tilde{X}_{33}, U) &\leftrightarrow (B_1, B_2). \end{aligned} \tag{3.110}$$

Substituting this mapping into the superpotential (3.109) we obtain

$$W = \mathcal{T}_4\Big(B_I\phi_{IJ}B_J + M^2\phi_{12}^2 + \det B_{\alpha\beta}\Big), \tag{3.111}$$

which is the same superpotential of [277], where this confining duality was anticipated. The remaining fields ϕ_I map to free decoupled fields in the original theory associated to the traces of the two anti-symmetric tensors $\mathrm{Tr}\, A_I$.

3.3.1.2 3d partition function

We complete our analysis by reproducing the derivation of the duality from supersymmetric localization on the squashed three sphere. Such procedure gives rise to the identity between the partition function of $\mathrm{USp}(4)$ with with two anti-symmetric and two fundamentals and the partition function of the WZ model for the gauge singlets. The global symmetry enters these identities in terms of real masses, that from the field theory side are associated to vevs of the reals scalars in the vector multiplets of the weakly gauged background flavor symmetries.

Before studying the deconfinement of two rank-2 anti-symmetric tensors from the three sphere partition function we briefly review the necessary definitions. The partition function on the squashed three sphere S_b^3, obtained from localization in [212] (see also [211, 240, 247] for the round case) is a matrix integral over the reals scalar in the vector multiplet in the Cartan of the gauge group. There is a classical term corresponding to the CS action (global and local) and the matter and the gauge multiplet contribute with their one loop determinant. These last can be associated to hyperbolic Gamma functions, formally defined as

$$\Gamma_h(z;\omega_1,\omega_2) = \prod_{n_1,n_2\geq 0}^{\infty} \frac{(n_1+1)\omega_1+(n_2+1)\omega_2-z}{n_1\omega_1+n_2\omega_2+z} \tag{3.112}$$

The argument of such Gamma functions is physically interpreted as a holomorphic combination between the real masses for the gauge and the global symmetries and the R-charges (or mass dimensions). The purely imaginary parameters $\omega_1 = ib$ and $\omega_2 = i/b$ are related to the squashing parameter of the three sphere S_b^3.

Here we will only focus on the case of symplectic gauge group. Let us consider the partition function of an $\mathrm{USp}(2N_c)$ gauge theory with $2N_f$ fundamentals. It is given by

$$Z_{\mathrm{USp}(2N_c),N_f}(\mu) \quad = \quad \frac{1}{2^n n!\sqrt{(-\omega_1\omega_2)^n}} \int \prod_{i=1}^{N_c} \mathrm{d}z_i \, \frac{\prod_{a=1}^{2N_f}\Gamma_h(\pm z_i+\mu_a)}{\Gamma_h(\pm 2z_i)} \prod_{i<j} \frac{1}{\Gamma_h(\pm z_i \pm z_j)} \tag{3.113}$$

In our analysis we will use an identity involving this partition function and its dual Aharony phase [6]. The identity is (see Theorem 5.5.9 of [123])

$$\begin{aligned} Z_{\mathrm{USp}(2N_c),N_f}(\mu) \quad &= \quad \Gamma_h\left(2\omega(N_f-N_c)-\sum_{a=1}^{2N_f}\mu_a\right) \\ &\times \prod_{a<b}\Gamma_h(\mu_a+\mu_b) Z_{\mathrm{USp}(2(N_f-N_c-1)),N_f}(\omega-\mu) \end{aligned} \tag{3.114}$$

with $2\omega \equiv \omega_1+\omega_2$. Observe that the identity (3.114) remains valid for $N_f = N_c+1$, that corresponds to the confining case of Aharony duality [6], where only the meson M and the minimal $\mathrm{USp}(2N_c)$ monopole Y survive in the WZ model and they interact through the superpotential $W = Y\,\mathrm{Pf}\,M$.

We start considering the original model, adding also the flippers β_I arising from the superpotential (3.97). The partition function is

$$\begin{aligned} Z = &\frac{\prod_{A=1,2}\Gamma_h(2\omega-n_A)}{8\sqrt{-\omega_1\omega_2}^2} \\ &\times \int \prod_{i=1,2} \mathrm{d}z_i \, \frac{\prod_{a=1,2}\Gamma_h(\pm z_i+m_a)}{\Gamma_h(\pm 2z_i)} \frac{\prod_{A=1,2}\Gamma_h(\pm z_1 \pm z_2+n_A)}{\Gamma_h(\pm z_1 \pm z_2)} \end{aligned} \tag{3.115}$$

In this formula $m_{1,2}$ are the real masses of the two fundamental fields and $n_{1,2}$ are the real masses of the two anti-symmetric fields. We can also use a different basis

$$m_1 = \rho + \sigma, \quad m_2 = \rho - \sigma, \quad n_1 = \mu + \nu, \quad n_2 = \mu - \nu \tag{3.116}$$

giving an explicit parameterization it terms of the Cartan of the $\mathrm{U}(2)^2$ flavor symmetry. Indeed in this way σ and ν parameterize the Cartan of $\mathrm{SU}(2)_a$ and $\mathrm{SU}(2)_A$ respectively. We then proceed by deconfining the two totally anti-symmetric tensors. This step produces two $\mathrm{USp}(2)$ gauge nodes, two bifundamentals, each connecting one of these $\mathrm{USp}(2)$ gauge groups to the original $\mathrm{USp}(4)$. The partition function of the model becomes

$$\begin{aligned} Z &= \frac{\prod_{A=1,2}\prod_{j=1,2}\Gamma_h(2\omega - jn_A)}{32\sqrt{(-\omega_1\omega_2)^4}} \int \frac{\mathrm{d}z_1\,\mathrm{d}z_2\,\mathrm{d}w_1\,\mathrm{d}w_2}{\Gamma_h(\pm 2z_1)\Gamma_h(\pm 2z_2)\Gamma_h(\pm 2w_1)\Gamma_h(\pm 2w_2)} \\ &\times \prod_{i=1,2}\left(\prod_{a=1,2}\Gamma_h(\pm z_i + m_a)\cdot\prod_{A=1,2}\Gamma_h(\pm z_i \pm w_A + n_A/2)\right) \end{aligned} \tag{3.117}$$

As a check we can see that (3.115) is obtained by applying (3.114) to the two $\mathrm{USp}(2)$ gauge groups in (3.117). The partition function (3.117) corresponds to the one for the model in represented in (3.100).

The next step consists of Aharony duality on $\mathrm{USp}(4)$. At the level of the partition function it corresponds to use the identity (3.114) on the gauge theory identified by the variables $z_{1,2}$. The partition function becomes

$$\begin{aligned} Z &= \frac{\prod_{A=1,2}\Gamma_h(2\omega - 2n_A)}{4\sqrt{(-\omega_1\omega_2)^2}}\Gamma_h(2\omega - m_1 - m_2 - n_1 - n_2)\Gamma_h(m_1 + m_2) \\ &\times \int \mathrm{d}w_1\,\mathrm{d}w_2\,\frac{\prod_{A=1,2}\prod_{a=1,2}\Gamma_h(m_a \pm w_A + n_A/2)\cdot\Gamma_h(\pm w_1 \pm w_2 + (n_1+n_2)/2)}{\Gamma_h(\pm 2w_1)\Gamma_h(\pm 2w_2)} \end{aligned} \tag{3.118}$$

The partition function (3.118) corresponds to the one for the model in represented in (3.101).

The next step consists of a confining limit of Aharony duality on one of the $\mathrm{USp}(2)$ factor. Choosing one of the two $\mathrm{USp}(2)$ nodes has the effect of making the $\mathrm{SU}(2)_A \times \mathrm{SU}(2)_a$ global symmetry not manifest in the integrand of the partition function. Following the discussion on the field theory side here we choose to dualize the $\mathrm{USp}(2)_1$ gauge group, such that the partition function becomes

$$\begin{aligned} Z &= \frac{1}{2}\Gamma_h(2\omega - m_1 - m_2 - n_1 - n_2)\Gamma_h(m_1 + m_2)\prod_{A=1,2}\Gamma_h(2\omega - 2n_A) \\ &\times \Gamma_h(m_1 + m_2 + n_1)\Gamma_h(n_1 + n_2)\Gamma_h(2\omega - 2n_1 - n_2 - m_1 - m_2) \\ &\times \int \mathrm{d}w_2\,\frac{\prod_{a=1,2}\Gamma_h(m_a \pm w_2 + n_2/2 + n_1)\cdot\Gamma_h(m_a \pm w_2 + n_2/2)}{\Gamma_h(\pm 2w_2)} \end{aligned} \tag{3.119}$$

The partition function (3.119) corresponds to the one for the model in represented in (3.104).
The last step of the procedure requires a confining limit of Aharony duality on the left-over $\text{USp}(2)_2$ gauge group. This gives the final partition function

$$\begin{aligned} Z &= \Gamma_h(2\omega - m_1 - m_2 - n_1 - n_2)\Gamma_h(m_1 + m_2) \prod_{A=1,2} \Gamma_h(2\omega - 2n_A) \\ &\times \Gamma_h(m_1 + m_2 + n_1)\Gamma_h(n_1 + n_2)\Gamma_h(2\omega - 2n_1 - n_2 - m_1 - m_2) \\ &\times \Gamma_h(m_1 + m_2 + 2n_1 + n_2)\Gamma_h(m_1 + m_2 + n_1) \\ &\times \Gamma_h(2\omega - 2m_1 - 2m_2 - 2n_1 - 2n_2) \prod_{a,b=1,2} \Gamma_h(m_a + m_b + n_1 + n_2) \end{aligned} \tag{3.120}$$

This expression still needs some massage. First we can integrate out the massive fields, as done on the field theory approach. Here this integration corresponds to take advantage of the formula $\Gamma_h(2\omega - x)\Gamma_h(x) = 1$. After this step we can also write down (3.120) in a manifestly $\text{SU}(2)_A \times \text{SU}(2)_a$ invariant form. We arrive to the expression

$$\begin{aligned} Z &= \Gamma_h(m_1 + m_2)\Gamma_h(n_1 + n_2) \prod_{A=1,2} \Gamma_h(m_1 + m_2 + n_A)\Gamma_h(2\omega - 2n_A) \\ &\times \Gamma_h(2\omega - 2m_1 - 2m_2 - 2n_1 - 2n_2) \prod_{a \le b} \Gamma_h(m_a + m_b + n_1 + n_2) \end{aligned} \tag{3.121}$$

or using (3.116)

$$\begin{aligned} Z &= \Gamma_h(2\rho)\Gamma_h(2\mu)\Gamma_h(2\mu \pm 2\nu)\Gamma_h(2\rho + \mu \pm \nu) \\ &\times \Gamma_h(2\omega - 4\mu - 4\rho)\Gamma_h(2\rho \pm 2\sigma + 2\mu, 2\rho + 2\mu) \end{aligned} \tag{3.122}$$

This is the final expression that matches with (3.115). We can see that all the fields B_I, ϕ_{IJ}, $B_{\alpha\beta}$, M and the monopole $\mathcal{T}_4$ appear in the partition function with the expected real masses. Explicitly we can associate these Gamma functions to the singlets of the confined phase using the mapping

$$\begin{aligned} B_{\alpha\beta} &\leftrightarrow \Gamma_h(2\rho \pm 2\sigma + 2\mu, 2\rho + 2\mu) \qquad & \phi_{IJ} &\leftrightarrow \Gamma_h(2\mu \pm 2\nu, 2\mu) \\ B_I &\leftrightarrow \Gamma_h(2\rho + \mu \pm \nu) & M &\leftrightarrow \Gamma_h(2\rho) \\ \mathcal{T}_4 &\leftrightarrow \Gamma_h(2\omega - 4\mu - 4\rho) \end{aligned} \tag{3.123}$$

Indeed the arguments of hyperbolic Gamma functions correspond to the real masses that can be read from the charges in formula (3.96).

3.4 Discussion and Conclusions

In this Section we have derived, using field theory arguments, a confining dualities that has been proposed in the literature from supersymmetric localization. Here the duality has been derived by combining the technique of rank-2 tensor deconfinement of [93] together with the sequential application of ordinary dualities and/or confining dualities. There are many interesting directions that would be worth to explore. As discussed in [278] the field content in this case corresponds to the one of the dualities studied in [37, 44] with D-type superpotential. Nevertheless, as observed in [278] there are differences in the operator mapping and in the charge spectrum. Furthermore, the $\mathrm{USp}(4)$ duality discussed here appears sporadic and its generalization to $\mathrm{USp}(2N_c)$ does not seem straightforward. For example, we did not find any confining duality by increasing the rank of the gauge group and keeping fixed the field content (i.e. keeping two rank-2 antisymmetric tensors and possibly increasing the number of fundamentals). It is nevertheless possible that further fields and interactions should be considered in order to have an $\mathrm{USp}(2N_c)$ confining theory with two rank-2 anti-symmetric tensors.

A last, related, question regards the existence of 4d confining dualities with two rank-2 tensors. Beyond the case of $\mathrm{USp}(2N_c)$ with two rank-2 anti-symmetric tensors, one can imagine also cases with unitary or orthogonal gauge groups or cases with more general rank-2 tensor matter fields.

CHAPTER 4

Duality Orbits from String Theory

Contents

The relevance of dualities is undeniable, as hopefully evidenced by the preceding Chapters. Typically, dualities grant access to the non-perturbative, strongly coupled sectors of field theories by offering alternative descriptions in terms of another theory with perturbative dynamics. In the following sections, we will delve into the role played by and generalizations of electromagnetic duality in $\mathcal{N} = 4$ SYM (supersymmetric Yang-Mills) and in more exotic non-Lagrangian theories with $\mathcal{N} = 3$ in 4 dimensions. The latter theories are inherently non-Lagrangian due to the representation theory of $\mathcal{N} = 3$, which can be shown to always enhance to $\mathcal{N} = 4$. They can emerge from brane setups in string theory and from certain compactifications of M-theory.

Electromagnetic duality, in conjunction with the gauging of finite higher-form symmetries [189], furnishes these theories with a suite of new symmetries known as non-invertible symmetries. These symmetries, in the broader sense of [189], are not described by groups due to the absence of inverses for any given element. Rather, they are given by categories, where the concept of a symmetry category arises as a generalization of a symmetry group. Particularly, we are interested in duality symmetries, i.e., symmetries that are partially implemented by an electromagnetic duality transformation and partially by the gauging of some finite symmetry. Such symmetries are expected since electromagnetic duality relates two theories with the same local theories but with distinct global structures, and gauging essentially does the same, thereby a suitable combination of the two will transform one theory into itself, thus constituting a symmetry of said theory.

In this Chapter, we will start in section 4.1 by revisiting $\mathcal{N} = 4$ SYM in 4 dimensions, a model in which these concepts naturally arise, and where electromagnetic duality behaves well. Subsequently, in section 4.1.1, employing electromagnetic duality, we will illustrate how the different global structures of this theory organize into closed orbits based on the topological properties of $\mathcal{N} = 4$. Building upon the example of $\mathcal{N} = 4$, in section 4.2 we will present the findings of [38], where the one-form symmetries of $\mathcal{N} = 3$ S-fold field theories were established, and the existence of non-invertible symmetries was conjectured, hypothesizing their existence in analogy to $\mathcal{N} = 4$.

4.1 A Useful Detour: 4d $\mathcal{N} = 4$ SYM

In this section we are going to give a simple review of the maximally supersymmetric Yang-Mills theory in four dimensions, which is going to be the basic toy model to understand the forthcoming discussion.

As introduced in section 1.1 the $\mathcal{N} = 4$ algebra in four dimensions has only one representation containing, in $\mathcal{N} = 1$ language, a vector multiplet and three complex chiral multiplets in the adjoint representation of the gauge group. This theory has an $\mathrm{SU}(4)_R$ R-symmetry under which the six real scalars transform in the rank-2 anti-symmetric representation, which is nothing but the fundamental representation of $\mathrm{SO}(6)$. The $\mathcal{N} = 1$ lagrangian for this theory is

$$\begin{aligned}\mathcal{L} = \frac{1}{32\pi}\,\mathrm{Im}\,\tau \int \mathrm{d}^2\theta\,\mathrm{Tr}\,W^\alpha W_\alpha + \int \mathrm{d}^2\theta\,\mathrm{d}^2\bar{\theta}\,\mathrm{Tr}\sum_A \bar{\Phi}_A e^{2gV}\Phi_A \\ - \int \mathrm{d}^2\theta\,\sqrt{2}g\,\mathrm{Tr}\,\Phi_1[\Phi_2, \Phi_3] + \mathrm{h.c.}\end{aligned} \tag{4.1}$$

where $\tau = \frac{\theta_I}{2\pi} + \frac{4\pi \mathrm{i}}{g^2}$ is the holomorphic coupling and θ_I is a topological term called instanton angle. More explicitly the lagrangian (4.1) takes the form

$$\begin{aligned}\mathcal{L} = \mathrm{Tr}\Big(- \frac{1}{2g_{\mathrm{YM}}^2} F_{\mu\nu}F^{\mu\nu} + \frac{\theta_I}{8\pi^2} F_{\mu\nu}\tilde{F}^{\mu\nu} - i\sum_a \bar{\lambda}^a \bar{\sigma}^\mu D_\mu \lambda_a - \sum_i D_\mu \phi^i D^\mu \phi^i \\ + g_{\mathrm{YM}} \sum_{a,b,i} C_i^{ab} \lambda_a [\Phi, \lambda_b] + g_{\mathrm{YM}} \sum_{a,b,i} \bar{C}_{iab} \bar{\lambda}^a \left[\phi^i, \bar{\lambda}^b\right] + \frac{g_{\mathrm{YM}}^2}{2} \sum_{i,j} \left[\phi^i, \phi^j\right]^2 \Big)\end{aligned} \tag{4.2}$$

where now ϕ are the real counterpart of the scalar components of the three complex Φ and C_i^{ab} are related to the Clifford matrices of $\mathrm{SO}(6)_R \sim \mathrm{SU}(4)_R$.

This theory is classically conformal invariant since the coupling constant g_{YM} has mass dimension zero. Thus, being also supersymmetric, the theory is actually superconformal and enjoys an $\mathrm{SU}(2, 2|4)$ symmetry. Remarkably, the theory is also believed to be UV finite, since upon perturbative renormalization no ultraviolet divergences arise. Also, instanton contributions are finite. Consequently, the β-function of this theory vanishes at all orders of perturbation theory and therefore the theory is conformal invariant also at the quantum level. It turns out that it is the only maximally superconformal theory in four dimensions[1].

This theory enjoys another symmetry acting on the holomorphic coupling τ. Indeed, the quantum theory is invariant under the shift $\theta_I \to \theta_I + 2\pi$, or equivalently $\tau \to \tau + 1$.

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[1] In general, it is believed that in any dimension there is only one maximally supersymmetric conformal field theory.

Montonen and Olive [270] conjectured that the quantum theory is also invariant under $\tau \to -1/\tau$. Combining these two symmetry transformations yields to the group $\mathrm{SL}(2,\mathbb{Z})$, known as S-duality group, acting projectively on the parameter τ

$$\tau \to \frac{a\tau + b}{c\tau + d}, \qquad ad - bc = 1, \qquad a, b, c, d \in \mathbb{Z}. \tag{4.3}$$

This result is quite interesting when looked through the eye of String Theory and M-theory. Indeed as breafily mentioned in section 2.2, $\mathcal{N} = 4$ SYM is the worldvolume theory of a stack of D3-branes and it should be evident that this $\mathrm{SL}(2,\mathbb{Z})$ transformation is inherited by the $\mathrm{SU}(1,1) \sim \mathrm{SL}(2,\mathbb{R})$ symmetry of type-IIB supergravity. In superstring theory the range of $\theta_I = 2\pi C$ is quantized so that one can identify $\theta_I \sim \theta_I + 2\pi$ and therefore the only allowed elements of $\mathrm{SL}(2,\mathbb{R})$ are the ones in the $\mathrm{SL}(2,\mathbb{Z})$ subgroup.
From the point of view of M-theory, $\mathcal{N} = 4$ SYM can be constructed as the dimensional reduction on a T^2 of 6d $\mathcal{N} = (2,0)$ SCFT which is the worldvolume theory of an M5-branes. Many of the properties of $\mathcal{N} = 4$ SYM can be understood from the geometry of M-theory on a 2-torus. In particular, S-duality is realized as the invariance of the holomorphic coupling τ under the action of $\mathrm{SL}(2,\mathbb{Z})$ which is the moduli space of complex structure of T^2. Not only that, but the holomorphic coupling τ is exactly related to the complex moduli of the torus.

4.1.1 Global Forms and S-duality Orbits

The discussion in the previous section does not reveal the full story. While we've established that $\mathrm{SL}(2,\mathbb{Z})$ acts as a symmetry of $\mathcal{N} = 4$ SYM, we've overlooked a crucial aspect: our discussion has been confined to the local dynamics of the theory. Although correlation functions remain invariant under modular transformations, focusing solely on local correlation functions overlooks the global structure of the gauge group. Delving into topological features, we find that S-duality takes on a more intricate form. The action of S-duality can map theories with distinct global structures but identical local dynamics, forming closed orbits known as S-duality orbits [16].
To understand global structures, we must explore extended operators, those supported on higher co-dimension manifolds, which act as probes for non-local dynamics. This section delves into the realm of line operators, namely Wilson, 't Hooft, and dyonic lines, that in modern language are to be considered as the operators carrying charge of some one-form symmetry.
For a gauge theory with a simply connected gauge group G, the classification of line operators relies on the structure of the universal cover of G, denoted $\tilde{G}$. This arises from the quotient $G = \tilde{G}/H$, where $H \subseteq Z(\tilde{G})$, the center of $\tilde{G}$. While local operators are governed solely by the gauge algebra $\mathfrak{g}$ and are thus invariant under H, Wilson line operators, tied to representations of G, are in one-to-one correspondence with the highest weight vectors of the algebra, which in turn are in bijection with elements of the Weyl chamber $\Lambda_w^G/\mathrm{Weyl}(G)$. Magnetic operators, known as 't Hooft lines, are more subtle. They are generally defined as singularities in the gauge field and are implemented, at the level of the partition function, by constraining the sum over gauge configurations

to those exhibiting such specific singular behavior. These can be understood in parallel to Wilson lines when transitioning to the dual "magnetic" frame, where they are described by functionals of the dual gauge field and are labeled by representations of the Langlands dual of the gauge group ${}^{\vee}G$ [199]. Consequently, 't Hooft lines are in one-to-one correspondence with elements of the Weyl chamber of the Langland dual algebra $\Lambda_{cw}/\text{Weyl}(G)$[2]. Dyonic lines, carrying both electric and magnetic charges, arise from combinations of Wilson and 't Hooft lines. Their representations are labeled by [245]

$$(\lambda_e, \lambda_m) \in \mathcal{L} = (\Lambda_w \times \Lambda_{cw})/\text{Weyl}(G). \tag{4.4}$$

However, not all dyonic lines manifest in a theory with a given gauge group G, as they are subject to various constraints

- The set of genuine lines has to form a commutative algebra [246], so that if (λ_e, λ_m) and (λ'_e, λ'_m) are allowed operators, then so is $(\lambda_e + \lambda'_e, \lambda_m + \lambda'_m)$. Additionally, if (λ_e, λ_m) is present, so is its orientation reversal $(-\lambda_e, -\lambda_m)$. This can be understood as the decomposition of representations into irreducible blocks, which contains the highest weight representation labeled by the sum.
- Gauge fields, defined by the adjoint representation of $\tilde{G}$, ensure that electric line operators corresponding to them are always present, being their parallel transport. These are labeled by $(r_e, 0)$ where r_e is a root (the weight of the adjoint representation). Similarly, purely magnetic lines $(0, r_m)$ with r_m a root of ${}^{\vee}\mathfrak{g}$ must also exist. Hence, inequivalent lines are labeled by elements of the weight lattice modulo the root lattice. This quotient defines the center of $\tilde{G}$, allowing genuine lines to be organized into pairs $(z_e, e_m) \in Z(\tilde{G}) \times Z(\tilde{G})$, given that $\tilde{G}$ and ${}^{\vee}\tilde{G}$ share the same center.
- Mutual locality, thought of as to the non-abelian analogue of the Dirac quantization condition [190, 245], imposes further constraints. Correlation functions of two lines should only depend on their support and charges; hence, since moving one line around the other results in an overall phase, it has to be set equal to one. This yields the condition

  $$\langle (z_e, z_m), (z'_e, z'_m) \rangle := \langle z_e, z'_m \rangle - \langle z'_e, z_m \rangle \equiv 0 \text{ mod } Z(\tilde{G}), \tag{4.5}$$

  where

  $$\langle \cdot\,, \cdot \rangle : Z(\tilde{G}) \times Z(\tilde{G}) \to \mathbb{Z} \tag{4.6}$$

  is the Dirac pairing. Here, $0 \text{ mod } Z(\tilde{G})$ signifies zero as an element of the center, i.e., when $Z(\tilde{G}) = \mathbb{Z}_N$, it becomes

  $$\langle (z_e, z_m), (z'_e, z'_m) \rangle = z_e z'_m - z_m z'_e \equiv 0 \text{ mod } N. \tag{4.7}$$

In the language of [189], a pure gauge theory with gauge group G has a one-form symmetry labelled by its center $Z(G)$ acting on the Wilson lines of the theory. Also, it enjoys

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[2] The Langlands dual algebra ${}^{\vee}\mathfrak{g}$ is generated by the co-roots $\frac{\alpha}{(\alpha,\alpha)}$.

a magnetic $(d-3)$-form symmetry, which in 4d is still a one-form, labelled by $\pi_1(G)$, acting on 't Hooft lines. In the presence of matter, the symmetry groups may be partially of totally broken due to screening phenomena.
We are now ready to consider the possible global structures of these theories. In particular, we will focus on $\mathcal{N}=4$ SYM with gauge algebra $\mathfrak{su}(N)$, but similar considerations can be made for cases with all other simply connected groups. The group $\mathrm{SU}(N)$ has center $Z(\mathrm{SU}(N))=\mathbb{Z}_N$ whose non-trivial subgroups are $\mathbb{Z}_k$ with $k|N$.

- $\mathrm{SU}(N)$: When the gauge group is $\mathrm{SU}(N)$, the maximal lattice is $\mathcal{L}\subset\mathbb{Z}_N^2$ and comprises of purely electric lines $(n,0)$ and dyonic lines with charge constrained by mutual locality

  $$\langle(n,0),(n',m)\rangle = nm \equiv 0 \bmod N, \qquad \forall n\in\mathbb{Z}_N, \tag{4.8}$$

  implying that the only possible 't Hooft lines are the ones with charge $m\equiv 0 \bmod N$, or kN with $k\in\mathbb{Z}$. These are the magnetic lines associated to the adjoint representation.

- $\mathrm{PSU}(N)\simeq\mathrm{SU}(N)/\mathbb{Z}_N$: In this case the center of the group is trivial and therefore the only purely electric line is the trivial dyon $(0,0)$. By mutual locality we can also see that the lines $(n,1)$ with $n=0,\dots,N-1$ are allowed and therefore all the dyonic lines of the form (nm,m) with $m=0,\dots,N-1$ are in the spectrum. For a fixed $n\neq0$ the only possible lines are the ones (nm,m), therefore we have N different possible theories labelled by n and they are all related by a shift of the θ-angle by the Witten effect

  $$(nm,m)\to(nm+m,m)=((n+1)m,m). \tag{4.9}$$

  This means that the shift $\theta\to\theta+2\pi$ changes the underlying line bundle

  $$\mathrm{PSU}(N)_n^{\theta+2\pi}=\mathrm{PSU}(N)^{\theta}_{(n+1)\bmod N}, \tag{4.10}$$

  and is therefore not a symmetry of the theory anymore. This enlarges the periodicity to $2\pi N$, which can be distinguished by the presence of a non-trivial class $w_b\in\mathrm{H}^2(\mathcal{M}^{(4)},\mathbb{Z}_N)$ that arises as a topological action in the partition function of the theory. The easiest example of this is the case of $\mathrm{SO}(3)\simeq\mathrm{SU}(2)/\mathbb{Z}_2$. Here we can distinguish two different theories, depicted in figure 4.1, usually denoted as $\mathrm{SO}(3)_\pm$. At the level of the bundle, this is a consequence of the presence of a non-trivial Stieffel-Witney class $w_2\in\mathrm{H}^2(\mathcal{M}^{(4)},\mathbb{Z}_2)$ of SO(3)-bundles. In partition function one has, schematically

  $$\mathcal{Z}_{\mathrm{SO}(3)_\pm}[\mathcal{M}^{(4)},\tau]=\sum_{E\to\mathcal{M}}\int\mathcal{D}A\mathcal{D}\Phi\; \mathrm{e}^{-S^{\mathcal{N}=4}[A,\Phi;\tau]}(\pm1)^{\int\frac{\mathcal{P}(w_2)}{2}}, \tag{4.11}$$

  where $\mathcal{P}(w_2)$ is the Pontryagin square.

- $\mathrm{SU}(N)/\mathbb{Z}_k$: Since $k|N$, consider k' an integer such that $kk'=N$. Then the al-

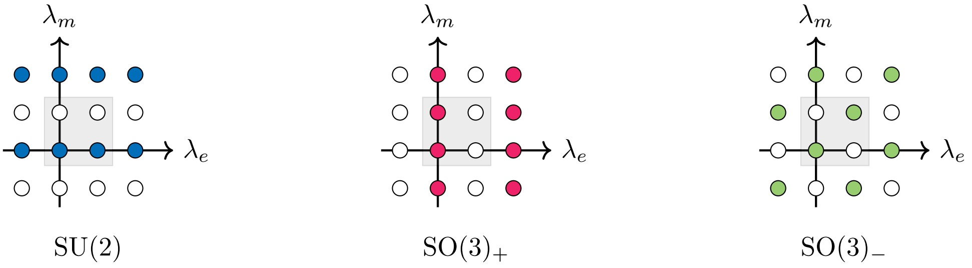


Figure 4.1: Possible line lattices for the different global structures in theories with gauge algebra $\mathfrak{su}(2)$. The shaded region highlights the $\mathbb{Z}_2$ charges.

lowed purely electric lines are $(n_e k \bmod N, 0)$ for every integer n_e. Mutual locality constraints dyons to be

$$\big(n_e(k,0) + n_m\,(n,k')\,\big) \bmod N, \tag{4.12}$$

with $n_m \in \mathbb{Z}$ and $n = 0, \cdots, k-1$ fixed. As before, we have a family of bundles $(\mathrm{SU}(N)/\mathbb{Z}_k)_n$ related by the Witten effect in a non-trivial way

$$(\mathrm{SU}(N)/\mathbb{Z}_k)_n^{\theta+2\pi} = (\mathrm{SU}(N)/\mathbb{Z}_k)^{\theta}_{(n+k') \bmod k}\,. \tag{4.13}$$

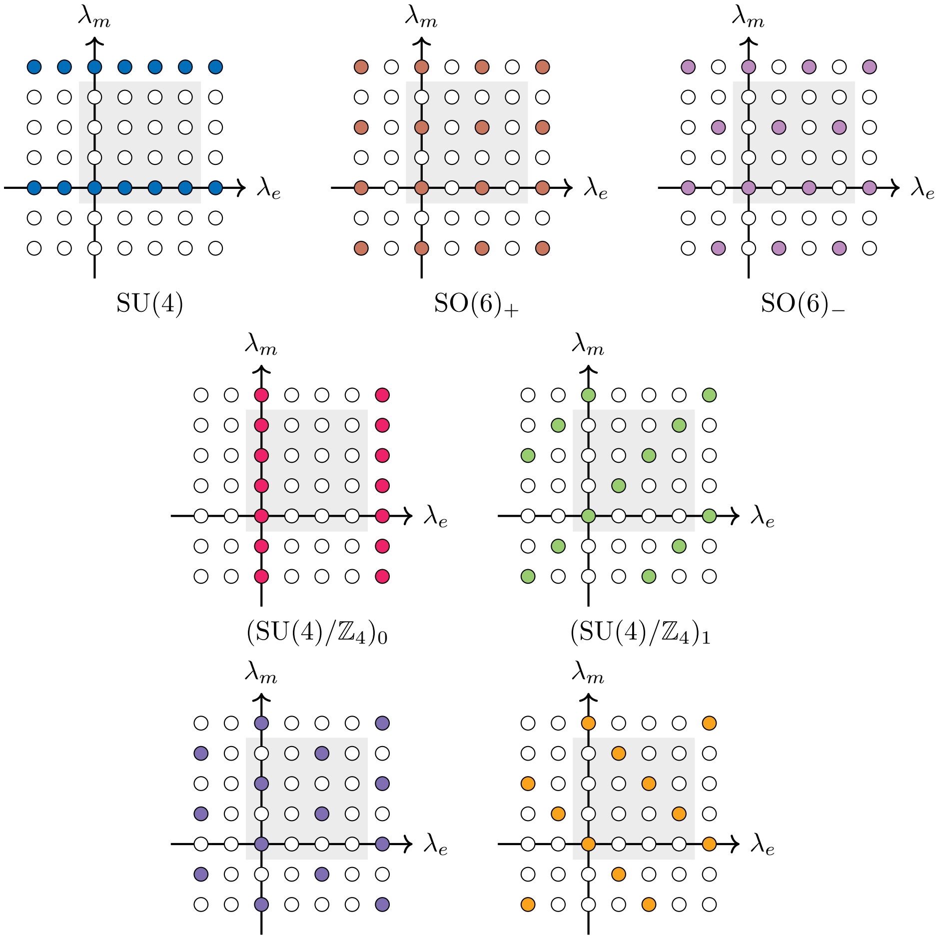
Yet, this is qualitatively different from the previous case. In fact, starting from a given n and shifting the θ angle we can only reach theories with $n' = n \bmod (\gcd(k,k'))$. This means that if $\gcd(k,k') = 1$, one can simply extend the θ angle periodicity. But when N is not square-free, i.e. some of its prime factors appear more than once in its prime decomposition, then we might have $\gcd(k,k') \neq 1$ and we have distinct orbits under shifts of θ. In this case, enlarging the periodicity of θ is not enough to cover all the configurations and the theory has discrete θ-angles. As an example of this is $\mathrm{SU}(4)/\mathbb{Z}_2 = \mathrm{SO}(6)$. We can verify the theory has 4 bundles, yet neither $(\mathrm{SU}(4)/\mathbb{Z}_2)_0$ and $(\mathrm{SU}(4)/\mathbb{Z}_2)_1$ nor $(\mathrm{SU}(4)/\mathbb{Z}_2)_2$ and $(\mathrm{SU}(4)/\mathbb{Z}_2)_3$ are connected through a θ periodicity, while $0 \leftrightarrow 2$ and $1 \leftrightarrow 3$ are. Therefore, the theory has 2 distinct theta angles θ_1 and θ_2 both with 4π periodicity. Again, at the bundle level this is the result of a non trivial $w_b \in H^2(\mathcal{M}, \mathbb{Z}_4)$

As concrete examples, in figures 4.1 and 4.2, we give the line lattices for theories with gauge algebra $\mathfrak{su}(2)$ and $\mathfrak{su}(4)$ respectively.

With this discussion in mind, we are now ready to understand how the elements of $\mathrm{SL}(2,\mathbb{Z})$ act on different global structures of $\mathcal{N} = 4$ SYM. For ease of discussion we will still focus on theories with gauge algebra $\mathfrak{g} = \mathfrak{su}(N)$. Here the Langland's dual algebra is $\mathfrak{su}(N)$ itself. The S generator of $\mathrm{SL}(2,\mathbb{Z})$ acts on the conjugacy classes of charges as

$$S : (z_e, z_m) \to (z_m, -z_e) \tag{4.14}$$

so that a given a lattice $\mathcal{L}_{n,k}$ of the theory $(\mathrm{SU}(N)/\mathbb{Z}_k)_n$, gets mapped to the lattice $\mathcal{L}_{k',n'}$



SU(4) $\quad$ $\mathrm{SO}(6)_+$ $\quad$ $\mathrm{SO}(6)_-$

$(\mathrm{SU}(4)/\mathbb{Z}_4)_0$ $\quad$ $(\mathrm{SU}(4)/\mathbb{Z}_4)_1$

$(\mathrm{SU}(4)/\mathbb{Z}_4)_2$ $\quad$ $(\mathrm{SU}(4)/\mathbb{Z}_4)_3$

Figure 4.2: Possible line lattices for the different global structures in theories with gauge algebra $\mathfrak{su}(4)$. The shaded region highlights the $\mathbb{Z}_4$ charges.

where the two factor are determined as the minimal charges of the form

$$(0, k'), (-N/k', n') \in \mathcal{L}_{k,n}. \tag{4.15}$$

The T generator, as it acts as a 2π shift in the θ-angle, maps theories as in (4.13). The two generators describe orbits in the space of theories, connecting different global structures. To illustrate the discussion we give the orbits of theories with $\mathfrak{g} = \mathfrak{su}(3)$ and $\mathfrak{g} = \mathfrak{su}(4)$ in figure 4.3.

Therefore, if one considers the global structure of these theories, the $\mathrm{SL}(2,\mathbb{Z})$ action is not a symmetry anymore since it changes the line lattice. But here comes the catch: one could generate a symmetry operation by going around these duality orbits at specific values of the gauge coupling. This is achieved by supplementing the action of either

$$\begin{tikzcd}
 & & (\mathrm{SU}(3)/\mathbb{Z}_3)_1 \arrow[dd, "T"'] \\
\mathrm{SU}(3) \arrow[loop left, "T"] \arrow[r, leftrightarrow, "S"] & (\mathrm{SU}(3)/\mathbb{Z}_3)_0 \arrow[ur, "T"] & \\
 & & (\mathrm{SU}(3)/\mathbb{Z}_3)_2 \arrow[ul, "T"] \arrow[uu, leftrightarrow, bend right, "S"']
\end{tikzcd}$$

$$\begin{tikzcd}
 & & (\mathrm{SU}(4)/\mathbb{Z}_4)_1 \arrow[dr, "T"] \arrow[dd, leftrightarrow, "S"'] & & & \\
\mathrm{SU}(4) \arrow[loop below, "T"] \arrow[r, leftrightarrow, "S"] & (\mathrm{SU}(4)/\mathbb{Z}_4)_0 \arrow[ur, "T"] & & (\mathrm{SU}(4)/\mathbb{Z}_4)_2 \arrow[dl, "T"] \arrow[r, leftrightarrow, "S"] & (\mathrm{SU}(4)/\mathbb{Z}_2)_+ & (\mathrm{SU}(4)/\mathbb{Z}_2)_+ \arrow[loop below, "{T,S}"] \\
 & & (\mathrm{SU}(4)/\mathbb{Z}_4)_3 \arrow[ul, "T"] & & &
\end{tikzcd}$$

Figure 4.3: S-duality orbits for $\mathcal{N} = 4$ SYM with gauge algebra $\mathfrak{g} = \mathfrak{su}(3)$ and $\mathfrak{g} = \mathfrak{su}(4)$ and different global structures.

S, T with a suitable gauging of a $\mathbb{Z}_k$ subgroups of the center [244]. In modern language what we want to do is to gauge subgroups of the one-form symmetry of the theory. By gauging we mean making the symmetry dynamical by turning on an associated background field, minimally couple it to the bulk theory and then sum over configurations of the background. Consider for example SYM with SU(2) gauge, and gauge the $\mathbb{Z}_2$ electric one-form symmetry of the theory. On the partition function this is implemented, schematically, as adding a background gauge field $B^{(e)}$ to it and then summing over bundles

$$\begin{aligned}
\sum_{[B^{(e)}]} \mathcal{Z}_{\mathrm{SU}(2)}[\tau; B^{(e)}] &= \sum_{[B^{(e)}]} \sum_{w_2(E)=B^{(e)}} \int \mathcal{D}A\mathcal{D}\Phi \; \mathrm{e}^{-S[A,\Phi,B^{(e)}]} \\
&= \sum_{E} \int \mathcal{D}A\mathcal{D}\Phi \; \mathrm{e}^{-S[A,\Phi]} = \mathcal{Z}_{\mathrm{SO}(3)_+}[\tau; 0].
\end{aligned} \tag{4.16}$$

Now one can imagine that the joint action of gauging the one-form symmetry, named σ in [244], followed by an S transformation will act as

$$\mathrm{SU}(2)[\tau] \xrightarrow{\sigma S} \mathrm{SU}(2)\left[-\frac{1}{\tau}\right], \tag{4.17}$$

which is going to be a symmetry of the theory when the coupling is $\tau = \mathrm{i}$. This is one example of a non-invertible self-duality defect. The non-invertibility comes from the gauging action which acts non-trivially on line operators.

This discussion can be further generalized to theories with general gauge groups by considering an additional operator. In the SU(2) case we can also get the $\mathrm{SO}(3)_-$ theory by stacking the partition function with a topological action[3], which can be considered as

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[3] In the literature this is known as Symmetry Protected Topological phase or SPT for short.

a new operator τ acting on $\mathrm{SU}(2)$ as

$$\tau : \mathrm{SU}(2) \rightarrow \mathrm{SU}(2) \otimes \mathrm{SPT}, \tag{4.18}$$

which just acts by stacking the partition function, in this case, with

$$\mathcal{Z}[\mathrm{SPT}] = \exp\left(\frac{\pi \mathrm{i}}{2} \int \mathcal{P}(\mathrm{B}^{(\mathrm{e})})\right) = (-1)^{\int \frac{\mathcal{P}(\mathrm{w}_2)}{2}} . \tag{4.19}$$

This is exactly the topological term we introduced in (4.11). This operator, together with gauging, are generators of an additional $\mathrm{SL}(2, \mathbb{Z}_2)$. By using combinations of these operators with the S, T transformations, one can construct various non-invertible symmetry generators [244].

4.2 Higher-form and Non-invertible Symmetries in $\mathcal{N} = 3$ S-folds

4.2.1 Introduction

In the recent past, a new paradigm for the notion of symmetry in QFTs became dominant. It is based on the necessity to include higher-form symmetries and the corresponding extended objects in the description of quantum field theories [189]. Restricting to four-dimensional QFTs, the simplest way to proceed consists in classifying the one-form symmetries in supersymmetric and conformal theories (SCFTs). A seminal paper that allowed for such a classification has been [16] where a general prescription was given in terms of the spectrum of mutually local Wilson and 't Hooft lines [245]. Such a prescription was initially based on the existence of a Lagrangian description for the SCFT under investigation. In absence of a Lagrangian description it is nevertheless possible to use other tools, coming from supersymmetry, holography and/or branes. These constructions have allowed to figure out the one-form symmetry structure of many different QFTs, including some 4d non-Lagrangian SCFTs, see [18, 51, 60, 68, 97, 100–102, 122, 153, 154, 171, 173, 174, 191, 223, 271].

A class of theories that has not been deeply investigated so far are SCFTs with 24 supercharges, i.e. $\mathcal{N} = 3$ conformal theories. Such models have been predicted in [10], and then found in [192]. Many generalizations have been then studied by using various approaches [17, 28, 58, 59, 117, 193, 242, 260]. A key role in the analysis of [192] is based on the existence, in the string theory setup, of non-perturbative extended objects that generalizes the notion of orientifolds, the S-folds (see [172, 229] for their original definition). From the field theory side, the projection implied by such S-folds on $\mathcal{N} = 4$ SYM has been associated to the combined action of an R-symmetry and an S-duality twist on the model at a fixed value of the holomorphic gauge coupling, where the global symmetry is enhanced by opportune discrete factors. Four possible $\mathbb{Z}_k$ have been identified, corresponding to $k = 2, 3, 4$ and 6. While the $\mathbb{Z}_2$ case corresponds to the original case of the orientifolds [47, 105, 106, 282, 283, 287], where actually the holomorphic gauge coupling does not require to be fixed, the other values of k correspond to new projections that can break supersymmetry down to $\mathcal{N} = 3$. The analysis has been further refined in [17], where the discrete torsion, in analogy with the case of orientifolds, has been added to this description. In this way, it has been possible to achieve a classification of such $\mathcal{N} = 3$ S-folds SCFT in terms of the Shephard–Todd complex reflection groups.

The goal of this Chapter consists in classifying one-form symmetries for such theories, constructing the lattices of lines and identifying which models possess non-invertible symmetries. The main motivation behind this expectation is that for the rank-2 S-folds, in absence of discrete torsion, the SCFTs enhance to $\mathcal{N} = 4$ SYM [17] where these properties are present. The existence of non-trivial one-form symmetries in some exceptional $\mathcal{N} = 3$ theories has also been argued in [136].

Our strategy adapts the one presented in [173] to S-fold setups. There, the spectrum of lines is built from the knowledge of the electromagnetic charges of massive states in a generic point of the Coulomb branch. These charges are read from the BPS quiver, under the assumption that the BPS spectrum is a good representative of the whole spectrum of electromagnetic charges. In the case of S-folds however such a BPS quiver description has not been worked out, and we extract the electromagnetic charges of dynamical particles from the knowledge of the (p, q)-strings configurations in the Type IIB setup [288, 311]. The main assumption behind the analysis is that such charges are a good representative of the electromagnetic spectrum.

We proceed as follows. First we choose an $\mathcal{N} = 3$ theory constructed via an S-fold projection of Type IIB. This consists in having N D3-branes, together with their images, on the background of an S-fold. At a generic point of the Coulomb branch, the corresponding low energy gauge dynamics corresponds to a $\mathrm{U}(1)^N$ gauge theory where each $\mathrm{U}(1)$ is associated to a D3. Then we list all (p, q)-strings that can be stretched between D3-branes and their images. They have electric and magnetic charges with respect to $\mathrm{U}(1)^N$. Eventually we run the procedure of [173]. This consist in finding all the lines that are genuine, i.e. have integer Dirac pairing with the local particles, modulo screening by the dynamical particles. This gives the lattice of possible charges, then the different global structures correspond to maximal sub–lattices of mutually local lines.

Table 4.1: Summary of our results. $\mathbb{1}$ represents a trivial group.

S-fold	One-form symmetry	# of inequivalent lattices	Non-invertible symmetry
$S_{3,1}$	$\mathbb{Z}_3$	2	Yes
$S_{3,3}$	$\mathbb{1}$	1	No
$S_{4,1}$	$\mathbb{Z}_2$	2	Yes
$S_{4,4}$	$\mathbb{1}$	1	No
$S_{6,1}$	$\mathbb{1}$	1	No

Our results are summarized in Table 4.1. In the first column, one finds the type of S-fold projection that has been considered. Such projections are identified by the two integers k and ℓ in $S_{k,\ell}$. The integer k corresponds to the $\mathbb{Z}_k$ projection while the second integer ℓ is associated to the discrete torsion. Then, when considering an $S_{k,\ell}$ S-fold on a stack of N D3-branes the complex reflection group associated to such a projection is $G(k, k/\ell, N)$. In the second column, we provide the one-form symmetry that we found in our analy-

sis, and in the third, the number of inequivalent line lattices that we have obtained. The last column specifies whether there exist cases that admit non-invertible symmetries. Indeed, here we find that in some of the cases there exists a zero-form symmetry mapping some of the different line lattices, that are therefore equivalent. Furthermore in such cases we expect the existence of non-invertible symmetries obtained by combining the zero-form symmetry with a suitable gauging of the one-form symmetry.
A remarkable observation strengthening our results regards the fact that our analysis reproduces the limiting $G(k,k,2)$ cases, where supersymmetry enhances to $\mathcal{N}=4$ with $\mathfrak{su}(3)$, $\mathfrak{so}(5)$ and $\mathfrak{g}_2$ gauge groups for $k=3$, 4 and 6 respectively. Another check of our result is that it matches with the cases $G(3,1,1)$ and $G(3,3,3)$, where an $\mathcal{N}=1$ Lagrangian picture has been worked out in [320].

4.2.2 Generalities

4.2.2.1 Global Structures from the IR

The strategy adopted here, as already discussed in the introduction, is inspired by the one of [173]. The main difference is that instead of using BPS quivers, not yet available for our S-folds, we take advantage of the type IIB geometric setups and probe the charge spectrum with (p,q)-strings – the bound state of p fundamental strings F1 and q Dirichlet strings D1.[4]
Despite this difference, the rest of the procedure is the one of [173] which we now summarize. Denote as

$$\gamma^i = (e_1^{(i)}, m_1^{(i)}; \ldots ; e_r^{(i)}, m_r^{(i)}) \tag{4.20}$$

a basis vector of the electromagnetic lattice of dynamical state charges under the $\mathrm{U}(1)_e^r \times \mathrm{U}(1)_m^r$ gauge symmetry on the Coulomb branch. The spectrum of lines can be determined by considering a general line $\mathcal{L}$ with charge

$$\ell = (e_1^{(l)}, m_1^{(l)}; \ldots ; e_r^{(l)}, m_r^{(l)})\,. \tag{4.21}$$

This is a genuine line operator if the Dirac pairings with all dynamical states Ψ are integer

$$\langle \Psi, \mathcal{L} \rangle \in \mathbb{Z} \qquad \forall\, \Psi\,. \tag{4.22}$$

This can be rephrased as the condition

$$\sum_{j=1}^{r} e_j^{(i)} m_j^{(l)} - m_j^{(i)} e_j^{(l)} \in \mathbb{Z} \qquad \forall\, i\,. \tag{4.23}$$

Furthermore, inserting a local operator with charge γ_i on the worldline of a line with charge ℓ shifts its charge by γ_i. Therefore if a line with charge ℓ appears in the spectrum then a line with charges $\ell + \sum k_i \gamma_i$ with $k_i \in \mathbb{Z}$ must also appear. When classifying the spectrum of charges of the line operators of a QFT it is then useful to consider the

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[4] In order to provide the IR spectrum of line operators of the SCFTs from this UV perspective, we assume the absence of wall-crossing. While such an assumption is *a priori* motivated by the high degree of supersymmetry, *a posteriori* it is justified by the consistency of our results with the literature

charges ℓ modulo these insertions of local states. This gives rise to equivalence classes of charges with respect to the relation

$$\ell \sim \ell + \gamma_i \qquad \forall\, i\,. \tag{4.24}$$

Borrowing the nomenclature of [173], we will refer to such identification as screening and we will work with each equivalence class by picking one representative. The genuine lines after screening form a lattice. In general two such lines are not mutually local and a choice of global structure corresponds to a choice of a maximal sublattice of mutually local lines.

4.2.2.2 Charged States in $S_{k,l}$-folds

We aim to determine the electromagnetic charges of the local states generated by (p, q)-strings stretched between (images of) D3-branes in presence of an S-fold. The S-fold background of Type IIB string theory consist of a spacetime $\mathbb{R}^4 \times (\mathbb{R}^6/\mathbb{Z}_k)$ where the $\mathbb{Z}_k$ quotient involves an S-duality twist by an element $\rho_k \in \mathrm{SL}(2,\mathbb{Z})$ of order k, where $k = 2, 3, 4, 6$. For $k > 2$ the value of the axio-dilaton vev is fixed by the requirement that it must be invariant under the modular transformation associated to ρ_k. The matrices ρ_k and the corresponding values[5] of τ are given in Table 4.2.

Table 4.2: Elements ρ_k of $\mathrm{SL}(2,\mathbb{Z})$ of order k used in S-fold projections, and the corresponding fixed coupling τ.

$\mathrm{SL}(2,\mathbb{Z})$	$S^2 = -\mathbb{I}_2$	$(ST)^{-1}$	S	$(S^3T)^{-1}$
k	2	3	4	6
ρ_k	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$
ρ_k^{-1}	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$
τ	any τ	$\mathrm{e}^{2\mathrm{i}\pi/3}$	i	$\mathrm{e}^{2\mathrm{i}\pi/3}$

A stack of N D3-branes probing the singular point of the S-fold background engineer an $\mathcal{N} = 3$ field theory on the worldvolume of the stack of D3-branes. It is useful to consider the k-fold cover of spacetime, and visualize the N D3-branes together with their $(k-1)N$ images under the S_k-fold projection. We are going to label the m-th image of the i-th D3-brane with the index i_m, where $i = 1, \dots, N$ and $m = 1, \dots, k$.
Under the S-fold projection, the two-form gauge fields of the closed string sector B_2 and

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[5] In our convention, an $\mathrm{SL}(2,\mathbb{Z})$ transformation of the axio-dilaton $\tau \to (a\tau + b)/(c\tau + d)$ relates to a matrix $\rho_k = \begin{pmatrix} d & c \\ b & a \end{pmatrix}$. We also have $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$.

C_2 transform in the fundamental representation

$$\begin{pmatrix} B_2 \\ C_2 \end{pmatrix} \to \rho_k \begin{pmatrix} B_2 \\ C_2 \end{pmatrix} . \tag{4.25}$$

Consistently, the (p, q) strings charged under these potentials are mapped to (p', q') where

$$(p'\ q') = (p\ q) \cdot \rho_k^{-1} . \tag{4.26}$$

We denote a state associated to a (p, q) connecting the i_m-th D3-brane and the j_n D3-brane as

$$|p, q\rangle_{i_m, j_n} = |-p, -q\rangle_{j_n, i_m} , \tag{4.27}$$

where we identity states with both opposite charges and orientation.
First, strings linking branes in the same copy of $\mathbb{R}^6/\mathbb{Z}_k$ transform as follows

$$|p, q\rangle_{i_m, j_m} \to \zeta_k^{-1} |p', q'\rangle_{i_{m+1}, j_{m+1}} , \tag{4.28}$$

where (p', q') are related to (p, q) by (4.26) and ζ_k is the primitive k-th root of unity. These states always collectively give rise to a single state in the quotient theory, with charges

$$\mathrm{D3}_i \mathrm{D3}_j \ : \ (0, 0; \ldots ; \overbrace{p, q}^{i\text{-th}} ; \ldots ; \overbrace{-p, -q}^{j\text{-th}} ; \ldots ; 0, 0) . \tag{4.29}$$

An important ingredient we need to add to our picture is the discrete torsion for B_2 and C_2 [17, 310]. In presence of such a discrete torsion, a string going from the i_m-th brane to the j_{m+1}-th brane should pick up an extra phase which depends only on its (p, q)-charge and the couple $(\theta_{\mathrm{NS}}, \theta_{\mathrm{RR}})$. More precisely, one expects that the S-fold action can be written as follows [219]:[6]

$$|p, q\rangle_{i_m j_{m+1}} \to \zeta_k^{-1} \mathrm{e}^{2\pi \mathrm{i}(p\theta_{\mathrm{NS}} + q\theta_{\mathrm{RR}})} \, |p', q'\rangle_{i_{m+1} j_{m+2}} , \tag{4.30}$$

where again (p', q') are related to (p, q) by (4.26). For $i \neq j$, this always leads to the following state in the projected theory [213, 233]:[7]

$$\mathrm{D3}_i \mathrm{D3}_j^{\rho} \ : \ (0, 0; \ldots ; \overbrace{p, q}^{i\text{-th}} ; \ldots ; \overbrace{-(p\ q) \cdot \rho_k}^{j\text{-th}} ; \ldots ; 0, 0) . \tag{4.31}$$

Note that this is the only case that might not lead to any state in the quotient theory when $i = j$, i.e. when a string links a brane and its image. When the quotient state exists, it has

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[6] We thank Shani Meynet for pointing out [219] to us.

[7] The action on (p, q) involves ρ_k^{-1}, see (4.26). In writing (4.31) however, we measure the charge with respect to the brane in the chosen fundamental domain, hence the appearance of ρ_k instead of its inverse.

charges

$$\mathrm{D3}_i\mathrm{D3}_i^{\rho} \;:\; (0,0;\ldots;\overbrace{(p\,q)-(p\,q)\cdot\rho_k}^{i\text{-th}};\ldots;0,0)\,. \tag{4.32}$$

Analogously, strings twisting around the S-fold locus n-times pick up n-times the phase in (4.30).

A last remark is that discrete torsion allows some strings to attach to the S-fold if the latter has the appropriate NS and/or RR charge. If this is the case, the state is mapped as in (4.28)

$$|p,q\rangle_{S_k i_m} \to |p',q'\rangle_{S_k i_{m+1}}\;, \tag{4.33}$$

and provides the following charge in the projected theory

$$S_k\mathrm{D3}_i \;:\; (0,0;\ldots;\overbrace{\,p,q\,}^{i\text{-th}};\ldots;0,0)\,. \tag{4.34}$$

These rules are illustrated and details on discrete torsion are provided in the remaining of this section for orientifolds and S-folds separately.

The case with $k=2$: orientifolds In this subsection we apply the formalism described above for orientifolds and reproduce the spectrum of strings known in the literature. The matrix ρ_2 is diagonal, therefore the two p and q factors can be considered independently. In this case the field theory obtained after the projection is Lagrangian and can be studied in perturbative string theory with unoriented strings. Discrete torsion takes value in $(\theta_{\mathrm{NS}},\theta_{\mathrm{RR}})\in\mathbb{Z}_2\oplus\mathbb{Z}_2$, giving four different choices of O3-planes related by $\mathrm{SL}(2,\mathbb{Z})$ actions [310], see Table 4.3.

Table 4.3: Different discrete torsions on O3-planes.

$O3$-planes	$O3^-$	$O3^+$	$\widetilde{O3}^-$	$\widetilde{O3}^+$
$(\theta_{\mathrm{NS}},\theta_{\mathrm{RR}})$	$(0,0)$	$(1/2,0)$	$(0,1/2)$	$(1/2,1/2)$

The orientifold action is then recovered from (4.28) and (4.30) with $\zeta_2=-1$. First, we have

$$|p,q\rangle_{i_1j_1} \to -|-p,-q\rangle_{i_2j_2} = -|p,q\rangle_{j_2i_2}\,. \tag{4.35}$$

For the strings that stretch from one fundamental domain of $\mathbb{R}^6/\mathbb{Z}_2$ to the next, there are four cases depending on the values of θ_{NS} and θ_{RR}

$$\begin{aligned}
O3^- &\;:\; |p,q\rangle_{i_1j_2} \to -|p,q\rangle_{j_1i_2}\,,\\
O3^+ &\;:\; |p,q\rangle_{i_1j_2} \to -\mathrm{e}^{p\pi\mathrm{i}}|p,q\rangle_{j_1i_2}\,,\\
\widetilde{O3}^- &\;:\; |p,q\rangle_{i_1j_2} \to -\mathrm{e}^{q\pi\mathrm{i}}|p,q\rangle_{j_1i_2}\,,\\
\widetilde{O3}^+ &\;:\; |p,q\rangle_{i_1j_2} \to -\mathrm{e}^{(p+q)\pi\mathrm{i}}|p,q\rangle_{j_1i_2}\,.
\end{aligned} \tag{4.36}$$

It is interesting to consider strings connecting one brane to its image, $i = j$. In the case of trivial discrete torsion, corresponding to the $O3^-$-plane, all such strings are projected out. On the contrary, in the $O3^+$ case, an F1-string linking mirror branes survives the projection, while a D1-string similarly positioned is projected out. We also find strings that can attach to the different orientifolds following [213]

$$O3^- \ : \ \text{none}\,, \quad O3^+ \ : \ |0,1\rangle_{O3^+ i_m}\,, \quad \widetilde{O3}^- \ : \ |1,0\rangle_{\widetilde{O3}^- i_m}\,, \quad \widetilde{O3}^+ \ : \ |1,1\rangle_{\widetilde{O3}^+ i_m}\,, \tag{4.37}$$

as well as bound states of these.

The cases with $k > 2$: S-folds The construction discussed above can be applied to $S_{k>2}$ in order to obtain the string states in the quotient theory. For $k > 2$, the discrete torsion groups have been computed in [17], the result being $\theta_{\text{NS}} = \theta_{\text{RR}} \in \mathbb{Z}_3$ for the S_3-case and $\theta_{\text{NS}} = \theta_{\text{RR}} \in \mathbb{Z}_2$ for the S_4-case. The S_6-fold does not admit non-trivial discrete torsion. It was also pointed out that, for the S_3-case, the choices $\theta_{\text{NS}} = \theta_{\text{RR}} = 1/3$ and $\theta_{\text{NS}} = \theta_{\text{RR}} = 2/3$ are related by charge conjugation; therefore everything boils down to whether the discrete torsion is trivial or not. Following the notation of [17], we denote as $S_{k,1}$ the S-folds with trivial discrete torsion and as $S_{k,k}$ the S-folds with non-trivial discrete torsion.

As before, the only states that might not lead to any state in the quotient theory are the strings linking different covers of $\mathbb{R}^6/\mathbb{Z}_k$. (4.30) generalizes in the following way [219]: a state $|p,q\rangle_{i_m j_n}$ is mapped to $\zeta_k^{-1} \mathrm{e}^{2\pi \mathrm{i}(p\theta_{\text{NS}} + q\theta_{\text{RR}})}\, |p',q'\rangle_{i_{m+1} j_{n+1}}$ with (p',q') obtained as in (4.26). In more details

$$\begin{array}{lcl}
S_{3,1} & : & |p,q\rangle_{i_1 j_{m+1}} \to \mathrm{e}^{-\mathrm{i}2\pi/3} |q-p,-p\rangle_{i_2 j_{m+2}}\,, \\
S_{3,3} & : & |p,q\rangle_{i_1 j_{m+1}} \to \mathrm{e}^{-\mathrm{i}2\pi/3} \mathrm{e}^{\mathrm{i}m(p+q)2\pi/3} |q-p,-p\rangle_{i_2 j_{m+2}}\,, \\
S_{4,1} & : & |p,q\rangle_{i_1 j_{m+1}} \to \mathrm{e}^{-\mathrm{i}\pi/2} |-q,p\rangle_{i_2 j_{m+2}}\,, \\
S_{4,4} & : & |p,q\rangle_{i_1 j_{m+1}} \to \mathrm{e}^{-\mathrm{i}\pi/2} \mathrm{e}^{\mathrm{i}m(p+q)\pi} |-q,p\rangle_{i_2 j_{m+2}}\,, \\
S_{6,1} & : & |p,q\rangle_{i_1 j_{m+1}} \to \mathrm{e}^{-\mathrm{i}\pi/3} |p-q,p\rangle_{i_2 j_{m+2}}\,.
\end{array} \tag{4.38}$$

This shows that no state is projected out for $S_{3,1}$ and $S_{3,3}$. Analogously to the orientifold cases, we project out some strings linking mirror branes: $|p,q\rangle_{i_n i_{n+2}}$ in $S_{4,1}$ and $S_{4,4}$, and $|p,q\rangle_{i_n i_{n+3}}$ in $S_{6,1}$ respectively.

Finally, we get extra strings linking the S-fold to D-branes for the cases with discrete torsion. Following the discussion in [233], we know that these S-folds admit all kinds of p and q numbers

$$S_{3,3} \ : \ |p,q\rangle_{S_{3,3} i_n}\,, \qquad S_{4,4} \ : \ |p,q\rangle_{S_{4,4} i_n}\,. \tag{4.39}$$

4.2.2.3 Dirac Pairing from (p,q)-strings

Having determined the states associated to (p,q)-strings that survive the S-fold projection we now analyze the electromagnetic charges of these states. It is useful to consider the system of a stack of D3-branes and an $S_{k,\ell}$-fold on a generic point of the Coulomb branch. This corresponds to moving away the D3-branes from the S-plane. On a generic point of the Coulomb branch, the low energy theory on the D3-branes is a $\mathrm{U}(1)_i^N$ gauge symmetry, where each $\mathrm{U}(1)_i$ factor is associated to the i-th D3-brane. The theory includes massive charged states generated by the (p, q)-strings studied in the previous section. A (p, q)-string stretched between the i-th and j-th D3-brane has electric charge p and magnetic charge q under $\mathrm{U}(1)_i$ as well as electric charge $-p$ and magnetic charge $-q$ under $\mathrm{U}(1)_j$, and is neutral with respect to other branes. We organize the charges under the various $\mathrm{U}(1)$s in a vector

$$(e_1, m_1; e_2, m_2; \dots; e_N, m_N) \tag{4.40}$$

where e_i and m_i are the electric and magnetic charge under $\mathrm{U}(1)_i$, respectively. In this notation the charge of a string stretched between the i-th and j-th D3-brane in the same cover of $\mathbb{R}^6/\mathbb{Z}_2$ has charge

$$\mathrm{D3}_i\mathrm{D3}_j : (0, 0; \dots; \overbrace{p, q}^{i-th}; 0, 0; \dots; \overbrace{-p, -q}^{j-th}; \dots)\,, \tag{4.41}$$

where the dots stand for null entries. We will keep using this notation in the rest of the Chapter. A (p, q)-string stretched between the i-th D3-brane and the l-th image of the j-th D3-brane imparts electromagnetic charges (p, q) under $\mathrm{U}(1)_i$ and charges $-(p, q)\rho_k^l$ under $\mathrm{U}(1)_j$. In formulas

$$\mathrm{D3}_i\mathrm{D3}_j^{\rho^l} : (0, 0; \dots; \overbrace{p, q}^{i-th}; 0, 0; \dots; \overbrace{-(p\; q) \cdot \rho_k^l}^{j-th}; \dots)\,. \tag{4.42}$$

The last ingredient for our analysis is given by the Dirac pairing between two states. Consider a state Ψ with charges e_i, m_i under $\mathrm{U}(1)_i$ and a state Ψ' with charges e_i', m_i' under $\mathrm{U}(1)_i$. The pairing between F1 and D1-strings in Type IIB dictates that the Dirac pairing between these states is given by

$$\langle \Psi, \Psi' \rangle = \sum_{i=1}^{N} (e_i m_i' - m_i e_i')\,. \tag{4.43}$$

By using this construction we can reproduce the usual Dirac pairing of $\mathcal{N} = 4$ SYM with $ABCD$ gauge algebras. As an example we now reproduce the Dirac pairing of D_N, engineered as a stack of N D3-branes probing an $O3^-$-plane. In this case the allowed (p, q)-strings have the following charges

$$\begin{aligned} &\mathrm{D3}_i\mathrm{D3}_j : (0, 0; \dots; \overbrace{p, q}^{i-th}; 0, 0; \dots; \overbrace{-p, -q}^{j-th}; \dots) \\ &\mathrm{D3}_i\mathrm{D3}_j^{\rho} : (0, 0; \dots; \overbrace{p, q}^{i-th}; 0, 0; \dots; \overbrace{p, q}^{j-th}; \dots) \end{aligned} \tag{4.44}$$

The states associated to $(1, 0)$-strings correspond to the $\mathcal{W}$ bosons while the states associated to $(0, 1)$-strings correspond to magnetic monopoles $\mathcal{M}$. For each root $\mathcal{W}_i$ of D_N

let $\mathcal{M}_i$ be the corresponding coroot. More precisely if $\mathcal{W}_i$ is associated to a $(1,0)$-string connecting two D3-branes, then the coroot $\mathcal{M}_i$ corresponds to the string $(0,1)$ stretched between the same pair of D3-branes. The only non-vanishing Dirac pairing is the one between a $\mathcal{W}_i$ boson and an $\mathcal{M}_j$ monopole. This pairing between the simple (co)roots $\mathcal{W}_i$ and $\mathcal{M}_j$ is given by the intersection between $\mathcal{W}_i$ and $\mathcal{W}_j$, explicitly

$$\langle \mathcal{W}_i, \mathcal{M}_j \rangle = (A_{D_N})_{i,j} \, , \tag{4.45}$$

where A_{D_N} is the Cartan matrix of the D_N algebra, corresponding to an $\mathfrak{so}(2N)$ gauge theory. Indeed the intersection between F1 strings in the background of an $O3^-$ reproduces the intersection of the roots of D_N. The Dirac pairing (4.45) reproduces the Dirac pairing of $\mathfrak{so}(2N)$ $\mathcal{N}=4$ SYM. Similar constructions for $O3^+$, $\widetilde{O3}^-$, and $\widetilde{O3}^+$ lead to the B and C cases (while branes in absence of orientifold would give A). The corresponding gauge algebras are summarized in Table 4.4.

Table 4.4: F1-string, D1-string, and the F1-D1 bound state providing respectively the electric, magnetic, and dyonic charges of the projected $\mathcal{N}=4$ gauge theory.

O3-planes	F1-string	D1-string	F1-D1 bound state
$O3^-$	$\mathfrak{so}(2N)$	$\mathfrak{so}(2N)$	$\mathfrak{so}(2N)$
$O3^+$	$\mathfrak{usp}(2N)$	$\mathfrak{so}(2N+1)$	$\mathfrak{usp}(2N)$
$\widetilde{O3}^-$	$\mathfrak{so}(2N+1)$	$\mathfrak{usp}(2N)$	$\mathfrak{usp}(2N)$
$\widetilde{O3}^+$	$\mathfrak{usp}(2N)$	$\mathfrak{usp}(2N)$	$\mathfrak{so}(2N+1)$

4.2.2.4 Lines in O3-planes

Before moving to new results, we illustrate our method with well understood O3-planes. Specifically, we consider placing $N=2$ D3-branes in the background of an O3$^+$-plane.

In this specific example, the F1-strings corresponding to elementary dynamical states in the quotient theory can be chosen to be $|1,0\rangle_{1_2 1_1}$ and $|1,0\rangle_{1_1 2_1}$. The first links the $i=1$ brane to its mirror (D3$^\rho_1$D3$_1$) and the second links the $i=1$ to the $i=2$ brane (D3$_1$D3$_2$). A pictorial representation of this setup is shown in Figure 4.4. In the notation of the previous section, they lead to $\mathcal{W}_i$-bosons in the gauge theory with the following charge basis

$$\text{D3}^\rho_1\text{D3}_1 \, : \, w_1 = (2,0;0,0) \, , \quad \text{D3}_1\text{D3}_2 \, : \, w_2 = (-1,0;1,0) \, . \tag{4.46}$$

These generate the algebra $\mathfrak{usp}(4)$ of electric charges. The elementary magnetic monopoles $\mathcal{M}_i$ come from the D1-strings $|0,1\rangle_{O3^+1_1}$ and $|0,1\rangle_{1_1 2_1}$, and provide the following charges

$$O3^+\text{D3}_1 \, : \, m_1 = (0,1;0,0) \, , \quad \text{D3}_1\text{D3}_2 \, : \, m_2 = (0,-1;0,1) \, . \tag{4.47}$$

This generates the algebra $\mathfrak{so}(5)$ of magnetic charges. Finally, the elementary $(1,1)$-strings leading to states in the quotient theory can be chosen to be $|1,1\rangle_{1_2 1_1}$ and $|1,1\rangle_{1_1 2_1}$,

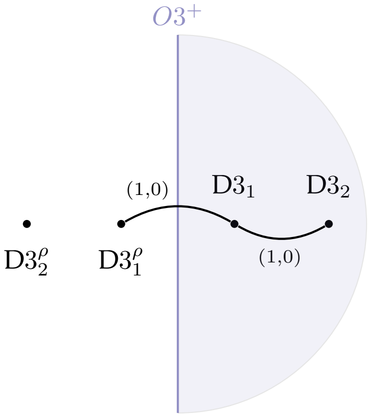

Figure 4.4: A pictorial representation of two D3-branes probing the $O3^+$ orientifold on a generic point of the Coulomb branch. The light blue shaded area is a possible choice of fundamental domain under the spacetime identification induced by the orientifold. Black (gray) dots represent (images of) D3-branes. Black lines correspond to (p, q)-strings stretched between D3-branes. In particular, we drew (p, q)-strings generating the $\mathcal{W}$-bosons corresponding to simple roots $\mathcal{N} = 4$ $\mathfrak{usp}(4)$ SYM.

i.e. $\mathrm{D3}_1^\rho\mathrm{D3}_1$ and $\mathrm{D3}_1\mathrm{D3}_2$ respectively. They provide dyons $\mathcal{D}_i$

$$\mathrm{D3}_1^\rho\mathrm{D3}_1 \,:\, d_1 = (2, 2; 0, 0)\,, \quad \mathrm{D3}_1\mathrm{D3}_2 \,:\, d_2 = (-1, -1; 1, 1)\,, \tag{4.48}$$

which reproduces an $\mathfrak{usp}(4)$ algebra. We will limit ourselves to considering the $\mathcal{W}$-bosons and magnetic monopoles $\mathcal{M}$. Indeed, they generate the full lattice of electromagnetic charges admissible in the orientifold theory. See that

$$d_1 = w_1 + 2m_1 \quad d_2 = w_2 + m_2\,. \tag{4.49}$$

Clearly, all other allowed (p, q)-charges can be reconstructed in this way. The Dirac pairing between these elementary electromagnetic charges reads

$$\begin{aligned} &\langle \mathcal{W}_1, \mathcal{W}_2 \rangle = \langle \mathcal{M}_1, \mathcal{M}_2 \rangle = 0\,, \\ &\langle \mathcal{M}_1, \mathcal{W}_2 \rangle = 1\,, \\ &\langle \mathcal{W}_1, \mathcal{M}_1 \rangle = \langle \mathcal{M}_2, \mathcal{W}_1 \rangle = \langle \mathcal{W}_2, \mathcal{M}_2 \rangle = 2\,. \end{aligned} \tag{4.50}$$

Now, introduce a line operator $\mathcal{L}$ with charge vector ℓ. It is convenient to express it in the basis of dynamical charges

$$\ell = \alpha_1 w_1 + \alpha_2 w_2 + \beta_1 m_1 + \beta_2 m_2\,, \tag{4.51}$$

where α_i and β_i to be determined. Screening with respect to $\mathcal{W}_1$ and $\mathcal{W}_2$ imposes

$$\alpha_1 \sim \alpha_1 + 1\,, \quad \alpha_2 \sim \alpha_2 + 1\,, \tag{4.52}$$

respectively, while screening with respect to $\mathcal{M}_1$ and $\mathcal{M}_2$ imposes

$$\beta_1 \sim \beta_1 + 1\,, \quad \beta_2 \sim \beta_2 + 1\,. \tag{4.53}$$

Mutual locality with respect to the dynamical charges requires the quantities

$$\begin{aligned} \langle \mathcal{L}, \mathcal{W}_1 \rangle &= -2\beta_1 + 2\beta_2\,, \qquad & \langle \mathcal{L}, \mathcal{W}_2 \rangle &= \beta_1 - 2\beta_2\,, \\ \langle \mathcal{L}, \mathcal{M}_1 \rangle &= 2\alpha_1 - \alpha_2\,, \qquad & \langle \mathcal{L}, \mathcal{M}_2 \rangle &= -2\alpha_1 + 2\alpha_2\,, \end{aligned} \tag{4.54}$$

to be integers. All these constraints set

$$\alpha_1 = \frac{e}{2} \quad \alpha_2 = 0\,, \quad \beta_1 = 0\,, \quad \beta_2 = \frac{m}{2} \quad \mod 1\,, \tag{4.55}$$

with $e, m = 0, 1$. Linearity of the Dirac pairing then guarantees mutual locality with respect to the full dynamical spectrum. Thus, the charge of the most general line (modulo screening) must read

$$\ell_{e,m} = \frac{1}{2}(2e, -m; 0, m)\,. \tag{4.56}$$

A choice of global structure consists in finding a set of mutually local lines. The mutual locality condition between two lines $\mathcal{L}$ and $\mathcal{L}'$ with charges $\ell_{e,m}$ and $\ell_{e',m'}$ is given by

$$\langle \mathcal{L}, \mathcal{L}' \rangle = \frac{1}{2}(-em' + e'm) \in \mathbb{Z}\,. \tag{4.57}$$

Equivalently

$$em' - me' = 0 \bmod 2\,. \tag{4.58}$$

We find three such sets, each composed of a single line with non-trivial charge: $\ell_{1,0}$, $\ell_{0,1}$, or $\ell_{1,1}$. In agreement with [16], we find that the line with charge $\ell_{1,0}$ transforms as a vector of $\mathfrak{usp}(4)$ and the theory is $\mathrm{USp}(4)$. The line with charge $\ell_{0,1}$ transforms as a spinor of $\mathfrak{so}(5)$ and corresponds to the global structure $(\mathrm{USp}(4)/\mathbb{Z}_2)_0$. The line with charge $\ell_{1,1}$ transforms both as a vector and a spinor, and the gauge group is $(\mathrm{USp}(4)/\mathbb{Z}_2)_1$. Motivated by the match between our results (obtained through the procedure described above) and the global structures of Lagrangian theories [16], in the next sections we use our method to analyze the line spectra of S-fold theories.

4.2.3 Lines in S-folds with $\mathcal{N} = 4$ Enhancement

We now derive the spectrum of mutually local lines for the gauge theories obtained with $N = 2$ D3-branes in the background of an $S_{k,1}$ plane, in each case $k = 3$, 4 and 6. More precisely, exploiting the strategy spelled out in Section 4.2.2, we first compute the electromagnetic charge lattice of local states generated by (p, q)-strings. From this we extract the possible spectra of lines and compare them with the ones obtained in an $\mathcal{N} = 4$ Lagrangian formalism [16], since these theories have been claimed to enhance to $\mathcal{N} = 4$ SYM [15]. Matching the spectra provides an explicit dictionary between the various lattices and corroborates the validity of our procedure. In section 4.2.4 we will then generalize the analysis to the pure $\mathcal{N} = 3$ $S_{k,\ell}$ projections for any rank, thus providing

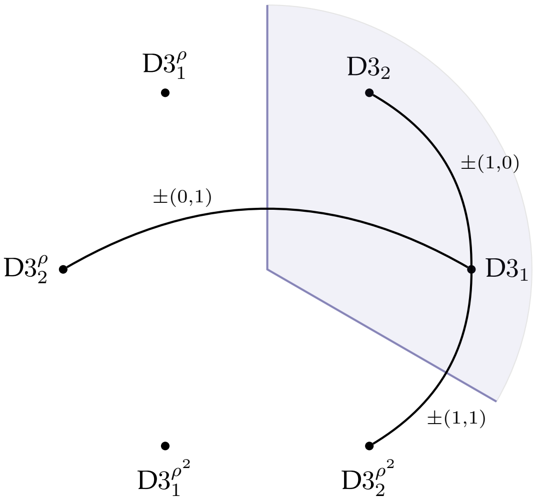

Figure 4.5: A pictorial representation of two D3-branes probing the $S_{3,1}$-fold. The transverse directions to the S-fold are shown. The light blue dot represents the position of the $S_{3,1}$-fold. The light blue shaded area is a possible choice of fundamental domain under the spacetime identification induced by the $S_{3,1}$-fold. Black (gray) dots represent (images of) D3-branes. Black lines correspond to (p,q)-strings stretched between D3-branes. In particular, we drew (p,q)-strings corresponding to $\mathcal{W}$-bosons of $\mathcal{N}=4$ $\mathfrak{su}(3)$ SYM.

the full classification for the one-form symmetries in all such cases.

4.2.3.1 Lines in $\mathfrak{su}(3)$ from $S_{3,1}$

Dynamical states and their charges Two D3-branes probing the singular point of the $S_{3,1}$-fold are claimed to engineer $\mathfrak{su}(3)$ $\mathcal{N}=4$ SYM. The charges of states generated by (p,q)-strings stretching between $D3_1$ and $D3_2$ or its first copy (see Figure 4.5) are

$$\mathrm{D3}_1\mathrm{D3}_2\,:\,(p,q;-p,-q)\,,\quad \mathrm{D3}_1\mathrm{D3}_2^{\rho}\,:\,(p,q;q,q-p)\,,\quad \mathrm{D3}_1\mathrm{D3}_2^{\rho^2}\,:\,(p,q;p-q,p)\,. \tag{4.59}$$

One may also consider copies of the strings listed in Equation 4.59 such as

$$\mathrm{D3}_1^{\rho}\mathrm{D3}_2^{\rho}\,:\,(-q,p-q;q,q-p)\,, \tag{4.60}$$

as well as the strings going from one D3-brane to its own copies, for instance[8]

$$\mathrm{D3}_1\mathrm{D3}_1^{\rho}\,:\,(2p-q,p+q;0,0)\,. \tag{4.61}$$

The charges of a generic string $\mathrm{D3}_1\mathrm{D3}_2^{\rho^2}$ in (4.59) can be expressed in terms of $\mathrm{D3}_1\mathrm{D3}_2$

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[8] In the absence of discrete torsion, these states have not been considered previously in the literature [4, 233], and we do here for the sake of consistency with the analysis of section 4.2.2. Note however that since their charge (which is the only feature that matters in order to derive line spectra) can be expressed as linear combinations of the charges of more conventional states, our results are independent of whether we consider them or not.

and $D3_1 D3_2^{\rho}$ charges

$$\begin{aligned} D3_1 D3_2^{\rho^2} \, : \, (p,q;p-q,p) \quad &= \quad q(1,0;-1,0) + (q-p)(0,1;0,-1) \\ &\quad +(p-q)(1,0;0,-1) + p(0,1;1,1)\,, \end{aligned} \tag{4.62}$$

where the first two vectors on the RHS come from $D3_1 D3_2$ with $p = 1$, $q = 0$ and $p = 0$, $q = 1$ respectively, and the last two come from $D3_1 D3_2^{\rho}$ with $p = 1$, $q = 0$ and $p = 0$, $q = 1$ respectively. Acting with ρ_3, one can express all $D3_1^{\rho} D3_2^{\rho}$ and $D3_1^{\rho^2} D3_2^{\rho^2}$ charges in terms of $D3_1 D3_2$ charges. The charges $D3_i D3_i^{\rho}$ can also be expressed as linear combinations of $D3_1 D3_2^{\rho}$ and $D3_2^{\rho} D3_1^{\rho}$ charges. All in all, we find that the charges of the strings $D3_1 D3_2$ and $D3_1 D3_2^{\rho}$ form a basis of the lattice of dynamical charges.
The states corresponding to the $\mathcal{W}$-bosons generate the $\mathfrak{su}(3)$ algebra. One can take the strings $D3_1 D3_2$ with $p = 1$ and $q = 0$ and $D3_1 D3_2^{\rho}$ with $p = 0$ and $q = 1$ as representing a choice of positive simple roots. Their electromagnetic charge w reads

$$w_1 = (1,0;-1,0)\,, \quad w_2 = (0,1;1,1)\,. \tag{4.63}$$

Furthermore, one can choose the strings $D3_1 D3_2$ with $p = 0$ and $q = 1$ and $D3_1 D3_2^{\rho}$ with $p = -1$ and $q = -1$ as generating the charge lattice of magnetic monopoles $\mathcal{M}$ of $\mathcal{N} = 4$ SYM with gauge algebra $\mathfrak{su}(3)$

$$m_1 = (0,1;0,-1)\,, \quad m_2 = (-1,-1;-1,0)\,. \tag{4.64}$$

The qualification of electric charges $\mathcal{W}$ and magnetic monopoles $\mathcal{M}$ of the $\mathcal{N} = 4$ theory makes sense since the Dirac pairing reads

$$\begin{aligned} \langle \mathcal{W}_1, \mathcal{W}_2 \rangle \quad &= \quad \langle \mathcal{M}_1, \mathcal{M}_2 \rangle \quad = \quad 0\,, \\ \langle \mathcal{W}_1, \mathcal{M}_1 \rangle \quad &= \quad \langle \mathcal{W}_2, \mathcal{M}_2 \rangle \quad = \quad 2\,, \\ \langle \mathcal{W}_1, \mathcal{M}_2 \rangle \quad &= \quad \langle \mathcal{W}_2, \mathcal{M}_1 \rangle \quad = \quad -1\,. \end{aligned} \tag{4.65}$$

In [4, 233], it has been shown that these states correspond indeed to BPS states, and this is a strong check of the claim of the supersymmetry enhancement in this case.

Line lattices Having identified the electromagnetic lattice of charges of (p,q)-strings we can now construct the spectrum of line operators and the corresponding one-form symmetries. It is useful to consider the charge $\ell = (e_1, m_1; e_2, m_2)$ of a general line $\mathcal{L}$ to be parameterized as follows

$$\begin{aligned} \ell \quad &= \quad \alpha_1 w_1 + \alpha_2 w_2 + \beta_1 m_1 + \beta_2 m_2 \\ &= \quad (\alpha_1 - \beta_2, \alpha_2 + \beta_1 - \beta_2; -\alpha_1 + \alpha_2 - \beta_2, \alpha_2 - \beta_1)\,. \end{aligned} \tag{4.66}$$

Screening with respect to w_i and m_i translates as the identifications

$$\alpha_i \sim \alpha_i + 1\,, \quad \beta_i \sim \beta_i + 1\,. \tag{4.67}$$

The Dirac pairing between the generic line $\mathcal{L}$ with charge ℓ given in (4.66) and the states $\mathcal{W}$ and $\mathcal{M}$ must be an integer, i.e.

$$\begin{array}{rclrcl} \langle \mathcal{L}, \mathcal{W}_1 \rangle & = & 2\beta_1 - \beta_2 \,, & \langle \mathcal{L}, \mathcal{W}_2 \rangle & = & -\beta_1 + 2\beta_2 \,, \\ \langle \mathcal{L}, \mathcal{M}_1 \rangle & = & -2\alpha_1 + \alpha_2 \,, & \langle \mathcal{L}, \mathcal{M}_2 \rangle & = & \alpha_1 - 2\alpha_2 \end{array} \in \mathbb{Z} \,. \tag{4.68}$$

Mutual locality with respect to the other states then follows by linearity as soon as (4.68) holds. Combining (4.67) and (4.68) we have

$$\alpha_1 = -\alpha_2 = \frac{e}{3} \,, \quad \text{and} \quad \beta_1 = -\beta_2 = \frac{m}{3} \,, \tag{4.69}$$

for $e, m = 0, 1, 2$. Then, the charge of the most general line compatible with the spectrum of local operators modulo screening reads

$$\ell_{e,m} = \frac{1}{3}(2e - m, \ e + m; \ -e - m, \ e - 2m) \,. \tag{4.70}$$

These charges form a finite 3×3 square lattice. The Dirac pairing between two lines $\mathcal{L}$ and $\mathcal{L}'$ with charges $\ell_{e,m}$ and $\ell_{e',m'}$ is

$$\langle \mathcal{L}, \mathcal{L}' \rangle = \frac{2}{3}(em' - e'm) \,. \tag{4.71}$$

Two lines $\mathcal{L}$ and $\mathcal{L}'$ are mutually local if their Dirac pairing is properly quantized. In our conventions this corresponds to the requirement that $\langle \mathcal{L}, \mathcal{L}' \rangle$ is an integer

$$e'm - em' = 0 \bmod 3 \,. \tag{4.72}$$

The lattice of lines together with the mutual locality condition obtained in (4.72) fully specifies the global structure of the $S_{3,1}$ SCFT of rank-2.

Our result is equivalent to the one obtained in [16] from the Lagrangian description of $\mathfrak{su}(3)$ $\mathcal{N} = 4$ SYM theory. Let us first write the charges in (4.70) as

$$\ell_{e,m} = e\frac{w_1 - w_2}{3} + m\frac{m_1 - m_2}{3} \,. \tag{4.73}$$

Note that $(w_1 - w_2)/3$ (respectively, $(m_1 - m_2)/3$) is a weight of the electric (respectively, magnetic) algebra $\mathfrak{su}(3)$ with charge 1 under the center $\mathbb{Z}_3$ of the simply-connected group $\mathrm{SU}(3)$. Therefore, the line $\ell_{e,m}$ corresponds to a Wilson-'t Hooft line of charge (e, m) under $\mathbb{Z}_3 \times \mathbb{Z}_3$.

As shown in [16], there are four possible lattices of mutually local Wilson-'t Hooft lines specified by two integers $i = 0, 1, 2$ and $p = 1, 3$. The corresponding gauge theories are denoted $(\mathrm{SU}(3)/\mathbb{Z}_p)_i$ and relate to the line spectra we have obtained as follows

$$\begin{array}{rcl} \mathrm{SU}(3) & \leftrightarrow & \{\ell_{0,0}, \ell_{1,0}, \ell_{2,0}\} \,, \\ (\mathrm{SU}(3)/\mathbb{Z}_3)_0 & \leftrightarrow & \{\ell_{0,0}, \ell_{0,1}, \ell_{0,2}\} \,, \\ (\mathrm{SU}(3)/\mathbb{Z}_3)_1 & \leftrightarrow & \{\ell_{0,0}, \ell_{1,1}, \ell_{2,2}\} \,, \\ (\mathrm{SU}(3)/\mathbb{Z}_3)_2 & \leftrightarrow & \{\ell_{0,0}, \ell_{2,1}, \ell_{1,2}\} \,. \end{array} \tag{4.74}$$

It follows from linearity and screening that each lattice in the S-fold picture is determined by a single non-trivial representative, that can itself be identified by two integers (e, m). For example, a possible choice is

$$(e, m) = (1, 0),\ (0, 1),\ (1, 1),\ (2, 1)\,. \tag{4.75}$$

4.2.3.2 Lines in $\mathfrak{so}(5)$ from $S_{4,1}$

Dynamical states and their charges Two D3-branes probing the singular point of the $S_{4,1}$-fold are claimed to engineer $\mathfrak{so}(5)$ $\mathcal{N} = 4$ SYM. Following a reasoning similar to one of the $S_{3,1}$-fold case, we can write all string charges as linear combinations of two kinds of strings, say

$$\mathrm{D3}_1\mathrm{D3}_2\ :\ (p, q; -p, -q)\,, \quad \mathrm{D3}_1\mathrm{D3}_2^\rho\ :\ (p, q; -q, p)\,. \tag{4.76}$$

States corresponding to the $\mathcal{W}$-bosons of $\mathcal{N} = 4$ SYM are generated by $\mathrm{D3}_1\mathrm{D3}_2$ with $p = 1$ and $q = 0$, and $\mathrm{D3}_1\mathrm{D3}_2^\rho$ with $p = -1$ and $q = -1$. Their charges are

$$w_1 = (1, 0; -1, 0)\,, \quad w_2 = (-1, -1; 1, -1)\,. \tag{4.77}$$

These states generate the algebra $\mathfrak{so}(5)$ with short and long positive simple roots w_1 and w_2, respectively. A possible choice of states corresponding to elementary magnetic monopoles $\mathcal{M}$ is $\mathrm{D3}_1\mathrm{D3}_2$ with $p = -1$ and $q = 1$, and $\mathrm{D3}_1\mathrm{D3}_2^\rho$ with $p = 1$ and $q = 0$. The charges of these strings are

$$m_1 = (-1, 1; 1, -1)\,, \quad m_2 = (1, 0; 0, 1)\,, \tag{4.78}$$

with m_1 the long and m_2 the short positive simple roots of the Langland dual algebra $\mathfrak{usp}(4)$. The Dirac pairings between $\mathcal{W}$ and $\mathcal{M}$ are as expected

$$\begin{array}{rcrcl} \langle \mathcal{W}_1, \mathcal{W}_2 \rangle & = & \langle \mathcal{M}_1, \mathcal{M}_2 \rangle & = & 0\,, \\ \langle \mathcal{W}_1, \mathcal{M}_1 \rangle & = & \langle \mathcal{W}_2, \mathcal{M}_2 \rangle & = & \langle \mathcal{M}_1, \mathcal{W}_2 \rangle = 2\,, \\ \langle \mathcal{M}_2, \mathcal{W}_1 \rangle & = & 1\,. & & \end{array} \tag{4.79}$$

Line lattices We begin by parametrizing the charge ℓ of a general line $\mathcal{L}$ as

$$\begin{aligned} \ell &= \alpha_1 w_1 + \alpha_2 w_2 + \beta_1 m_1 + \beta_2 m_2 \\ &= (\alpha_1 - \alpha_2 - \beta_1 + \beta_2,\ \beta_1 - \alpha_2;\ -\alpha_1 + \alpha_2 + \beta_1,\ -\alpha_2 - \beta_1 + \beta_2)\,. \end{aligned} \tag{4.80}$$

Screening with respect to the local states $\mathcal{W}$ and $\mathcal{M}$ translates as

$$\alpha_i \sim \alpha_i + 1\,, \quad \beta_i \sim \beta_i + 1\,. \tag{4.81}$$

Mutual locality with respect to the dynamical states generated by (p, q)-strings reads

$$\begin{array}{rcl} \langle \mathcal{L}, \mathcal{W}_1 \rangle & = & 2\beta_1 - \beta_2 \\ \langle \mathcal{L}, \mathcal{W}_2 \rangle & = & -2\beta_1 + 2\beta_2 \\ \langle \mathcal{L}, \mathcal{M}_1 \rangle & = & -2\alpha_1 + 2\alpha_2 \\ \langle \mathcal{L}, \mathcal{M}_2 \rangle & = & \alpha_1 - 2\alpha_2 \end{array} \quad \in \mathbb{Z} \,. \tag{4.82}$$

This imposes $\alpha_1 = \beta_2 = 0$ and $\alpha_2, \beta_1 \in \frac{1}{2}\mathbb{Z}$, and therefore the charge of the most general line compatible with the spectrum of local states can be written as

$$\ell_{e,m} = \frac{e}{2} w_2 + \frac{m}{2} m_1 = \frac{1}{2}(-e - m, -e + m; e + m, -e - m) \,. \tag{4.83}$$

The Dirac pairing between two lines $\mathcal{L}$ and $\mathcal{L}'$ with charges $\ell_{e,m}$ and $\ell_{e',m'}$ is

$$\langle \mathcal{L}, \mathcal{L}' \rangle = \frac{1}{2}(e'm - em') \,. \tag{4.84}$$

Two such lines are mutually local if their Dirac pairing if $\langle \mathcal{L}, \mathcal{L}' \rangle$ is an integer, i.e.

$$(e'm - em') = 0 \bmod 2 \,. \tag{4.85}$$

Therefore, the allowed lines form a finite 2×2 square lattice parametrized by $e, m = 0, 1$, where the mutual locality condition is given by (4.85). This reproduces the expected global structures of $\mathcal{N} = 4$ $\mathfrak{so}(5)$ SYM. There are three possible choices of maximal lattices of mutually local lines which correspond to the three possible global structures of $\mathfrak{so}(5)$. The explicit mapping can be obtained by comparing the electromagnetic charges of the lines with the charges of the $\mathcal{W}$ bosons and monopoles $\mathcal{M}$, along the lines of the analysis of above in the $\mathfrak{su}(3)$ case. We obtain the following global structures

$$\begin{array}{rcl} \mathrm{Spin}(5) & \leftrightarrow & \{\ell_{0,0}, \ell_{1,0}\} \,, \\ \mathrm{SO}(5)_0 & \leftrightarrow & \{\ell_{0,0}, \ell_{0,1}\} \,, \\ \mathrm{SO}(5)_1 & \leftrightarrow & \{\ell_{0,0}, \ell_{1,1}\} \,. \end{array} \tag{4.86}$$

4.2.3.3 Trivial Line in $\mathfrak{g}_2$ from $S_{6,1}$

Dynamical states and their charges Two $D3$-branes probing the singular point of the $S_{6,1}$-fold are claimed to engineer $\mathfrak{g}_2$ $\mathcal{N} = 4$ SYM. The charges of states generated by (p, q)-strings are

$$\begin{array}{llll} \mathrm{D3}_1\mathrm{D3}_2 & : \ (p, q; -p, -q) \,, & \mathrm{D3}_1\mathrm{D3}_2^{\rho} & : \ (p, q; -q, p - q) \,, \\ \mathrm{D3}_1\mathrm{D3}_2^{\rho^2} & : \ (p, q; p - q, p) \,, & \mathrm{D3}_1\mathrm{D3}_2^{\rho^3} & : \ (p, q; p, q) \,, \\ \mathrm{D3}_1\mathrm{D3}_2^{\rho^4} & : \ (p, q; q, -p + q) \,, & \mathrm{D3}_1\mathrm{D3}_2^{\rho^5} & : \ (p, q; -p + q, -p) \,, \\ \vdots & \quad \vdots & \vdots & \quad \vdots \end{array} \tag{4.87}$$

As shown in [17] and as before, one can choose a set of strings representing dynamical particles and generating the algebra $\mathfrak{g}_2$.

Line lattice The analysis of the charge spectrum in the case of the $S_{6,1}$-fold can be carried out along the lines of the previous sections. One can show that the only line that is mutually local with respect to the local states generated by (p,q)-strings modulo screening is the trivial line with charges $\ell = (0,0;0,0)$. This is consistent with the enhancement to $\mathcal{N} = 4$ with gauge algebra $\mathfrak{g}_2$ because the center of the simply-connected G_2 is trivial, which implies the absence of non-trivial lines [16]. There is only one possible global structure, and the one-form symmetry is trivial.

4.2.4 Lines in $\mathcal{N} = 3$ S-folds

In this section, we generalize the procedure spelled out in the previous sections to S-folds theories of arbitrary rank, and later to the cases with non-trivial discrete torsion for the B_2 and C_2 fields. This allows us to classify the line spectrum for every $\mathcal{N} = 3$ S-fold theory, and identify the one-form symmetry group as well as the allowed global structures for a given theory.

The basic ingredients needed in the analysis are the lattice of electromagnetic charges of local states and the Dirac pairing, both of which can be inferred from the Type IIB setup along the lines of the rank-2 cases studied in Section 4.2.3. As already emphasized, we work under the assumption that the states generated by (p,q)-string form a good set of representatives of the electromagnetic charge lattice of the full spectrum.

Note that it does not strictly make sense to talk about (p,q)-strings on the $\mathbb{R}^4 \times \mathbb{R}^6/\mathbb{Z}_k$ S-fold background because the S-fold projection involves an $\mathrm{SL}(2,\mathbb{Z})$ action which mixes F1 and D1 strings. This is analogous to the fact that in the orientifold cases it only makes sense to consider unoriented strings, since the orientifold action reverses the worldsheet parity (equivalently, it involves the element $-\mathbb{I}_2 \in \mathrm{SL}(2,\mathbb{Z})$). Nevertheless it makes sense to consider oriented strings (together with their images) on the double cover of the spacetime; this allows the computation of the electromagnetic charge lattice of local states and the Dirac pairing, as reviewed in Section 4.2.2. Similarly when dealing with S_k-folds we consider (p,q)-strings on the k-cover of the spacetime, and extract from this the charges of local states and the Dirac pairing. The spectrum of lines can then be obtained using the procedure of [173] reviewed in Section 4.2.2.

4.2.4.1 Lines in $S_{3,1}$-fold

Let us first determine the lattice of electromagnetic charges of dynamical states. The charges generated by (p,q)-strings on the background of an $S_{3,1}$ fold are given by

$$\mathrm{D3}_i\mathrm{D3}_j^{\rho^l} \ : \ (0,0;\dots;\overbrace{p,q}^{i-th};\dots;\overbrace{-(p\ \ q)\cdot\rho_3^l}^{j-th};\dots;0,0)\,. \tag{4.88}$$

This expression is obtained from a (p,q)-string stretched between the i-th D3-brane and the l-th image of the j-th D3-brane. Recall that ρ_3 generates a $\mathbb{Z}_3$ subgroup of $\mathrm{SL}(2,\mathbb{Z})$.

A possible basis for the lattice of charges generated by (p,q)-strings is given by

$$\begin{aligned} w_1 &= (1,0;-1,0;\dots)\,, \\ w_2 &= (0,1;1,1;\dots)\,, \\ m_1 &= (0,1;0,-1;\dots)\,, \\ m_2 &= (-1,-1;-1,0;\dots)\,, \\ P_i &= (1,0;0,0;\dots;\overbrace{-1,0}^{i-th};0,0;\dots)\,, \\ Q_i &= (0,1;0,0;\dots;\overbrace{0,-1}^{i-th};0,0;\dots)\,, \end{aligned} \tag{4.89}$$

where w_i and m_i are the charges of the corresponding states in the rank-2 case, with all other entries set to 0. Let $\mathcal{P}_i$ and $\mathcal{Q}_i$ be the states with charges P_i and Q_i respectively, for $i=3,\dots,N$. Note that when the rank is $N>2$, it does not make sense to talk about $\mathcal{W}$-bosons and magnetic monopoles $\mathcal{M}$ since the pure $\mathcal{N}=3$ theories are inherently strongly coupled and do not admit a Lagrangian description. Nevertheless, we will denote $\mathcal{W}_i$ and $\mathcal{M}_i$ the states with charges w_i and m_i respectively, by analogy with the above.

The charge ℓ of a general line $\mathcal{L}$ can be written as the linear combination

$$\ell = \alpha_1 w_1 + \alpha_2 w_2 + \beta_1 m_1 + \beta_2 m_2 + \sum_{i=3}^{N} (\delta_i P_i + \gamma_i Q_i)\,. \tag{4.90}$$

Besides, screening translates into the identifications

$$\alpha_i \sim \alpha_i + 1\,, \quad \beta_i \sim \beta_i + 1\,, \quad \delta_i \sim \delta_i + 1\,, \quad \gamma_i \sim \gamma_i + 1\,. \tag{4.91}$$

Let us now analyze the constraints imposed on this line given by mutual locality with respect to the dynamical states generated by (p,q)-strings. Our results are summarized in Table 4.5.

Table 4.5: The charges of allowed lines in the $S_{3,1}$-fold theories. The charges w_i, m_i, P and Q are given in (4.89), and $r,s = 0,1,2$. The mutual locality condition for two lines with charges $\ell_{r,s}$ and $\ell_{r',s'}$ is $rs' - sr' = 0 \bmod 3$.

Rank	Line charge
$3n$	$\ell_{r,s} = \frac{r}{3} w_1 + \frac{s}{3} w_2 - \frac{r}{3} m_1 - \frac{s}{3} m_2 + \frac{r+s}{3}(P-Q)$
$3n+1$	$\ell_{r,s} = \frac{r}{3} w_1 + \frac{r-s}{3} w_2 + \frac{s}{3} m_1 + \frac{r}{3} m_2 + \frac{r+s}{3}(P-Q)$
$3n+2$	$\ell_{r,s} = \frac{r}{3} w_1 - \frac{r}{3} w_2 + \frac{s}{3} m_1 - \frac{s}{3} m_2 - \frac{r+s}{3}(P-Q)$

Consider the mutual locality conditions

$$\langle \mathcal{L}, \mathcal{P}_i - \mathcal{P}_j \rangle = \delta_i - \delta_j \in \mathbb{Z} \quad \Rightarrow \quad \delta_i = \delta_j = \delta \qquad i,j = 3,\dots,N\,, \tag{4.92}$$

and

$$\langle \mathcal{L}, \mathcal{Q}_i - \mathcal{Q}_j \rangle = \gamma_j - \gamma_i \in \mathbb{Z} \quad \Rightarrow \quad \gamma_j = \gamma_i = \gamma \qquad i,j = 3,\dots,N\,. \tag{4.93}$$

Furthermore, there are dynamical states with charges

$$\begin{aligned}
(0,0;\dots;\overbrace{1,-1}^{i-th};\dots) &= (p,q;\dots;\overbrace{-p,-q}^{i-th};\dots)\Bigg|_{\substack{p=0\\q=1}} + (p,q;\dots;\overbrace{p-q,p}^{i-th};\dots)\Bigg|_{\substack{p=0\\q=-1}} \,,\\
(0,0;\dots;\overbrace{2,1}^{i-th};\dots) &= (p,q;\dots;\overbrace{-p,-q}^{i-th};\dots)\Bigg|_{\substack{p=-1\\q=0}} + (p,q;\dots;\overbrace{p-q,p}^{i-th};\dots)\Bigg|_{\substack{p=1\\q=0}} \,.
\end{aligned} \tag{4.94}$$

Mutual locality with respect to these impies

$$\gamma = -\delta\,, \qquad \delta \in \frac{1}{3}\mathbb{Z}\,. \tag{4.95}$$

Therefore, the charge of a general line can be rewritten as

$$\ell = \alpha_1 w_1 + \alpha_2 w_2 + \beta_1 m_1 + \beta_2 m_2 + \delta(P - Q)\,, \tag{4.96}$$

where

$$\begin{aligned}
P &= \sum_{i=3}^{N} p_i = (N-2,0;0,0;-1,0;-1,0;\dots;-1,0)\,,\\
Q &= \sum_{i=3}^{N} q_i = (0,N-2;0,0;0,-1;0,-1;\dots;0,-1)\,.
\end{aligned} \tag{4.97}$$

In (4.97), we have modified our notation slightly since the dots $\dots$ now represent a sequence of pairs $(-1,0)$ and $(0,-1)$ for P and Q respectively. Mutual locality between the line $\mathcal{L}$ and the generators of the charge lattice of dynamical states imposes the following constraints

$$\left.\begin{array}{rcl}
\langle \mathcal{L}, \mathcal{P}_i \rangle &=& (N-1)\delta - \alpha_2 - \beta_1 + \beta_2\,,\\
\langle \mathcal{L}, \mathcal{Q}_i \rangle &=& (N-1)\delta + \alpha_1 - \beta_2\,,\\
\langle \mathcal{L}, \mathcal{W}_1 \rangle &=& (N-2)\delta - 2\beta_1 + \beta_2\,,\\
\langle \mathcal{L}, \mathcal{W}_2 \rangle &=& (N-2)\delta - 2\beta_2 + \beta_1\,,\\
\langle \mathcal{L}, \mathcal{M}_1 \rangle &=& (N-2)\delta + 2\alpha_1 - \alpha_2\,,\\
\langle \mathcal{L}, \mathcal{M}_2 \rangle &=& -2(N-2)\delta - \alpha_1 + 2\alpha_2
\end{array}\right. \quad \in \mathbb{Z}\,. \tag{4.98}$$

One can compute the following

$$\begin{array}{lclcll}
\langle \mathcal{L}, \mathcal{W}_1 + 2\mathcal{W}_2 \rangle & = & 3(N-2)\delta - 3\beta_2 & \in \mathbb{Z} & \Rightarrow & \beta_2 \in \frac{1}{3}\mathbb{Z} \, , \\
\langle \mathcal{L}, \mathcal{M}_1 + 2\mathcal{M}_2 \rangle & = & -3\alpha_1 & \in \mathbb{Z} & \Rightarrow & \alpha_1 \in \frac{1}{3}\mathbb{Z} \, , \\
\langle \mathcal{L}, \mathcal{W}_1 - \mathcal{W}_2 \rangle & = & 3(\beta_2 - \beta_1) & \in \mathbb{Z} & \Rightarrow & \beta_1 \in \frac{1}{3}\mathbb{Z} \, , \\
\langle \mathcal{L}, \mathcal{M}_1 - \mathcal{M}_2 \rangle & = & 3(N-2)\delta + 3(\alpha_1 - \alpha_2) & \in \mathbb{Z} & \Rightarrow & \alpha_2 \in \frac{1}{3}\mathbb{Z} \, .
\end{array} \tag{4.99}$$

In brief, we have found that $\alpha_i, \beta_i, \delta \in \frac{1}{3}\mathbb{Z}$. It is now useful to treat separately three cases, depending on the value of N mod 3. In all these cases we find that the lines modulo screening can be arranged in a finite 3×3 lattice, the one-form symmetry group is $\mathbb{Z}_3$ and there are four choices of global structure.

Case $N = 3n$ The mutual locality conditions in (4.98) can be written as

$$\left.\begin{array}{lcl}
\langle \mathcal{L}, \mathcal{P}_i \rangle & = & -\delta - \alpha_2 - \beta_1 + \beta_2 \, , \\
\langle \mathcal{L}, \mathcal{Q}_i \rangle & = & -\delta + \alpha_1 - \beta_2 \, , \\
\langle \mathcal{L}, \mathcal{W}_1 \rangle & = & \delta - 2\beta_1 + \beta_2 \, , \\
\langle \mathcal{L}, \mathcal{W}_2 \rangle & = & \delta - 2\beta_2 + \beta_1 \, , \\
\langle \mathcal{L}, \mathcal{M}_1 \rangle & = & \delta + 2\alpha_1 - \alpha_2 \, , \\
\langle \mathcal{L}, \mathcal{M}_2 \rangle & = & \delta - \alpha_1 + 2\alpha_2
\end{array}\right. \in \mathbb{Z} \, . \tag{4.100}$$

One computes that

$$\begin{array}{lclclcl}
\langle \mathcal{L}, \mathcal{Q}_i + \mathcal{W}_1 \rangle & = & \alpha_1 + \beta_1 & \Rightarrow & \beta_1 & = & -\alpha_1 \, , \\
\langle \mathcal{L}, \mathcal{P}_i + \mathcal{W}_2 \rangle & = & -\alpha_2 - \beta_2 & \Rightarrow & \beta_2 & = & -\alpha_2 \, , \\
\langle \mathcal{L}, \mathcal{Q}_i \rangle & = & -\delta + \alpha_1 + \alpha_2 & \Rightarrow & \delta & = & \alpha_1 + \alpha_2 \, ,
\end{array} \tag{4.101}$$

and this implies

$$\alpha_1 = -\beta_1 = \frac{r}{3} \, , \qquad \alpha_2 = -\beta_2 = \frac{s}{3} \, , \qquad \delta = \frac{r+s}{3} \, , \qquad r, s = 0, 1, 2 \, . \tag{4.102}$$

Therefore the lines form a finite 3×3 lattice parametrized by r and s. Mutual locality between two general lines $\mathcal{L}$ and $\mathcal{L}'$ with charges $\ell_{r,s}$ and $\ell_{r',s'}$ reads

$$\langle \mathcal{L}, \mathcal{L}' \rangle = \frac{2}{3}(sr' - rs') \in \mathbb{Z} \, , \tag{4.103}$$

or equivalently

$$sr' - rs' = 0 \bmod 3 \, . \tag{4.104}$$

There are four possible choices of maximal lattices of mutually local lines. As in the rank-2 case discussed in section 4.2.3, each lattice is uniquely identified by one of its element, or equivalently by the pair (r, s) of one of its non-trivial elements

$$(r,s) = \begin{cases} (1,0) \leftrightarrow \{\ell_{0,0}, \ell_{1,0}, \ell_{2,0}\} \\ (0,1) \leftrightarrow \{\ell_{0,0}, \ell_{0,1}, \ell_{0,2}\} \\ (1,1) \leftrightarrow \{\ell_{0,0}, \ell_{1,1}, \ell_{2,2}\} \\ (1,2) \leftrightarrow \{\ell_{0,0}, \ell_{1,2}, \ell_{2,1}\} \end{cases} . \tag{4.105}$$

Case $N = 3n + 1$ In this case the mutual locality constraints (4.98) are

$$\left.\begin{array}{rcl} \langle \mathcal{L}, \mathcal{P}_i \rangle & = & -\alpha_2 - \beta_1 + \beta_2 \\ \langle \mathcal{L}, \mathcal{Q}_i \rangle & = & \alpha_1 - \beta_2 \\ \langle \mathcal{L}, \mathcal{W}_1 \rangle & = & -\delta - 2\beta_1 + \beta_2 \\ \langle \mathcal{L}, \mathcal{W}_2 \rangle & = & -\delta - 2\beta_2 + \beta_1 \\ \langle \mathcal{L}, \mathcal{M}_1 \rangle & = & -\delta + 2\alpha_1 - \alpha_2 \\ \langle \mathcal{L}, \mathcal{M}_2 \rangle & = & 2\delta - \alpha_1 + 2\alpha_2 \end{array} \right. \in \mathbb{Z} . \tag{4.106}$$

One computes that

$$\begin{array}{rcl} \alpha_2 & = & \alpha_1 - \beta_1 , \\ \delta & = & \alpha_1 + \beta_1 , \\ \alpha_1 & = & \beta_2 . \end{array} \tag{4.107}$$

Therefore the most general α_i, β_i and δ satisfy

$$\alpha_1 = \beta_2 = \frac{r}{3} , \quad \beta_1 = \frac{s}{3} , \quad \alpha_2 = \frac{r-s}{3} , \quad \delta = \frac{r+s}{3} , \quad r, s = 0, 1, 2 . \tag{4.108}$$

The lines again form a finite 3×3 lattice parametrized by r and s. Mutual locality between two general lines $\mathcal{L}$ and $\mathcal{L}'$ with charges $\ell_{r,s}$ and $\ell_{r',s'}$ reads

$$\langle \mathcal{L}, \mathcal{L}' \rangle = \frac{1}{3}(sr' - rs') \in \mathbb{Z} , \tag{4.109}$$

or equivalently

$$sr' - rs' = 0 \bmod 3 . \tag{4.110}$$

Similarly to the case $N = 3n$ there are four possible choices of maximal lattices of mutually local lines that can be indexed by one of their element, or equivalently by $(r, s) = (1, 0),\ (0, 1),\ (1, 1),\ (1, 2)$.

Case $N = 3n + 2$ In this case, the mutual locality constraints (4.98) are

$$\left.\begin{array}{rcl} \langle \mathcal{L}, \mathcal{P}_i \rangle & = & \delta - \alpha_2 - \beta_1 + \beta_2 \\ \langle \mathcal{L}, \mathcal{Q}_i \rangle & = & \delta + \alpha_1 - \beta_2 \\ \langle \mathcal{L}, \mathcal{W}_1 \rangle & = & -2\beta_1 + \beta_2 = \beta_1 + \beta_2 \\ \langle \mathcal{L}, \mathcal{W}_2 \rangle & = & -2\beta_2 + \beta_1 \\ \langle \mathcal{L}, \mathcal{M}_1 \rangle & = & 2\alpha_1 - \alpha_2 = -\alpha_1 - \alpha_2 \\ \langle \mathcal{L}, \mathcal{M}_2 \rangle & = & -\alpha_1 + 2\alpha_2 \end{array} \right. \in \mathbb{Z} . \tag{4.111}$$

One can compute that the solution is given by

$$\begin{aligned} \beta_2 &= -\beta_1 \, , \\ \alpha_2 &= -\alpha_1 \, , \\ \delta &= -\alpha_1 - \beta_1 \, . \end{aligned} \tag{4.112}$$

Therefore the most general α_i, β_i and δ satisfy

$$\alpha_1 = -\alpha_2 = \frac{r}{3} \, , \quad \beta_1 = -\beta_2 = \frac{s}{3} \, , \quad \delta = -\frac{r+s}{3} \, , \quad r, s = 0, 1, 2 \, . \tag{4.113}$$

Dirac pairing between two general lines $\mathcal{L}$ and $\mathcal{L}'$ with charges $\ell_{r,s}$ and $\ell_{r',s'}$ reads

$$\langle \mathcal{L}, \mathcal{L}' \rangle = \frac{2}{3}(sr' - rs') \in \mathbb{Z} \, . \tag{4.114}$$

Two such lines are mutually local if they satisfy the constraint

$$sr' - rs' = 0 \text{ mod } 3 \, . \tag{4.115}$$

As before, there are four possible choices of maximal lattices of mutually local lines that can be indexed by one of their element, or equivalently by

$$(r, s) = (1, 0), \ (0, 1), \ (1, 1), \ (1, 2) \, . \tag{4.116}$$

4.2.4.2 Lines in $S_{4,1}$-fold

We now study the spectrum of lines in theories engineered by a stack of D3-branes probing the $S_{4,1}$-fold. The charges of states generated by a (p, q)-string on the background of an $S_{4,1}$-fold read

$$\text{D3}_i \text{D3}_j^{\rho^l} \ : \ (0, 0; \ldots; \overbrace{p, q}^{i-th}; \ldots; \overbrace{-(p \ \ q) \cdot \rho_4^l}^{j-th}; \ldots; 0, 0) \tag{4.117}$$

for a (p, q)-strings stretched between the i-th D3-brane and the l-th image of the j-th D3-brane. One possible basis for the lattice of charges generated by (p, q)-strings is

$$\begin{aligned} w_1 &= (1, 0; -1, 0; 0, 0; \ldots) \, , \\ w_2 &= (-1, -1; 1, -1; 0, 0; \ldots) \, , \\ m_1 &= (-1, 1; 1, -1; 0, 0; \ldots) \, , \\ m_2 &= (1, 0; 0, 1; 0, 0; \ldots) \, , \\ P_i &= (1, 0; 0, 0; \ldots; \overbrace{-1, 0}^{i-th}; 0, 0; \ldots) \, , \\ Q_i &= (0, 1; 0, 0; \ldots; \overbrace{0, -1}^{i-th}; 0, 0; \ldots) \, , \end{aligned} \tag{4.118}$$

where w_i and m_i are the charges of the corresponding states in the rank-2 case, with all other entries set to 0. We denote $\mathcal{W}_i, \mathcal{M}_i, \mathcal{P}_i$ and $\mathcal{Q}_i$ the states with charges w_i, m_i, P_i and

Q_i, respectively.

The charge ℓ of a general line $\mathcal{L}$ can be written as the linear combination

$$\ell = \alpha_1 w_1 + \alpha_2 w_2 + \beta_1 m_1 + \beta_2 m_2 + \sum_{i=3}^{N} (\delta_i P_i + \gamma_i Q_i) \,. \tag{4.119}$$

Screening translates into the identifications

$$\alpha_i \sim \alpha_i + 1, \quad \beta_i \sim \beta_i + 1, \quad \delta_i \sim \delta_i + 1, \quad \gamma_i \sim \gamma_i + 1 \,. \tag{4.120}$$

In the remainder of this section we compute the constraints imposed by mutual locality between the general line $\mathcal{L}$ and dynamical states. Our results are summarized in Table 4.6.

Table 4.6: The charges of allowed lines in the $S_{4,1}$-fold theories. The charges w_i, m_i, P and Q are given in (4.118), (4.97), and $r, s = 0, 1$. The mutual locality condition for two lines with charges $\ell_{r,s}$ and $\ell_{r',s'}$ is $rs' - sr' = 0 \bmod 2$.

Rank	Line charge
$2n$	$\ell_{r,s} = \frac{r}{2} w_2 + \frac{s}{2} m_1 + \frac{r+s}{2}(P-Q)$
$2n+1$	$\ell_{r,s} = \frac{r}{2} w_1 + \frac{s}{2} w_2 + \frac{s}{2} m_1 + \frac{r}{2} m_2 + \frac{r}{2}(P-Q)$

Consider first the mutual locality conditions

$$\langle \mathcal{L}, \mathcal{P}_i - \mathcal{P}_j \rangle = \delta_i - \delta_j \in \mathbb{Z} \quad \Rightarrow \quad \delta_i = \delta_j = \delta \,, \tag{4.121}$$

$$\langle \mathcal{L}, \mathcal{Q}_i - \mathcal{Q}_j \rangle = \gamma_j - \gamma_i \in \mathbb{Z} \quad \Rightarrow \quad \gamma_j = \gamma_i = \gamma \,. \tag{4.122}$$

Furthermore, there are dynamical states with charges

$$\begin{aligned}
(0,0;\ldots;\overbrace{1,-1}^{i-th};\ldots) &= (p,q;\ldots;\overbrace{-p,-q}^{i-th};\ldots)\Big|_{\substack{p=0\\q=1}} + (p,q;\ldots;\overbrace{-q,p}^{i-th};\ldots)\Big|_{\substack{p=0\\q=-1}} \,, \\
(0,0;\ldots;\overbrace{1,1}^{i-th};\ldots) &= (p,q;\ldots;\overbrace{-p,-q}^{i-th};\ldots)\Big|_{\substack{p=-1\\q=0}} + (p,q;\ldots;\overbrace{-q,p}^{i-th};\ldots)\Big|_{\substack{p=1\\q=0}} \,.
\end{aligned}$$

and mutual locality with respect to these states implies

$$\gamma = -\delta, \qquad \delta \in \frac{1}{2}\mathbb{Z} \,. \tag{4.123}$$

Therefore, the charge of a general line can be rewritten as

$$\ell = \alpha_1 w_1 + \alpha_2 w_2 + \beta_1 m_1 + \beta_2 m_2 + \delta(P - Q) \, , \tag{4.124}$$

where P and Q are defined in (4.97). Mutual locality between the line $\mathcal{L}$ and the generators of the charge lattice of dynamical states implies

$$\begin{array}{lcl} \langle \mathcal{L}, \mathcal{P}_i \rangle & = & (N-1)\delta + \alpha_2 - \beta_1 \, , \\ \langle \mathcal{L}, \mathcal{Q}_i \rangle & = & (N-1)\delta + \alpha_1 - \alpha_2 - \beta_1 + \beta_2 \, , \\ \langle \mathcal{L}, \mathcal{W}_1 \rangle & = & (N-2)\delta - 2\beta_1 + \beta_2 \, , \\ \langle \mathcal{L}, \mathcal{W}_2 \rangle & = & 2(N-2)\delta - 2\beta_2 + 2\beta_1 \, , \\ \langle \mathcal{L}, \mathcal{M}_1 \rangle & = & 2\alpha_1 - 2\alpha_2 \\ \langle \mathcal{L}, \mathcal{M}_2 \rangle & = & (N-2)\delta - \alpha_1 + 2\alpha_2 \end{array} \in \mathbb{Z} \, . \tag{4.125}$$

One computes the following

$$\begin{array}{ll} \langle \mathcal{L}, \mathcal{W}_1 + \mathcal{W}_2 - \mathcal{M}_1 - \mathcal{M}_2 \rangle = -\beta_2 - \alpha_1 \in \mathbb{Z} & \Rightarrow \beta_2 = -\alpha_1 \, , \\ \langle \mathcal{L}, \mathcal{Q}_i + \mathcal{P}_i \rangle = -2\beta_1 \in \mathbb{Z} & \Rightarrow \beta_1 \in \frac{1}{2}\mathbb{Z} \, , \\ \langle \mathcal{L}, \mathcal{Q}_i - \mathcal{P}_i \rangle = -2\alpha_2 \in \mathbb{Z} & \Rightarrow \alpha_2 \in \frac{1}{2}\mathbb{Z} \, , \\ \langle \mathcal{L}, \mathcal{M}_1 \rangle = 2\alpha_1 \in \mathbb{Z} & \Rightarrow \alpha_1, \beta_2 \in \frac{1}{2}\mathbb{Z} \, . \end{array} \tag{4.126}$$

We have thus shown that $\alpha_i, \beta_i, \delta \in \frac{1}{2}\mathbb{Z}$ and $\alpha_1 = -\beta_2$. It is now useful to treat separately the cases of odd and even N. In both cases we find that the lines form a 2×2 lattice, the one-form symmetry is $\mathbb{Z}_2$ and there are three choices of global structure.

Case $N = 2n$ Mutual locality conditions (4.125) read

$$\begin{array}{lcl} \langle \mathcal{L}, \mathcal{P}_i \rangle & = & -\delta - \beta_1 + \alpha_2 \\ \langle \mathcal{L}, \mathcal{Q}_i \rangle & = & -\delta - \alpha_2 - \beta_1 \\ \langle \mathcal{L}, \mathcal{W}_1 \rangle & = & \beta_2 \\ \langle \mathcal{L}, \mathcal{W}_2 \rangle & = & 0 \\ \langle \mathcal{L}, \mathcal{M}_1 \rangle & = & 0 \\ \langle \mathcal{L}, \mathcal{M}_2 \rangle & = & -\alpha_1 \end{array} \in \mathbb{Z} \, , \tag{4.127}$$

and each solution can be written as

$$\alpha_2 = \frac{r}{2} \, , \quad \beta_1 = \frac{s}{2} \, , \quad \alpha_1 = \beta_2 = 0 \, , \quad \delta = \frac{r+s}{2} \, , \quad r, s = 0, 1 \, . \tag{4.128}$$

Therefore the lines form a 2×2 lattice parametrized by r, s. Mutual locality between two lines $\mathcal{L}$ and $\mathcal{L}'$ with charges $\ell_{r,s}$ and $\ell_{r',s'}$ respectively translates into

$$\langle \mathcal{L}, \mathcal{L}' \rangle = \frac{1}{2}(r's - rs') \in \mathbb{Z} \, , \tag{4.129}$$

or equivalently

$$r's - rs' = 0 \bmod 2 \, . \tag{4.130}$$

The one-form symmetry group is thus $\mathbb{Z}_2$ and there are three different choices of maximal lattices of mutually local lines parametrized by $(r, s) = (1, 0),\ (0, 1),\ (1, 1)$.

Case $N = 2n + 1$ The Dirac pairings (4.125) read

$$\left.\begin{array}{rcl} \langle \mathcal{L}, \mathcal{P}_i \rangle & = & \alpha_2 - \beta_1 \,, \\ \langle \mathcal{L}, \mathcal{Q}_i \rangle & = & -\alpha_2 - \beta_1 \,, \\ \langle \mathcal{L}, \mathcal{W}_1 \rangle & = & \delta + \beta_2 \,, \\ \langle \mathcal{L}, \mathcal{W}_2 \rangle & = & 0 \,, \\ \langle \mathcal{L}, \mathcal{M}_1 \rangle & = & 0 \,, \\ \langle \mathcal{L}, \mathcal{M}_2 \rangle & = & \delta - \alpha_1 \end{array}\right. \quad \in \mathbb{Z} \,, \tag{4.131}$$

and the general solution can be written as

$$\alpha_1 = \beta_2 = \delta = \frac{r}{2} \,, \qquad \alpha_2 = \beta_1 = \frac{s}{2} \,, \qquad r, s = 0, 1 \,. \tag{4.132}$$

Mutual locality between two lines $\mathcal{L}$ and $\mathcal{L}'$ with charges $\ell_{r,s}$ and $\ell_{r',s'}$ respectively translates into

$$\langle \mathcal{L}, \mathcal{L}' \rangle = \frac{1}{2}(r's - rs') \in \mathbb{Z} \,, \tag{4.133}$$

or equivalently

$$r's - rs' = 0 \bmod 2 \,. \tag{4.134}$$

As in the previous case, the one-form symmetry group is therefore $\mathbb{Z}_2$ and there are three different choices of maximal lattices of mutually local lines that can be parametrized by

$$(r, s) = (1, 0),\ (0, 1),\ (1, 1) \,. \tag{4.135}$$

4.2.4.3 Trivial Line in $S_{6,1}$-fold

The analysis of the spectrum of lines in the case of the $S_{6,1}$-fold can be carried out along the lines of the previous subsections. One finds that the integer lattice of charges associated to (p, q)-strings is fully occupied. To see this notice that there are two states with the following charges

$$\begin{array}{rcl} (1, 0; 0, 0; 0, 0; \dots) & = & (p, q; p - q, p; 0, 0; \dots)\Big|_{\substack{p=0\\ q=-1}} - (p, q; -q, p, 0, 0; \dots)\Big|_{\substack{p=1\\ q=0}} \,, \\ (0, 1; 0, 0; 0, 0; \dots) & = & (1, 0; 0, 0; 0, 0; \dots) - (p, q; -p - q; 0, 0; \dots)\Big|_{\substack{p=0\\ q=1}} \\ & & -(p, q; -q, p; 0, 0; \dots)\Big|_{\substack{p=0\\ q=-1}} \,. \end{array}$$

By combining these states with $\mathcal{P}_i$ and $\mathcal{Q}_i$ we can obtain states with electric or magnetic charge 1 with respect to the i-th brane, and all other charges set to zero. Let us now consider a general line $\mathcal{L}$ with charge $\ell = (e_1, m_1; e_2, m_2, \dots)$. Mutual locality with respect to the local states we have just discussed implies

$$e_i, m_i \in \mathbb{Z} \quad \forall i \,, \tag{4.136}$$

and the insertion of the same local states along the lines translates to the identification

$$e_i \sim e_i + 1, \qquad m_i \sim m_i + 1 \,. \tag{4.137}$$

Therefore, the only allowed line modulo screening is the trivial line, with charge $\ell = (0, 0; 0, 0; \dots)$. This implies that the one form symmetry group is trivial, and accordingly there is only one possible choice of global form.

4.2.4.4 Trivial Line in the Discrete Torsion Cases

We generalize the analysis discussed in the previous sections to the cases with non-trivial discrete torsion in the $S_{3,3}$-fold and $S_{4,4}$-fold.
As we argued in Section 4.2.2 all the strings states that are present when the discrete torsion is trivial are also allowed when the discrete torsion is non-zero. Furthermore, there are strings ending on the S-fold itself, as discussed in Section 4.2.2. Thus, the lattice of charges of local states in the case of the $S_{3,3}$-fold and $S_{4,4}$-fold are generated by strings stretched between (images of) D3-branes – as in the cases with trivial discrete torsion – together with those additional strings. One can show that the integer lattice of electromagnetic charges of dynamical states is then fully occupied. Therefore, by a similar argument to the one used in the case of the $S_{6,1}$-fold in Section 4.2.4.3, the only line that is allowed is the trivial one, and the one-form symmetry group is $\mathbb{1}$ for the $S_{3,3}$-fold and $S_{4,4}$-fold with non-zero discrete torsion.

4.2.5 Non-invertible Symmetries

We now discuss the possible presence of non-invertible symmetries in S-fold theories. In the case of $\mathcal{N} = 4$ theories, the presence of S-duality orbits can imply the existence of non-invertible duality defects which are built by combining the action of some element of $\mathrm{SL}(2, \mathbb{Z})$ and the gauging of a discrete one-form symmetry [48, 74, 75, 98, 99, 103, 104, 147, 148, 217, 243, 244].
Similar structures can be inferred from the S-fold construction. Consider moving one of the D3-brane along the non-contractible one-cycle of $S^5/\mathbb{Z}_k$ until it reaches its original position. The brane configurations before and after this are identical, and therefore the S-fold theories are invariant under this action. Going around the non-contractible one-cycle of $S^5/\mathbb{Z}_k$ in the case an $S_{k,l}$-fold involves an $\mathrm{SL}(2, \mathbb{Z})$-transformation on the electric and magnetic charges e_i, m_i associated to the D3-brane that has been moved. Let Σ_k^i denote the process of moving the i-th D3-brane along the non-contractible cycle of an

$S_{k,l}$-fold. The action of Σ_k^i on the charges is

$$\Sigma_k^i : \begin{pmatrix} e_j \\ m_j \end{pmatrix} \to \begin{cases} \rho_k \cdot \begin{pmatrix} e_j \\ m_j \end{pmatrix} & j = i \\ \begin{pmatrix} e_j \\ m_j \end{pmatrix} & j \neq i \end{cases} . \tag{4.138}$$

The charge lattice of dynamical states is invariant under Σ_k^i, while the set of line lattices can be shuffled. Consider for example the $S_{3,1}$-case with rank $N = 2$. One can compute explicitly the following orbits

$$\begin{tikzcd} (1,0) \arrow[r, leftrightarrow] & (0,1) \arrow[r, leftrightarrow] & (1,1) & (1,2) \arrow[loop above] \end{tikzcd} , \tag{4.139}$$

where the pairs (e, m) parametrize the maximal sub-lattice of mutually local lines as discussed in section (4.2.3.1). Two line lattices connected by an arrow in (4.139) are mapped to each other under proper combinations of Σ_3^i.

This theory enhances to $\mathfrak{su}(3)$ $\mathcal{N} = 4$ SYM. Using the mapping (4.74) between the line lattices parametrized by (e, m) and the global structures of $\mathfrak{su}(3)$, the formula (4.139) reproduces the $\mathcal{N} = 4$ orbits under the element $ST \in \mathrm{SL}(2, \mathbb{Z})$. As shown in the literature [147, 148, 243, 244], this transformation can be combined with a proper gauging of the one-form symmetry to construct the non-invertible self-duality defects of $\mathfrak{su}(3)$ at $\tau = \mathrm{e}^{2\pi \mathrm{i}/3}$. Therefore in our notation we expect the existence of non-invertible symmetries involving Σ_k^i for the lattices labeled by $(e, m) = (1, 0), (0, 1), (1, 1)$, and none in the $(e, m) = (1, 2)$ case.

Similarly, one can consider the orbits in the case of $S_{4,1}$ with $N = 2$, where the SCFT enhances to $\mathfrak{so}(5)$ $\mathcal{N} = 4$ SYM. By using the transformations Σ_4^i as above we find the following orbits

$$\begin{tikzcd} (0,1) \arrow[r, leftrightarrow] & (1,0) & (1,1) \arrow[loop above] \end{tikzcd} , \tag{4.140}$$

where the pairs (e, m) parametrize the maximal sub-lattices of mutually local lines as discussed in section (4.2.3.2).

These reproduce the $\mathcal{N} = 4$ orbits under the element $S \in \mathrm{SL}(2, \mathbb{Z})$. Again this transformation can be combined with a proper gauging of the one-form symmetry to construct the non-invertible self-duality defects of $\mathfrak{so}(5)$ at $\tau = i$.

Motivated by this match, one can expect that in the case of general rank, non-invertible symmetries will be present when multiple choices of maximal sub-lattices of mutually local lines are related by the transformations Σ_k^i, as above. The orbits are

$$S_{3,1} : \quad \begin{tikzcd} (1,0) \arrow[r, leftrightarrow] & (0,1) \arrow[r, leftrightarrow] & (1,1) & (1,2) \arrow[loop right] \end{tikzcd} , \tag{4.141}$$

$$S_{4,1} : \left\{ \begin{tikzcd} (0,1) \arrow[r, leftrightarrow] & (1,0) & (1,1) \arrow[loop right] & N = 0 \bmod 2 \\ (1,0) \arrow[r, leftrightarrow] & (1,1) & (0,1) \arrow[loop right] & N = 1 \bmod 2 \end{tikzcd} \right. , \tag{4.142}$$

where the pairs (r, s) parametrize the maximal sub-lattices of mutually local lines as in section 4.2.4.

In the $S_{6,1}$, $S_{3,3}$ and $S_{4,4}$-cases, there is only one possible global structure that is mapped to itself by the Σ_k^i transformations.

By analogy with the cases where there is $\mathcal{N} = 4$ enhancement, we expect the existence of non-invertible symmetries when the transformations Σ_k^i map different line lattices, built by combining this Σ_k^i-action with a suitable gauging of the one-form symmetry.

4.3 Discussion and Conclusions

In this Chapter, we have exploited the recipe of [173] for arranging the charge lattice of genuine lines modulo screening by dynamical particles. We have adapted such strategy, originally designed for BPS quivers, to the case of (p, q)-strings, in order to access to the electromagnetic charges of non-Lagrangian $\mathcal{N} = 3$ S-fold SCFTs. This procedure has allowed us to provide a full classification of the one-form symmetries of every S-fold SCFT. We singled out two cases with a non-trivial one-form symmetry, corresponding to the $\mathbb{Z}_3$ and the $\mathbb{Z}_4$ S-folds in absence of discrete torsion, denoted here as $S_{3,1}$ and $S_{4,1}$ respectively. Our results are consistent with the supersymmetry enhancement that takes place when two D3-branes are considered. Lastly, we discuss the possibility of non-invertible duality defects, by recovering the expected results for the cases with supersymmetry enhancement and proposing a generalization at any rank.

We left many open questions that deserve further investigations. It would for example be interesting to study in more details the projection of the states generated by the (p, q)-configurations in an S-fold background. In the present article, the only relevant information was the electromagnetic charges carried by these states, but a deeper analysis of the dynamics of these S-fold theories requires more work. This would in turn improve our understanding of their mass spectrum. For instance, a comparison of the BPS spectrum could be made exploiting the Lagrangian descriptions of [320]. This could also help find the origin of the mapping between the multiple lattices found in the $S_{3,1}$ and $S_{4,1}$-cases. Further investigations in this direction would deepen our geometric understanding of the non-invertible symmetries expected in this class of theories, along the lines of the brane analysis of [50, 194, 218].

It would also be of interest to generalize the analysis to other $\mathcal{N} = 3$ SCFTs that are not constructed from S-fold projections, such as the exceptional $\mathcal{N} = 3$ theories [193, 242]. These theories can be obtained from M-theory backgrounds and one may study the charge lattice with probe M2-branes. One could therefore apply an analysis similar to the one spelled in [34, 39, 40, 176, 318].

Regarding the S-fold constructions, the cases of S-folds with $\mathcal{N} = 2$ supersymmetry [52, 195] also deserve further investigations (see [49, 76] for similar analysis in class S theories). In the absence of BPS quivers, one needs to adapt the UV analysis of [173]. In general, one would like to find a stringy description that avoids wall crossing and allows reading the charge lattices and the one-form symmetries for such theories.

CHAPTER 5

Geometric Engineering and Compactifications: The Spindle

Contents

A prediction of the AdS/CFT correspondence is the matching of exact quantities of a CFT with their gravitational counterparts. An ancestor result in this direction was obtained in [120], where the central charges of a 2d CFT was computed in terms of an AdS_3 gravitational background. Furthermore, in absence of a Lagrangian description of an interacting fixed point the correspondence represents a definition of the desired CFT. Another way to produce superconformal field theories consists of compactifying higher dimensional theories on curved manifolds, preserving some supersymmetry by turning on quantized magnetic background fluxes for the global symmetries. Such mechanism, commonly referred to as (partial) topological twist [94, 95, 316], has been vastly studied in many stringy and holographic setups.

The prototypical example was discussed in [265] in terms of branes wrapped on Riemann surfaces. From the gravitational side the mechanism is usually referred as a (gravitational) flow across dimensions. Then in [83] such flows have been generalized and related to the c-extremization principle of [82]. The c-extremization principle in this case is related to a gravitational attractor mechanism (see [45, 84, 109, 249, 251] for related works in this direction).

Recently it has been observed that one can extend the notion of the topological twist on manifolds with orbifold singularities [182]. The explicit orbifold considered in [183] is the spindle, topologically a two sphere with deficit angles at the poles. Supersymmetry in this case is preserved such that the Killing spinors are neither constant nor chiral on the orbifold. Furthermore, there are two ways to preserve supersymmetry, denoted as the twist and the anti-twist. Many field theoretical and gravitational constructions have been proposed in the recent years by considering compactifications on orbifolds [42, 113–115, 140, 141, 161–163, 165–167, 178–181, 183, 184, 196, 226, 228, 234, 250, 293, 295, 296]. This Chapter is based on [35] and is organized as follows. In section 5.2.1 we study the spindle compactification of the 4d non-lagrangian theories obtained in [67]. First, in subsection 5.2.1.1, we review the relevant aspects of the construction of [67] focusing on the ’t Hooft anomalies and on the distinction between the trial R-symmetry emerging from the higher dimensional picture and the exact one due to a-maximization. This distinction indeed plays a crucial role in the analysis. Then in sub-section 5.2.1.2 we study the compactification on the spindle and we compute the central charge of the emerging two-dimensional theory. In the computation of the exact 2d R-symmetry we observe that the result can be formulated (when the conditions of integerness on the fluxes is satisfied) in terms of the 4d trial R-symmetry or in terms of the 4d exact one. As a bonus we also study in subsection 5.2.1.3 the case of the spindle compactification of 4d models associated to negative degree bundles, corresponding to the models obtained in [274]. The in section 5.2.2 we review the supergravity truncation of [132] in order to fix the notations and the conventions that we use in subsequents sections of the Chapter. In section 5.2.3 we study the compactification of the spindle of these 5d $\mathcal{N} = 2$ gauged supergravities, obtaining the relevant BPS and Maxwell equations. In section 5.2.4 we focus on the calculation of the conserved charges and of the integer fluxes. In this way we can fix most of the scalars at their boundary values on the spindle and from these results we extract the exact central charges form the gravitational perspective. We eventually observe that these results agree with the ones obtained from the field theoretical analysis. In section 5.2.5 we complete our analysis by studying the gravitational solution. First, in sub-section 5.2.5.1 we look for an analytical solution, finding that it exists for the universal twist, for choices of p and q that correspond to a rational 4d R-symmetry. Then in sub-section 5.2.5.2 we look for numerical solutions for more generic values of p and q, by turning on also the magnetic charge associated to the flavor symmetry. We find numerical solutions only in the case of the anti-twist class for Riemann surfaces of positive curvature.

5.1 Anomaly Polynomials and Reduction

In this section, we will review the definition and some basic properties of anomaly polynomials and their behavior under dimensional reduction. This will be the basis for the computation of the 2d central charge of the compactified model discussed in this Chapter.

Given a 2d-dimensional quantum field theory (QFT), the anomaly polynomial is a (2d+2)-form I_{2d+2} from which anomalies can be extracted through the descent formalism [19–21, 307]. One can give a more physical interpretation of this in the following way. If one considers the partition function of a QFT $\mathcal{Z}[g_{\mu\nu}, A^I]$, this is actually a section of a certain line bundle. The first Chern class of this bundle is given by the integral of the anomaly polynomial over the whole spacetime. The argument goes as follows: the partition function of a gauge theory must be a well-defined function over the space of connections modulo gauge transformations $\mathcal{A}/G$. Any line bundle on $\mathcal{A}$ is trivializable, but not on

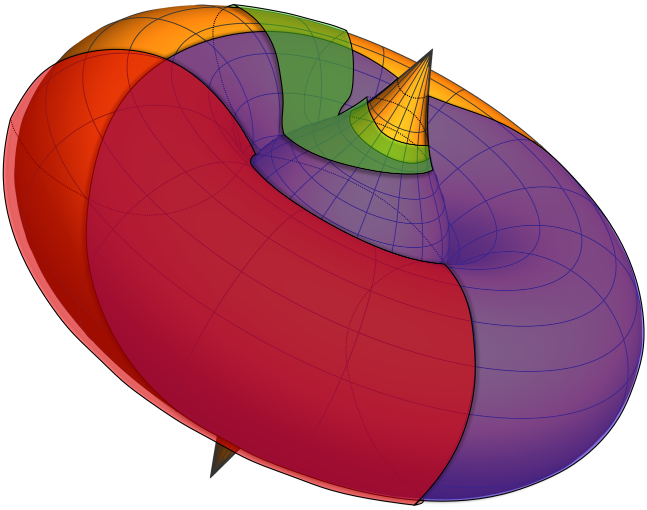

Figure 5.1: ”The forbidden everything bagel”. An artist depiction of M5-branes wrapped around a geometry resembling $T^2 \times \mathbb{\Sigma}$.

$\mathcal{A}/G$. Consider then $\mathcal{Z} \in \Gamma(\mathcal{L}, \mathcal{A}/G)$, a section on $\mathcal{L} \to \mathcal{A}/G$. Any trivialization of $\mathcal{A}/G$ allows one to promote $\mathcal{Z}$ to an actual function (a globally defined section). Anomalies are obstructions to such trivialization, i.e., a class in $\mathrm{H}^2(\mathcal{A}/G, \mathbb{Z})$.

In general, the anomaly polynomial is a characteristic class constructed from the fiber bundle and the tangent bundle. In the cases of our interest, the fiber bundle corresponds to the global symmetry group. If $\mathcal{S}$ is the set of chiral fields in the theory, then the total anomaly polynomial is defined as [108]

$$I_{2d+2} = \left(\sum_{\psi_i \in \mathcal{S}} c(\psi_i) \right) P_{d+1} = \sum_{\psi_i \in \mathcal{S}} I^{(i)}_{2d+2}, \tag{5.1}$$

where $c(\psi_i)$ are coefficients depending on the chiral field content of the theory and P_{d+1} is the characteristic class constructed out of polynomials in the curvature of the bundle

$$P_{d+1} = \mathrm{Tr}\, F^d := \mathrm{Tr}\, F \wedge \underbrace{F}_{d-\text{times}} \wedge \cdots \wedge F. \tag{5.2}$$

5.1.1 Anomaly Polynomial of a Stack of M5-branes

The anomaly polynomial relevant to our discussion is that of the 6d $\mathcal{N} = (2,0)$ SCFT, the worldvolume theory of an M5-brane. The M5-brane of M-theory has chiral world-volume fields that lead to potential anomalies in diffeomorphisms of the five-brane world-volume TW, as well as in diffeomorphisms that act as $\mathrm{SO}(5)$ gauge transformations of the connection on the normal bundle NW. More precisely, the field content of this theory comprises a three-form with self-dual field strength, five scalars, and four real Weyl fermions [215, 236, 313, 319]. The chiral spinors contribute to the I_8 anomaly polynomial by the following factor

$$I_D = \frac{1}{2}\mathrm{ch}S(NW)\hat{A}(TW). \tag{5.3}$$

The chiral spinors are sections of a rank-four spinor bundle constructed from the normal bundle NW using the spinor representation of $\mathrm{SO}(5)$, which is the remaining isometry from M-theory after the insertion of the M5-brane. The class $\hat{A}(TW)$ is the A-roof genus of the tangent bundle, generalizing the index of the Dirac operator on it. Up to order four, this is given by

$$\hat{A}(TW) = 1 - \frac{p_1(TW)}{24} + \frac{7p_1(TW)^2 - 4p_2(TW)}{5760}. \tag{5.4}$$

By expanding the Chern character up to the required order, the contribution from the Weyl spinors amounts to

$$I_D = \frac{1}{2}\left(\frac{p_2(NW)}{24} + \frac{p_1(NW)^2}{96} - \frac{p_1(NW)p_1(TW)}{48} + \frac{7p_1(TW)^2 - 4p_2(TW)}{1440}\right). \tag{5.5}$$

The chiral two-form propagates on the worldvolume and does not “see” the normal bundle. The standard anomaly of such a field is given by

$$I_A = \frac{1}{5760}\left(16p_1(TW)^2 - 112p_2(TW)\right). \tag{5.6}$$

There is an additional contribution term arising from a careful treatment of the Chern-Simons term in 11d supergravity in the presence of an M5-brane, which amounts to [185]

$$I_{CS} = -\frac{1}{24}p_2(NW). \tag{5.7}$$

When considering a stack of M5-branes, the situation is a bit trickier. The contributions from the worldvolume fields scale linearly with the number of branes, but the Chern-Simons interaction scales cubically [215]. Therefore, the final anomaly polynomial for a stack of M5-branes of type A_N is given by

$$I_8 = \frac{N-1}{48}\left[p_2(NW) - p_2(TW) + \frac{1}{4}\left(p_1(TW) - p_1(NW)\right)^2\right] + \frac{N^3 - N}{24}p_2(NW). \tag{5.8}$$

5.1.2 Reducing the Anomaly Polynomial

Following [224], we review how to relate the anomaly polynomial of a d-dimensional[1] theory to the one a dimensionally reduced $(\mathrm{d}-\mathrm{s})$-dimensional theory. We first suppose

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[1] To ease the discussion we are going to always consider even dimensional spaces as anomalies are absent in odd dimensions.

that we can neglect the isometries of the compactification manifold $\mathcal{M}^{(s)}$ and then see how the come into play.
Intuitively, if we consider a 2d-dimensional theory with anomaly polynomial I_{d+2}, then by compactifying the theory on a $\mathcal{M}^{(s)}$ manifold, the new anomaly polynomial is given by

$$I_{d-s+2} = \int\limits_{\mathcal{M}^{(s)}} I_{d+2}. \tag{5.9}$$

To see how this relation comes about, let $\mathcal{X}^{(d)}$ be the space-time manifold on which the initial theory is defined. Let $\mathcal{Y}^{(d+1)} = \mathcal{X}^{(d)} \times S^1$. We introduce a background metric and background gauge fields on $\mathcal{Y}^{(d+1)}$ by gluing the ones of $\mathcal{X}^{(d)} \times [0,1]$ on the boundaries $\mathcal{X}^{(d)}|_0$ and $\mathcal{X}^{(d)}|_1$ by the aforementioned diffeomorphism and gauge transformation that modify the phase of the partition function of the theory. Such phase is given by

$$\int\limits_{\mathcal{Y}^{(d+1)}} \mathcal{CS}_{d+1}, \quad \text{where } \mathrm{d}\mathcal{CS}_{d+1} = I_{d+1}. \tag{5.10}$$

Compactifying on a manifold $\mathcal{M}^{(s)}$ means taking, at least locally, $\mathcal{X}^{(d)} = \mathcal{X}^{(d-s)} \times \mathcal{M}^{(s)}$, which implies $\mathcal{Y}^{(d+1)} = \mathcal{Y}^{(d-s+1)} \times \mathcal{M}^{(d)}$. Therefore

$$\int\limits_{\mathcal{Y}^{(d+1)}} \mathcal{CS}_{d+1} = \int\limits_{\mathcal{Y}^{(d-s+1)}} \mathcal{CS}_{d-s+1}, \quad \text{where } \mathcal{CS}_{d-s+1} = \int\limits_{\mathcal{M}^{(s)}} \mathcal{CS}_{d+1}. \tag{5.11}$$

By considering a manifold $\mathcal{Z}^{(d-s+2)}$ such that $\partial\mathcal{Z}^{(d-s+2)} = \mathcal{Y}^{()}d - s + 1$ and set $\mathcal{Z}^{(d+2)} = \mathcal{Z}^{(d-s+2)} \times \mathcal{M}^{(s)}$, then (5.11) implies

$$\int\limits_{\mathcal{Z}^{(d+2)}} I_{d+2} = \int\limits_{\mathcal{Z}^{(d-s+2)}} I_{d-s+2} \implies I_{d-s+2} = \int\limits_{\mathcal{M}^{(s)}} I_{d+2}. \tag{5.12}$$

To include isometries in this discussion, one needs to add a background gauge field for such isometry. This is done by taking $\mathcal{Z}^{(d+2)}$ to be a non-trivial $\mathcal{M}^{(s)}$ bundle over $\mathcal{Z}^{(d-s+2)}$ with a non-trivial G-connection, so that

$$\mathcal{M}^{(s)} \hookrightarrow \mathcal{Z}^{(d+2)} \xrightarrow{\pi} \mathcal{Z}^{(d-s+2)}. \tag{5.13}$$

5.2 Spindle Compactification of 4d SCFTs from Riemann Surfaces

In this Chapter we will focus on the case of M5 branes wrapped on a complex curve $\mathcal{C}_\mathfrak{g}$ in a Calabi-Yau three-fold $\mathcal{X}$ [66, 67]. These models are a generalization of the ones obtained in [265] where M5 branes wrapped on a Riemann surface were considered. The construction of [66, 67] generates an infinite family of 4d SCFTs obtained by gluing T_N theories [188]. The setup is specified by two integers that depend on the local geometry of $\mathcal{X}$, corresponding to a decomposable $\mathbb{C}^2$ bundle over $\mathcal{C}_\mathfrak{g}$. The (non-negative) integers, denoted as p and q, are the Chern numbers of the line bundles $\mathcal{L}_{1,2}$ that specify $\mathcal{L}_1 \oplus$

$\mathcal{L}_2 \to \mathcal{C}_\mathfrak{g}$. For $p = q$ the $\mathcal{N} = 1$ case studied in [265] is recovered, while $p = 0$ (or $q = 0$) corresponds to the $\mathcal{N} = 2$ case of [265] . For other choices of p and q the 4d SCFT corresponds to a different $\mathcal{N} = 1$ SCFT.

While M5 branes and the theories of [265] have been already studied on the spindle in various setups [114, 140, 181, 295] a general analysis for the models introduced in [66, 67] has not been pursued so far. Here we are interested in generic choices of p and q from the supergravity perspective. Our starting point are the 5d consistent truncations obtained in full generality by [132] (see also [133, 177, 268, 299] for earlier results in this direction). Such truncations have the advantage to hold for any choice of p and q, but the price to pay in this case is the presence of hypermultiplets. Anyway, by exploiting the general recipe of [56], we can analyze the reduction on the spindle of the consistent truncations of [132] even in presence of hypermultiplets. The reason is that in this case one hyperscalar triggers an Higgs mechanism that gives a mass to one of the vector multiplets. The Higgsing simplifies the analysis of the BPS equations and of the fluxes at the poles of the spindle, allowing to find the boundary conditions that most of the scalars have to satisfy at the poles in order to compute the central charges in the twist and in the anti-twist class. While this analysis makes the calculation of the central charges possible, it does not guarantee the existence of a solution. Furthermore, it does not fix the boundary condition for the hyperscalar.

However, by restricting to the graviton sector, the universal analytic solution of the type discussed in [180, 182] is found. In this case the scalars are fixed to their AdS$_5$ value. Observe that the universal twist is consistent only if the 4d superconformal R-charge is rational, and this limits the amount of accessible truncations. For more general twists, beyond the universal one, we solved numerically the BPS equations for various values of the hyperscalar at one of the poles of the spindle. When the (unique) value of the hyperscalar that solves the BPS equation, at such pole of the spindle, is found, the existence of the solution is guaranteed. The procedure fixes also the boundary condition for the hyperscalar at the other pole and the finite distance between the poles.

In the following we will exploit such procedure for the consistent truncations of [132] and we will compare our results with the one found on the field theory side by integrating the anomaly polynomial.

5.2.1 The 4d SCFT on the Spindle

In sub-section 5.2.1.1 we are going to review the M-theory construction of $\mathcal{N} = 1$ SCFTs in 4d of [67], which is going to be the starting point for our effective 2d theories compactified on the spindle. These models turn out to be dual to $\mathcal{N} = 1$ SCFT built by opportunely gluing T_N blocks [188]. Then in sub-section 5.2.1.2 we construct the theory compactified on the spindle $\mathbb{\Sigma}$, closely following [56, 182] mutatis mutandis. Eventually in sub-section 5.2.1.3 we study the case of negative degree bundles, obtained in [274], on the spindle.

5.2.1.1 The 4d Model

The worldvolume theory of stack of N M5-branes is well known to be a $6d$ $\mathcal{N} = (2,0)$ SCFT. One can construct effective 4d theories by wrapping the branes on some specific geometry. In this particular case, we are interested in effective 4d theories obtained by wrapping the M5-branes on a complex Riemann curve of genus $\mathfrak{g}$ $\mathcal{C}_{\mathfrak{g}}$ in a Calabi-Yau three-fold. This geometric construction gives rise to an infinite family of 4d effective theories which are parametrized by two integers depending on the local geometry of the Calabi-Yau three-fold X which in the case of interest is just a holomorphic $\mathbb{C}^2$ bundle over $\mathcal{C}_{\mathfrak{g}}$

$$\mathbb{C}^2 \hookrightarrow X \xrightarrow{\pi} \mathcal{C}_{\mathfrak{g}} . \tag{5.14}$$

Crucially, when X is decomposable it will take the simpler form $X = \mathcal{L}_1 \oplus \mathcal{L}_2$. This structure has a manifest $\mathrm{U}(1)^2$ isometry, one factor for each fiber in the line bundle. The two isometries give rise to two abelian symmetries, one being the R-symmetry $\mathrm{U}(1)_R$ and the other being an additional flavor symmetry $\mathrm{U}(1)_F$.
The integers describing the families of IR $\mathcal{N} = 1$ SCFTs are just the Chern numbers labelling the possible bundle decomposition

$$c_1(\mathcal{L}_1) = p, \qquad c_1(\mathcal{L}_2) = q, \tag{5.15}$$

subject to the Calabi-Yau condition $p + q = 2(\mathfrak{g} - 1)$. Depending on the choices of these two integers, the fields in the M5-brane theory transform in different representation of the $\mathrm{U}(1)_F$ symmetry, leading to different IR fixed points. A solution to the constraint of the Chern numbers is given by the following parametrization

$$p = (1 + \mathbf{z})(\mathfrak{g} - 1), \qquad q = (1 - \mathbf{z})(\mathfrak{g} - 1) \tag{5.16}$$

where $\mathbf{z}(\mathfrak{g} - 1) \in \mathbb{Z}$.
An explicit field theory construction for these theories can be given when the integers p, q in (5.15) are positive. For these cases in fact the theories can be described, from class-$\mathcal{S}$, as opportune gluing of $2(\mathfrak{g} - 1)$ T_N building blocks to create a Riemann surface with no punctures. In the next section we are going also to consider the cases of negative p, q whose explicit construction was given in [274] but for which no dual gravity solution is known.
In this setup the key observables are the central charges c and a, determined by the following combinations of R-symmetry anomalies

$$\begin{aligned} c &= \frac{1}{32}\left(9 \operatorname{Tr} R^3 - 5 \operatorname{Tr} R\right) , \\ a &= \frac{3}{32}\left(3 \operatorname{Tr} R^3 - \operatorname{Tr} R\right) . \end{aligned} \tag{5.17}$$

Note that in the large N limit, for holographic SCFTs $a = c$. The central charges can be recovered from the known anomaly polynomial of the M5-brane theory integrated over $\mathcal{C}_{\mathfrak{g}}$, assuming that no accidental symmetries are generated along the flow. Since the abelian

symmetries $\mathrm{U}(1)_R$ and $\mathrm{U}(1)_F$ mix together, the exact superconformal R-symmetry is found by a-maximization [239].
One finds that the 't Hooft anomalies of the trial R-charge $R(\epsilon_{4d}) = R + \epsilon_{4d} F$, for theories of type $G = A_N, D_N, E_N$, are given by

$$\begin{aligned} &\mathrm{Tr}\, R(\epsilon_{4d})^3 = (\mathfrak{g}-1)[(r_G + d_G h_G)(1 + \mathbf{z}\epsilon_{4d}^3) - d_G h_G(\epsilon_{4d}^2 + \mathbf{z}\epsilon)] \,, \\ &\mathrm{Tr}\, R(\epsilon_{4d}) = (\mathfrak{g}-1) r_G (1 + \mathbf{z}\epsilon_{4d}) \,, \end{aligned} \tag{5.18}$$

where r_G, d_G and h_G are the rank, dimension and Coxeter number of G respectively, while ϵ is the mixing parameter.
We are interested in the A_{N-1} case at large N. By plugging (5.18) into (5.17) we can use a-maximisation to find the superconformal R-charge. This is given by the mixing $R(\epsilon_{4d}^*) \equiv R^* = R + \epsilon_{4d}^* F$ where the mixing parameter at large N, fixed by a-maximisation, is given by

$$\epsilon_{4d}^* = \frac{1 + \mathbf{k}\sqrt{1 + 3\mathbf{z}^2}}{3\mathbf{z}} \,, \tag{5.19}$$

where $\mathbf{k}$ is half of the scalar curvature of $\mathcal{C}_\mathfrak{g}$[2]. Choosing $\mathbf{k} = -1$ for later purposes, the 't Hooft anomalies for the superconformal R-symmetry read

$$\begin{aligned} &k_{R^*R^*R^*} = \frac{2(\mathfrak{g}-1)}{27\mathbf{z}^2}\left[9\mathbf{z}^2 - 1 + (3\mathbf{z}^2+1)^{3/2}\right] N^3, \qquad k_{R^*R^*F} = 0, \\ &k_{R^*FF} = -\frac{(\mathfrak{g}-1)}{3}\sqrt{3\mathbf{z}^2+1}N^3, \qquad k_{FFF} = (\mathfrak{g}-1)\mathbf{z}N^3. \end{aligned} \tag{5.20}$$

The mixed 't Hooft anomalies between the R-symmetry R and the flavor symmetry F can be computed from (5.18) and they read

$$\begin{aligned} &k_{RRR} = (\mathfrak{g}-1)N^3, \qquad && k_{RRF} = -\frac{1}{3}(\mathfrak{g}-1)\mathbf{z}N^3, \\ &k_{RFF} = -\frac{1}{3}(\mathfrak{g}-1)N^3, \qquad && k_{FFF} = (\mathfrak{g}-1)\mathbf{z}N^3. \end{aligned} \tag{5.21}$$

5.2.1.2 BBBW on the Spindle

Consider the 4d SCFT reviewed above, whose anomaly polynomial in the large N limit reads

$$I_6 = \frac{1}{6}\sum_{i,j,k=R,F} k_{ijk}\, c_1(F_i)c_1(F_j)c_1(F_k) \tag{5.22}$$

where the coefficients k_{ijk} are given by the mixed 't Hooft anomalies (5.21) and the $c_1(F_{R,F})$ are the first Chern-classes for the $\mathrm{U}(1)$-bundles over the total space X_4 with

---

[2] The Ricci scalar curvature is normalized such that $\mathbf{k} = 1$ for $\mathfrak{g} = 0$, $\mathbf{k} = 0$ for $\mathfrak{g} = 1$ and $\mathbf{k} = -1$ for $\mathfrak{g} > 1$.

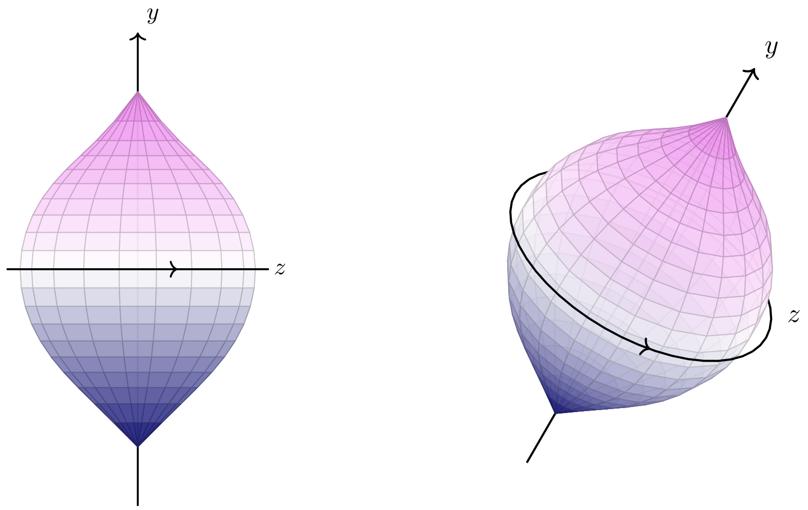

Figure 5.2: A Spindle from two different view angles. The proper coordinate system on $\mathbb{\Sigma}$ is shown.

gauge curvature R and F. We proceed to compactify further the 4d theory over the spindle $\mathbb{\Sigma} \equiv \mathbb{WCP}^1_{[n_N,n_S]}$, where n_N, n_S label the deficit angles at the north and south pole of the orbifold respectively, with background magnetic fluxes for the two abelian $\mathrm{U}(1)_R$ and $\mathrm{U}(1)_F$ symmetries of the 4d theory. In order to do that, we need to take into account the azimuthal $\mathrm{U}(1)_J$ isometry of the spindle which is generated by rotations about the axis passing through the poles. Geometrically, this is given by considering the total space X_4 as a X_2 orbibundle fibered over $\mathbb{\Sigma}$. In the field theory, this can be achieved by turning on a connection A_J for the $\mathrm{U}(1)_J$ isometry, so that we can write the following gauge connections

$$A^{(I)} = \rho_I(y)(\mathrm{d}z + A_J) \qquad I = R, F \tag{5.23}$$

where $\rho_I(y)$ are the background fluxes for the abelian symmetries, and $(y,\, z)$ are respectively the longitundal and azimuthal coordinates over $\mathbb{\Sigma}$, with $y \in [y_N,\, y_S]$ and $z \sim z + 2\pi$. The curvatures for the fields (5.23) are given by

$$F^{(I)} = \rho_I'(y)\,\mathrm{d}y \wedge (\mathrm{d}z + A_J) + \rho_I(y) F_J \qquad I = R, F \tag{5.24}$$

where $F_J = \mathrm{d}A_J$. These fields are consistent with the flux condition

$$\frac{1}{2\pi}\int_{\mathbb{\Sigma}} F^{(I)} = [\rho_I]^{y_S}_{y_N} = \frac{p_I}{n_S n_N}. \tag{5.25}$$

The curvature forms $F^{(I)}$ define a U(1)-line bundle $\mathcal{L}_I$ over X_4, and the associated first Chern classes are[3]

$$c_1(\mathcal{L}_I) \equiv \left[\frac{F^{(I)}}{2\pi}\right] \in \mathrm{H}^2(X_4,\mathbb{R}), \qquad c_1(J) \equiv \left[\frac{F_J}{2\pi}\right] \in \mathrm{H}^2(X_2,\mathbb{R}). \tag{5.26}$$

To obtain the 2d anomaly polynomial, we make the following substitution

$$c_1(R) \to c_1(R) + \frac{1}{2}c_1(\mathcal{L}_R), \qquad c_1(F) \to c_1(F) + c_1(\mathcal{L}_F) \tag{5.27}$$

where $c_1(R)$ and $c_1(F)$ are the pull-back of the $\mathrm{U}(1)_R$ and $\mathrm{U}(1)_F$ bundles over X_2 respectively. The choice of normalization is such that the R-symmetry generators give charge 1 to the supercharges. Thus, we shift the curvatures in Eq. (5.24) accordingly, compute the anomaly polynomial in Eq. (5.22) and integrate it over Σ. The result is a combination of the four non-zero mixed 't Hooft anomalies given in sec. 5.2.1.1. In the following, as a working example we show only the computation for the terms proportional to k_{RRR}

$$\int_\Sigma \left(c_1(R) + \frac{1}{2}c_1(\mathcal{L}_R)\right)^3 = \int_\Sigma \left(\frac{3}{2}c_1(R)^2 c_1(\mathcal{L}_R) + \frac{3}{4}c_1(R)c_1(\mathcal{L}_R)^2 + \frac{1}{8}c_1(\mathcal{L}_\mathcal{R})^3\right) , \tag{5.28}$$

where the product of forms is understood. Notice that the $c_1(R)$ does not depend on the spindle, so they can be factorized out of the integral. Let us consider the first term in (5.28)

$$\int_\Sigma \frac{3}{2}c_1(R)^2 c_1(\mathcal{L}_R) = \frac{3}{2}c_1(R)^2 \int_\Sigma \frac{F^{(R)}}{2\pi} = \frac{3}{2}c_1(R)^2 [\rho_R]_{y_N}^{y_S}. \tag{5.29}$$

The second term reads

$$\begin{aligned}
\int_\Sigma \frac{3}{4}c_1(R)c_1(\mathcal{L}_R)^2 &= \frac{3}{4}c_1(R)\int_\Sigma \frac{1}{4\pi^2}F^{(R)}\wedge F^{(R)} \\
&= \frac{3}{4}c_1(R)\int_\Sigma \frac{2}{4\pi^2}\rho_R(y)\rho_R'(y)\,\mathrm{d}y\wedge(\mathrm{d}z + A_J)\wedge F_J \\
&= \frac{3}{4}c_1(R)\int_\Sigma \frac{1}{4\pi^2}\,\mathrm{d}\rho_R^2\wedge(\mathrm{d}z\wedge F_J + A_J\wedge F_J) \\
&= \frac{3}{4}c_1(R)c_1(J)\int_\Sigma \frac{1}{2\pi}\,\mathrm{d}\rho_R^2\wedge \mathrm{d}z \\
&= \frac{3}{4}c_1(R)c_1(J)[\rho_R^2]_{y_N}^{y_S}
\end{aligned} \tag{5.30}$$

where we used the fact that $A_J\wedge F_J$ is just a total derivative and that F_J does not depend on the spindle as stated in (5.26). In the second to last step we went back from forms to cohomology classes. The last term in (5.28) evaluates to

$$\frac{1}{8}\int_\Sigma c_1(\mathcal{L}_R)^3 = \frac{1}{8}\int_\Sigma \frac{1}{(2\pi)^3}F^{(R)}\wedge F^{(R)}\wedge F^{(R)} = \frac{1}{8}c_1(J)^2[\rho_R^3]_{y_N}^{y_S}. \tag{5.31}$$

---

[3] Note that the gauge curvature of J is only defined on X_2. It's Chern class will not contribute in the integral.

The complete anomaly 4-form of the 2d theory reads

$$\begin{aligned} I_4 = &\frac{1}{4}\left(k_{RRR}[\rho_R]^{y_S}_{y_N} + 2k_{RRF}[\rho_F]^{y_S}_{y_N}\right) c_1(R)^2 + \frac{1}{4}\left(k_{RFF}[\rho_R]^{y_S}_{y_N} + 2\,k_{FFF}[\rho_F]^{y_S}_{y_N}\right) c_1(F)^2 \\ &+ \frac{1}{48}\left(k_{RRR}[\rho_R^3]^{y_S}_{y_N} + 8k_{FFF}[\rho_F^3]^{y_S}_{y_N} + 6k_{RRF}[\rho_F\rho_R^2]^{y_S}_{y_N} + 12k_{RFF}[\rho_R\rho_F^2]^{y_S}_{y_N}\right) c_1(J)^2 \\ &+ \frac{1}{2}\left(k_{RRF}[\rho_R]^{y_S}_{y_N} + 2\,k_{RFF}[\rho_F]^{y_S}_{y_N}\right) c_1(F)c_1(R) \\ &+ \frac{1}{8}\left(k_{RRR}[\rho_R^2]^{y_S}_{y_N} + 4k_{RRF}[\rho_R\rho_F]^{y_S}_{y_N} + 4k_{RFF}[\rho_F^2]^{y_S}_{y_N}\right) c_1(J)c_1(R) \\ &+ \frac{1}{8}\left(4\,k_{FFF}[\rho_F^2]^{y_S}_{y_N} + k_{RRF}[\rho_R^2]^{y_S}_{y_N} + 4k_{RFF}[\rho_R\rho_F]^{y_S}_{y_N}\right) c_1(J)c_1(F) \end{aligned} \tag{5.32}$$

To compute the exact central charge we allow a mixing between the various $\mathrm{U}(1)$ factors $c_1(J) = \epsilon_{2d}\, c_1(R)$ and $c_1(F) = x_{2d}\, c_1(R)$, extremizing the function

$$c^{2d}_{\text{trial}}(\epsilon_{2d}, x_{2d}) = \frac{6I_4}{c_1(R)^2}. \tag{5.33}$$

The background magnetic fluxes are fixed to be

$$\int \frac{F^{(R)}}{2\pi} = \frac{p_R}{n_S n_N}, \qquad \int \frac{F^{(F)}}{2\pi} = \frac{p_F}{n_S n_N} \tag{5.34}$$

where $p_R, p_F \in \mathbb{Z}$. For the R-symmetry, we have two possible choices of fluxes consistent with supersymmetry

$$\rho_R(y_N) = \frac{(-1)^{t_N}}{n_N}, \qquad \rho_R(y_S) = \frac{(-1)^{t_S+1}}{n_S} \tag{5.35}$$

where $t_N = 0, 1$, while t_S is fixed by the twisting procedure, namely $t_S = t_N$ for the twist, while $t_S = t_N + 1$ for the anti-twist. For the flavor symmetry, the flux can be fixed to

$$\rho_F(y_N) = \mathbf{z}_0, \qquad \rho_F(y_S) = \frac{p_F}{n_S n_N} + \mathbf{z}_0 \tag{5.36}$$

where $\mathbf{z}_0$ is an arbitrary constant.
Let us consider the following parametrization of the on-shell central charge

$$c^{2d}_{\text{trial}}(\epsilon^*_{2d}, x^*_{2d}) \equiv c^*_{2d} = \frac{f(n_S, n_N, p_F; \mathbf{z})}{g(n_S, n_N, p_F; \mathbf{z})}(\mathfrak{g} - 1)N^3. \tag{5.37}$$

In the case of the twist we have

$$
\begin{aligned}
f(n_S,n_N,p_F;\mathbf{z}) &= \left((n_N+n_S)^2-4p_F^2\right)\left(2\mathbf{z}p_F+(-1)^{t_N}(n_N+n_S)\right)\\
&\times\Big((-1)^{t_N}(n_N+n_S)\left(16\mathbf{z}p_F+\left(\mathbf{z}^2+3\right)(-1)^{t_N}(n_N+n_S)\right)\\
&+4\left(3\mathbf{z}^2+1\right)p_F^2\Big),\\
g(n_S,n_N,p_F;\mathbf{z}) &= 2n_Nn_S\Big(8p_F^2\left(-2n_Nn_S+3\mathbf{z}^2(n_S^2+n_N^2)\right)-32\mathbf{z}p_F^3(-1)^{t_N}(n_N+n_S)\\
&+8\mathbf{z}p_F(-1)^{t_N}(n_N+n_S)\left(3n_N^2-2n_Nn_S+3n_S^2\right)\\
&-48\mathbf{z}^2p_F^4+(n_N+n_S)^2\left(-2\left(\mathbf{z}^2+2\right)n_Nn_S+\left(\mathbf{z}^2+4\right)n_S^2\right.\\
&\left.+\left(\mathbf{z}^2+4\right)n_N^2\right)\Big).
\end{aligned}
\tag{5.38}
$$

The central charge is extremized by the mixing $\epsilon^*_{2d}, x^*_{2d}$ for which we give the exact, albeit quite cumbersome, result

$$
\epsilon^*_{2d}=\frac{\varepsilon(n_S,n_N,p_F;\mathbf{z})}{d(n_S,n_N,p_F;\mathbf{z})},\qquad x^*_{2d}=\frac{\chi(n_S,n_N,p_F;\mathbf{z})}{d(n_S,n_N,p_F;\mathbf{z})}-\mathbf{z}_0\epsilon^*_{2d} \tag{5.39}
$$

where

$$
\begin{aligned}
\varepsilon(n_S,n_N,p_F;\mathbf{z}) &= 4n_Nn_S(-1)^{t_N}(n_N-n_S)(2n_N(-1)^{t_N}(8\mathbf{z}p_F+(\mathbf{z}^2+3)n_S(-1)^{t_N})\\
&+16\mathbf{z}p_Fn_S(-1)^{t_N}+4(3\mathbf{z}^2+1)p_F^2+(\mathbf{z}^2+3)n_S^2+(\mathbf{z}^2+3)n_N^2)
\end{aligned}
\tag{5.40}
$$

$$
\begin{aligned}
\chi(n_S,n_N,p_F;\mathbf{z}) &= -2n_S^2\left(2\left(\mathbf{z}^2-3\right)p_Fn_N(-1)^{t_N}-20\mathbf{z}p_F^2+3\mathbf{z}n_N^2\right)\\
&-4n_S^3(-1)^{t_N}\left(\mathbf{z}n_N(-1)^{t_N}-2p_F\right)\\
&-4\mathbf{z}n_S(-1)^{t_N}\left(n_N^2-4p_F^2\right)\left(2\mathbf{z}p_F+n_N(-1)^{t_N}\right)\\
&-16\left(\mathbf{z}^2+1\right)p_F^3n_N(-1)^{t_N}-4\left(\mathbf{z}^2+1\right)p_Fn_N^3(-1)^{t_N}\\
&-24\mathbf{z}p_F^2n_N^2-16\mathbf{z}p_F^4-\mathbf{z}n_S^4-\mathbf{z}n_N^4
\end{aligned}
\tag{5.41}
$$

$$
\begin{aligned}
d(n_S,n_N,p_F;\mathbf{z}) &= 24\mathbf{z}^2p_F^2n_S^2+4n_N^3(-1)^{t_N}\left(6\mathbf{z}p_F+(-1)^{t_N}n_S\right)\\
&+2\mathbf{z}n_N^2\left(4p_Fn_S(-1)^{t_N}+12\mathbf{z}p_F^2-\mathbf{z}n_S^2\right)\\
&+4n_N(-1)^{t_N}\left(n_S^2-4p_F^2\right)\left(2\mathbf{z}p_F+n_S(-1)^{t_N}\right)\\
&+24\mathbf{z}p_Fn_S^3(-1)^{t_N}-32\mathbf{z}p_F^3n_S(-1)^{t_N}\\
&-48\mathbf{z}^2p_F^4+\left(\mathbf{z}^2+4\right)n_S^4+\left(\mathbf{z}^2+4\right)n_N^4
\end{aligned}
\tag{5.42}
$$

Notice that there is no explicit $\mathbf{z}_0$ dependence in the central charge.
We can check the validity of the result, by considering the S^2 limiting case, where $n_S = n_N = 1$, $p_F = 0$ and comparing with the result of [83]. As expected the two results match[4]. One can see that in this limit the mixing parameter $\epsilon^*_{2d} = 0$. This is to be expected since in this limit the spindle becomes a $\mathbb{P}^1$, and the abelian $\mathrm{U}(1)_J$ is enhanced to the

---

[4] From the result of [83], one fixes $\eta_1 = 2(\mathfrak{g}-1), \eta_2 = -2, \kappa_1 = -1, \kappa_2 = 1, z_1 = z_2 = \mathbf{z}$ to find the matching.

SU(2) isometry of the $\mathbb{P}^1$, thus does not mix anymore with the R-symmetry.
Instead, for the anti-twist case the on-shell central charge is given by

$$\begin{aligned} f(n_S,n_N,p_F;\mathbf{z}) &= \left((n_S-n_N)^2-4p_F^2\right)\left(2\mathbf{z}p_F+(-1)^{t_N}(n_N-n_S)\right)\\ &\times\Big((-1)^{t_N}(n_N-n_S)\left(16\mathbf{z}p_F+\left(\mathbf{z}^2+3\right)(-1)^{t_N}(n_N-n_S)\right)\\ &+4\left(3\mathbf{z}^2+1\right)p_F^2\Big)\\ g(n_S,n_N,p_F;\mathbf{z}) &= 2n_Nn_S\Big(8p_F^2\left(2n_Nn_S+3\mathbf{z}^2n_S^2+3\mathbf{z}^2n_N^2\right)+32\mathbf{z}p_F^3(-1)^{t_N}(n_S-n_N)\\ &-8\mathbf{z}p_F(-1)^{t_N}(n_S-n_N)\left(3n_N^2+2n_Nn_S+3n_S^2\right)\\ &-48\mathbf{z}^2p_F^4+(n_S-n_N)^2\left(2\left(\mathbf{z}^2+2\right)n_Nn_S+\left(\mathbf{z}^2+4\right)n_S^2\right.\\ &\left.+\left(\mathbf{z}^2+4\right)n_N^2\right)\Big) \end{aligned} \tag{5.43}$$

where the extremum, using the same parametrization as in (5.39), is reached for the following mixing

$$\begin{aligned} \varepsilon(n_S,n_N,p_F;\mathbf{z}) &= -4n_Nn_S(-1)^{t_N}(n_N+n_S)\Big(2n_N(-1)^{t_N}\left(8\mathbf{z}p_F-(-1)^{t_N}\left(\mathbf{z}^2+3\right)n_S\right)\\ &-16\mathbf{z}p_Fn_S(-1)^{t_N}+4\left(3\mathbf{z}^2+1\right)p_F^2+\left(\mathbf{z}^2+3\right)\left(n_S^2+n_N^2\right)\Big)\\ \chi(n_S,n_N,p_F;\mathbf{z}) &= -2n_S^2\left(2(-1)^{t_N}\left(\mathbf{z}^2-3\right)p_Fn_N-20\mathbf{z}p_F^2+3\mathbf{z}n_N^2\right)\\ &+4(-1)^{t_N}n_S^3\left((-1)^{t_N}\mathbf{z}n_N-2p_F\right)\\ &+4(-1)^{t_N}\mathbf{z}n_S\left(n_N^2-4p_F^2\right)\left(2\mathbf{z}p_F+(-1)^{t_N}n_N\right)\\ &-16(-1)^{t_N}\left(\mathbf{z}^2+1\right)p_F^3n_N-4(-1)^{t_N}\left(\mathbf{z}^2+1\right)p_Fn_N^3\\ &-24\mathbf{z}p_F^2n_N^2-16\mathbf{z}p_F^4-\mathbf{z}(n_S^4+n_N^4)\\ d(n_S,n_N,p_F;\mathbf{z}) &= 24\mathbf{z}^2p_F^2n_S^2+4(-1)^{t_N}n_N^3\left(6\mathbf{z}p_F-(-1)^{t_N}n_S\right)\\ &+2\mathbf{z}n_N^2\left(-4(-1)^{t_N}p_Fn_S+12\mathbf{z}p_F^2-\mathbf{z}n_S^2\right)\\ &+4(-1)^{t_N}n_N\left(n_S^2-4p_F^2\right)\left(2\mathbf{z}p_F-(-1)^{t_N}n_S\right)\\ &-24(-1)^{t_N}\mathbf{z}p_Fn_S^3+32(-1)^{t_N}\mathbf{z}p_F^3n_S\\ &-48\mathbf{z}^2p_F^4+\left(\mathbf{z}^2+4\right)\left(n_S^4+n_N^4\right) \end{aligned} \tag{5.44}$$

Once again, the on-shell central charge does not depend on $\mathbf{z}_0$ as expected.
The central charge calculated from the R^*, F anomalies (5.20) instead of R, can be computed in the same manner as just described. The two exact central charges will then match as follows

$$\begin{aligned} &c_{2d}^*\Big(\epsilon_{2d}^{(1)*},x_{2d}^{(1)*};R,F,n_S(-1)^{t_N}+n_N(-1)^{t_S},p_F\Big)\\ &=c_{2d}^*\Big(\epsilon_{2d}^{(2)*},x_{2d}^{(2)*};R^*,F,n_S(-1)^{t_N}+n_N(-1)^{t_S},p_F+\epsilon_{4d}^*\frac{n_S(-1)^{t_N}+n_N(-1)^{t_S}}{2}\Big) \end{aligned} \tag{5.45}$$

where ϵ^*_{4d} is the 4d mixing parameter found in (5.19) with $\mathbf{k} = -1$, and we specified which symmetries we are considering as well as their fluxes. Namely, the former is obtained from the anomaly polynomial considering the 't Hooft anomalies (5.21) and their fluxes, while the latter is obtained considering the anomalies (5.20) and their fluxes are related with the other by a shift.

Observe that the universal twist is consistent only if the exact 4d R-symmetry is rational. From the second line in (5.45) it follows that this choice requires to set the combination $p_F + \epsilon^*_{4d}\frac{n_S(-1)^{t_N}+n_N(-1)^{t_S}}{2}$ to zero. The integerness conditions on p_F, n_S and n_N then restrict the allowed values of p and q admitting the universal twist.

5.2.1.3 Negative Degree Bundles

Here we further generalize the construction of [66, 67] by gluing $2(\mathfrak{g}-1)$ together copies of $T_N^{(m)}$ theories [5]. This construction reproduces the model of [66, 67] when $m = 0$ [274] and generalizes it for generic m. The construction of [66, 67] in fact allows only for positive $p, q \geq 0$, while in the construction of [274], one can allow also for negative degree bundles. Although these theories have no known supergravity description at this time, we give the field theory calculation for completeness.

The cubic anomalies of the model of [66, 67] can be recovered from the ones of the $T_N^{(m)}$ blocks by linear combination of the $\mathrm{U}(1)_i$ isometries of the line bundles. Namely, $R = (J_+ + J_-)/2$ and $F = (J_- - J_+)/2$, following the naming convention of [274]. Therefore, in the large-N limit

$$\begin{aligned} k_{RRR} &= \frac{N^3}{2}, & k_{RRF} &= -\frac{1}{6}(1+2m)N^3, \\ k_{RFF} &= -\frac{N^3}{6}, & k_{FFF} &= \frac{1}{2}(1+2m)N^3 \end{aligned} \tag{5.46}$$

where the integer m parametrizes the degree of the line bundles $p = m+1$ and $q = -m$. Following the same arguments as before, we can compactify these theories on the spindle and find the central charge of a family of theories parametrized by m. By taking the anomaly polynomial constructed from the anomalies (5.46), we find the following central charge in the case of the twist

$$\begin{aligned} f(n_S, n_N, p_F; m) = {} & 2\left(4p_F^2 - (n_N+n_S)^2\right)\left(2(2m+1)p_F + (-1)^{t_N}(n_S+n_N)\right) \\ & \times \bigg((-1)^{t_N}\left(n_N+n_S\right)\left(4(2m+1)p_F + \left(m^2+m+1\right)(-1)^{t_N}\left(n_N+n_S\right)\right) \\ & + 4(3m(m+1)+1)p_F^2\bigg) \end{aligned} \tag{5.47}$$

$$\begin{aligned} g(n_S, n_N, p_F; m) = {} & n_N n_S \bigg((-1)^{t_N}\big(4n_S^3\left(6(2m+1)p_F + (-1)^{t_N}n_N\right) \\ & + 2(-1)^{t_N}(2m+1)n_S^2\big(12(2m+1)p_F^2 \\ & - (-1)^{t_N}n_N\left((-1)^{t_N}(2m+1)n_N - 4p_F\right)\big) \\ & + 4n_S\left(n_N^2 - 4p_F^2\right)\left(2(2m+1)p_F + (-1)^{t_N}n_N\right) \\ & + n_N\big((-1)^{t_N}n_N\big((-1)^{t_N}n_N\left(24(2m+1)p_F + (-1)^{t_N}(4m(m+1)+5)n_N\right) \end{aligned}$$

$$\begin{aligned}
&+ 24(2m+1)^2 p_F^2\big) - 32(2m+1)p_F^3\big) + (-1)^{t_N}(4m(m+1)+5)n_S^4\big) \\
&- 48(2m+1)^2 p_F^4\Big)
\end{aligned} \tag{5.48}$$

where we used the parametrization (5.37). The mixing is given by

$$\begin{aligned}
\varepsilon(n_S, n_N, p_F; m) = {}& 16 n_N n_S (-1)^{t_N} \Big(4(-1)^{t_N}(2m+1)p_F \left(n_N^2 - n_S^2\right) \\
&+ 4(3m(m+1)+1)p_F^2 \left(n_N - n_S\right) \\
&+ \left(m^2+m+1\right)\left(n_N - n_S\right)\left(n_N + n_S\right)^2 \Big)
\end{aligned} \tag{5.49}$$

$$\begin{aligned}
\chi(n_S, n_N, p_F; m) = {}& -4n_N^3(-1)^{t_N}\left(2\left(2m^2+2m+1\right)p_F + (-1)^{t_N}(2m+1)n_S\right) \\
&- 4n_N(-1)^{t_N}\Big(2\left(2m^2+2m-1\right)p_F n_S^2 + 8\left(2m^2+2m+1\right)p_F^3 \\
&- 4(-1)^{t_N}(2m+1)p_F^2 n_S + (-1)^{t_N}(2m+1)n_S^3\Big) \\
&- 2(2m+1)n_N^2\left(4(-1)^{t_N}(2m+1)p_F n_S + 12p_F^2 + 3n_S^2\right) \\
&+ 32(-1)^{t_N}(2m+1)^2 p_F^3 n_S + 40(2m+1)p_F^2 n_S^2 \\
&- 16(2m+1)p_F^4 + 8(-1)^{t_N} p_F n_S^3 - (2m+1)n_S^4 - (2m+1)n_N^4 \\
d(n_S, n_N, p_F; m) = {}& -32(-1)^{t_N}(2m+1)p_F^3\left(n_N + n_S\right) \\
&+ 8p_F^2\left(3(2m+1)^2 n_N^2 + 3(2m+1)^2 n_S^2 - 2n_N n_S\right) \\
&+ 8(-1)^{t_N}(2m+1)p_F\left(n_N + n_S\right)\left(-2n_N n_S + 3n_N^2 + 3n_S^2\right) \\
&- 48(2m+1)^2 p_F^4 + \left(n_N + n_S\right)^2\Big(-2(4m(m+1)+3)n_N n_S \\
&+ (4m(m+1)+5)(n_N^2 + n_S^2)\Big)
\end{aligned} \tag{5.50}$$

For the anti-twist case we get

$$\begin{aligned}
f(n_S, n_N, p_F; m) = {}& 2\left(\left(n_N - n_S\right)^2 - 4p_F^2\right)\left(2(2m+1)p_F + (-1)^{t_N}(n_S + n_N)\right) \\
&\times \Big((-1)^{t_N}\left(n_N - n_S\right)\left(4(2m+1)p_F + \left(m^2+m+1\right)(-1)^{t_N}\left(n_N - n_S\right)\right) \\
&+ 4(3m(m+1)+1)p_F^2\Big)
\end{aligned} \tag{5.51}$$

$$\begin{aligned}
g(n_S, n_N, p_F; m) = {}& -n_N n_S\Big((-1)^{t_N}\big(-4n_S^3\left(6(2m+1)p_F + (-1)^{t_N}n_N\right) \\
&+ 2(-1)^{t_N}(2m+1)n_S^2\big(12(2m+1)p_F^2 \\
&- (-1)^{t_N}n_N\left((-1)^{t_N}(2m+1)n_N - 4p_F\right)\big) \\
&- 4n_S\left(n_N^2 - 4p_F^2\right)\left(2(2m+1)p_F + (-1)^{t_N}n_N\right) \\
&+ n_N\big((-1)^{t_N}n_N\big((-1)^{t_N}n_N\left(24(2m+1)p_F + (-1)^{t_N}(4m(m+1)+5)n_N\right) \\
&+ 24(2m+1)^2 p_F^2\big) - 32(2m+1)p_F^3\big) + (-1)^{t_N}(4m(m+1)+5)n_S^4\big) \\
&- 48(2m+1)^2 p_F^4\Big)
\end{aligned} \tag{5.52}$$

where we used the parametrization (5.37). The mixing is given by

$$
\begin{align}
\varepsilon(n_S, n_N, p_F; m) &= -16 n_N n_S (-1)^{t_N} \Big(4(-1)^{t_N}(2m+1) p_F \left(n_N^2 - n_S^2\right) \notag\\
&\quad + 4(3m(m+1)+1) p_F^2 \left(n_N + n_S\right) \notag\\
&\quad + \left(m^2+m+1\right)\left(n_N+n_S\right)\left(n_N-n_S\right)^2 \Big) \notag\\
\chi(n_S, n_N, p_F; m) &= -4 n_N^3 (-1)^{t_N} \left(2\left(2m^2+2m+1\right) p_F - (-1)^{t_N}(2m+1) n_S\right) \notag\\
&\quad - 4 n_N (-1)^{t_N} \Big(2\left(2m^2+2m-1\right) p_F n_S^2 + 8\left(2m^2+2m+1\right) p_F^3 \notag\\
&\quad + 4(-1)^{t_N}(2m+1) p_F^2 n_S - (-1)^{t_N}(2m+1) n_S^3 \Big) \notag\\
&\quad - 2(2m+1) n_N^2 \left(-4(-1)^{t_N}(2m+1) p_F n_S + 12 p_F^2 + 3 n_S^2\right) \notag\\
&\quad - 32(-1)^{t_N}(2m+1)^2 p_F^3 n_S + 40(2m+1) p_F^2 n_S^2 \notag\\
&\quad - 16(2m+1) p_F^4 - 8(-1)^{t_N} p_F n_S^3 - (2m+1) n_S^4 - (2m+1) n_N^4 \tag{5.53}\\
d(n_S, n_N, p_F; m) &= -32(-1)^{t_N}(2m+1) p_F^3 \left(n_N - n_S\right) \tag{5.54}\\
&\quad + 8 p_F^2 \left(3(2m+1)^2 n_N^2 + 3(2m+1)^2 n_S^2 + 2 n_N n_S\right) \notag\\
&\quad + 8(-1)^{t_N}(2m+1) p_F \left(n_N - n_S\right)\left(2 n_N n_S + 3 n_N^2 + 3 n_S^2\right) \notag\\
&\quad - 48(2m+1)^2 p_F^4 + \left(n_N - n_S\right)^2 \Big(2(4m(m+1)+3) n_N n_S \notag\\
&\quad + (4m(m+1)+5)(n_N^2 + n_S^2) \Big) \tag{5.55}
\end{align}
$$

In in the limit of $m \to 0$ one recovers the same result of the compactified model of [66, 67], as expected.

5.2.2 The 5d Supergravity Truncation

The five-dimensional supergravity model we are working with is a consistent truncation from eleven-dimensional supergravity studied in [132]. It contains two vector multiplets and one hypermultiplet and it has gauge group $\mathrm{U}(1) \times \mathbb{R}$.

As we mentioned before, this truncation generalizes the structure associated with the solutions of [66, 67] and it completes the consistent truncation of seven-dimensional $\mathcal{N} = 4$ $\mathrm{SO}(5)$ gauged supergravity reduced on a Riemann surface $\mathcal{C}_\mathfrak{g}$ analyzed in [299]. There, the 5d model was obtained truncating the 7d supergravity to the $\mathrm{U}(1)^2$ sector, corresponding to the Cartan of $\mathrm{SO}(5)$. Besides enclosing the two $\mathrm{U}(1)$ gauge fields and the two scalars belonging to the vector multiplets, the bosonic sector of the construction made in [132] also includes all the scalar fields in the hypermultiplet, and furthermore it gives a direct derivation of the gauging. In the following we outline the construction made in [132]. The eleven-dimensional metric is

$$
ds_{11}^2 = e^{2\Delta} ds_{\mathrm{AdS}_5}^2 + ds_6^2, \tag{5.56}
$$

which corresponds to a warped product $\mathrm{AdS}_5 \times_{\mathrm{w}} \mathcal{M}$ with warp factor $e^{2\Delta}\ell^2 = e^{2f_0}\bar{\Delta}^{1/3}$, where ℓ is the AdS radius and $\bar{\Delta}$ and f_0 are constants. $\mathcal{M}_6$ is a six-dimensional manifold given by a fibration of a squashed-sphere $\mathcal{M}_4$ over the Riemann surface $\mathcal{C}_{\mathfrak{g}}$ and has metric

$$ds_6^2 = \bar{\Delta}^{1/3} e^{2g_0} ds^2_{\mathcal{C}_{\mathfrak{g}}} + \frac{1}{4}\bar{\Delta}^{-2/3} ds_4^2, \tag{5.57}$$

where g_0 is a constant. The Riemann surface has Ricci scalar curvature $\mathbf{k}$ as discussed after formula (5.19) and the metric on $\mathcal{M}_4$ is

$$ds_4^2 = X_0^{-1} d\mu_0^2 + \sum_{i=1,2} X_i^{-1}\big(d\mu_i^2 + \mu_i^2 (d\varphi_i + A^{(i)})^2\big), \tag{5.58}$$

with

$$\mu_0 = \cos\zeta, \qquad \mu_1 = \sin\zeta\cos\frac{\theta}{2}, \qquad \mu_2 = \sin\zeta\sin\frac{\theta}{2}. \tag{5.59}$$

The angles φ_1, φ_2 are in $[0, 2\pi]$, while ζ, θ are in $[0, \pi]$. $A^{(1)}$ and $A^{(2)}$ gauge two $\mathrm{U}(1)$ isometries of the squashed S^4. Furthermore,

$$\bar{\Delta} = \sum_{I=0}^{2} X_I \mu_I^2, \quad e^{f_0} = X_0^{-1}, \quad e^{2g_0} = -\frac{1}{8}\mathbf{k} X_1 X_2 \big[(1-\mathbf{z})X_1 + (1+\mathbf{z})X_2\big], \tag{5.60}$$

where $\mathbf{z}$, that can be read from (5.16) as

$$\mathbf{z} = \frac{p-q}{p+q}, \tag{5.61}$$

is a discrete parameter related to the Chern numbers p and q and

$$\begin{aligned} X_0 &= (X_1X_2)^{-2}, \\ X_1 X_2^{-1} &= \frac{1+\mathbf{z}}{2\mathbf{z} - \mathbf{k}\sqrt{1+3\mathbf{z}^2}}, \\ X_1^5 &= \frac{1 + 7\mathbf{z} + 7\mathbf{z}^2 + 33\mathbf{z}^3 + \mathbf{k}(1 + 4\mathbf{z} + 19\mathbf{z}^2)\sqrt{1+3\mathbf{z}^2}}{4\mathbf{z}(1-\mathbf{z})^2}. \end{aligned} \tag{5.62}$$

There is also a four-form flux, but we address the interested reader to [132] for its explicit form.

Notice that the $\mathcal{N} = 1$ and $\mathcal{N} = 2$ twistings studied in [265] can be recovered as special cases from this model: the first one arises from setting $p = q$ (corresponding to $\mathbf{z} = 0$), while the second one from $p = 0$ or $q = 0$ ($\mathbf{z} = \pm 1$).

5.2.2.1 $\mathcal{N} = 2$ Supergravity Structure

The reduction described above gives rise to an infinite family of $\mathcal{N} = 2$ gauged supergravity theories in five dimensions. Here we summarize the most salient features of the model and we refer the reader to appendix A of [42] for a short review of 5d $\mathcal{N} = 2$

gauged supergravity[5].
Focusing on the vector multiplet sector, the two real scalars Σ and ϕ parametrize the Very Special Real Manifold

$$\mathcal{M}_V = \mathbb{R}_+ \times \mathrm{SO}(1,1) \tag{5.64}$$

that has metric

$$g_{xy} = \begin{pmatrix} \frac{3}{\Sigma^2} & 0 \\ 0 & 1 \end{pmatrix} . \tag{5.65}$$

The homogeneous coordinates $h^I(\Sigma, \phi)$ (from now on we will omit the explicit dependence of the sections from the two real scalars Σ and ϕ) are given by

$$h^0 = \frac{1}{\Sigma^2}, \qquad h^1 = -\Sigma H^1, \qquad h^2 = -\Sigma H^2, \tag{5.66}$$

where

$$H^1 = \sinh\phi, \qquad H^2 = \cosh\phi \tag{5.67}$$

parametrize the unit hyperboloid $\mathrm{SO}(1,1)$, while Σ parametrizes $\mathbb{R}^+$. The metric g_{xy} is the pull-back of the metric a_{IJ} in the ambient space, which takes the form

$$a_{IJ} = \frac{2}{3\Sigma^2} \begin{pmatrix} \frac{\Sigma^6}{2} & 0 & 0 \\ 0 & 2(H^1)^2+1 & -2H^1H^2 \\ 0 & -2H^1H^2 & 2(H^2)^2-1 \end{pmatrix} . \tag{5.68}$$

The non-zero components of the totally symmetric tensor C_{IJK} are

$$C_{0\bar{I}\bar{J}} = C_{\bar{I}0\bar{J}} = C_{\bar{I}\bar{J}0} = \frac{1}{3}\eta_{\bar{I}\bar{J}}, \qquad \text{for } \bar{I}, \bar{J} = 1, 2, \tag{5.69}$$

with $\eta = \mathrm{diag}(-1, 1)$.
Moving to the hypermultiplet sector, the quaternionic manifold

$$\mathcal{M}_H = \frac{\mathrm{SU}(2,1)}{\mathrm{SU}(2) \times \mathrm{U}(1)} \tag{5.70}$$

is spanned by the scalars $q^X = \{\varphi, \Xi, \theta_1, \theta_2\}$ with line element[6]

$$g_{XY} dq^X dq^Y = -d\varphi^2 - \frac{1}{2}e^{2\varphi}(d\theta_1^2 + d\theta_2^2) - \frac{1}{4}e^{4\varphi}(d\Xi - \theta_1 d\theta_2 + \theta_2 d\theta_1)^2. \tag{5.71}$$

Only the hypermultiplet sector is gauged and the corresponding Killing vectors $k_I =$

---

[5] The Lagrangian in (B.10) of [132] that we are using here can be obtained from the one used in [42] by rescaling the gauge fields and the coupling constant as

$$A^I_{\text{there}} = -\sqrt{\frac{3}{2}} A^I_{\text{here}}, \qquad g_{\text{there}} = -\sqrt{\frac{2}{3}} g_{\text{here}}. \tag{5.63}$$

[6] We are using a different normalization w.r.t. [132]. This allows us to obtain a simplified version of the hyperino variation, as it was pointed out in [42].

$k_I^X \partial_X$ read

$$k_0 = \partial_\Xi, \qquad k_1 = \mathbf{zk}\partial_\Xi, \qquad k_2 = -\mathbf{k}\partial_\Xi + 2(\theta_2\partial_{\theta_1} - \theta_1\partial_{\theta_2}), \tag{5.72}$$

with associated Killing prepotentials

$$\begin{aligned} P_0^r &= \{0, 0, \frac{1}{4}e^{2\varphi}\}, \\ P_1^r &= \{0, 0, \frac{\mathbf{zk}}{4}e^{2\varphi}\}, \\ P_2^r &= \{\sqrt{2}e^\varphi\theta_1, \sqrt{2}e^\varphi\theta_2, -1 + \frac{1}{4}e^{2\varphi}(2\theta_1^2 + 2\theta_2^2 - \mathbf{k})\}. \end{aligned} \tag{5.73}$$

Thus, the bosonic part of the five-dimensional Lagrangian is

$$\begin{aligned} e^{-1}\mathcal{L} = \frac{1}{2}R - \frac{1}{\Sigma^2}\partial_\mu\Sigma\,\partial^\mu\Sigma - \frac{3}{4}a_{\bar{I}\bar{J}}\,\partial_\mu(\Sigma H^{\bar{I}})\,\partial^\mu(\Sigma H^{\bar{J}}) - \frac{1}{2}g_{XY}\mathcal{D}_\mu q^X\mathcal{D}^\mu q^Y \\ -\frac{\Sigma^4}{12}F^0_{\mu\nu}F^{0\mu\nu} - \frac{1}{4}a_{\bar{I}\bar{J}}F^{\bar{I}}_{\mu\nu}F^{\bar{J}\mu\nu} - \frac{e^{-1}}{12}\sqrt{\frac{2}{3}}\epsilon^{\mu\nu\rho\sigma\tau}\left(F^1_{\mu\nu}F^{\mu\nu1} - F^2_{\mu\nu}F^{\mu\nu2}\right)A^0_\tau - g^2V\,, \end{aligned} \tag{5.74}$$

where we recall the notation $\bar{I}, \bar{J} = 1, 2$ and V represents the scalar potential of the theory.

5.2.2.2 The Model

In the remainder of this paper we will work with a further truncation of the 5d supergravity model introduced above, which is obtained by setting

$$\theta_1 = \theta_2 = 0, \tag{5.75}$$

consistently with the AdS$_5$ vacuum of the model we started from. In this truncation, the Killing vectors (5.72) simplify to

$$k_0 = \partial_\Xi, \qquad k_1 = \mathbf{zk}\partial_\Xi, \qquad k_2 = -\mathbf{k}\partial_\Xi. \tag{5.76}$$

Notice that from (5.76) we can see that the field Ξ gets charged under the vector $A_\mu^{(0)} + \mathbf{zk}A_\mu^{(1)} - \mathbf{k}A_\mu^{(2)}$, that becomes massive. Furthermore, only the third SU(2)-components of the Killing prepotentials (5.73) survive and they reduce to

$$P_0^3 = \frac{1}{4}e^{2\varphi}, \qquad P_1^3 = \frac{\mathbf{zk}}{4}e^{2\varphi}, \qquad P_2^3 = -1 - \frac{\mathbf{k}}{4}e^{2\varphi}. \tag{5.77}$$

We can thus introduce a superpotential as

$$W = h^I P_I^3 = \frac{\Sigma^3\left(\left(\mathbf{k}e^{2\varphi} + 4\right)\cosh\phi - \mathbf{zk}e^{2\varphi}\sinh\phi\right) + e^{2\varphi}}{4\Sigma^2}. \tag{5.78}$$

Furthermore, the following AdS$_5$ vacuum is also a vev for the scalars Σ, ϕ, φ in this truncation:

$$\begin{aligned}
\varphi &= \frac{1}{2}\log\left(\frac{4}{\sqrt{3\mathbf{z}^2+1}-2\mathbf{k}}\right),\\
\phi &= \operatorname{arctanh}\left(\frac{1+\mathbf{k}\sqrt{1+3\mathbf{z}^2}}{3\mathbf{z}}\right),\\
\Sigma^3 &= \frac{\sqrt{2\left(3\mathbf{z}^2-1-\mathbf{k}\sqrt{1+3\mathbf{z}^2}\right)}}{\mathbf{z}\left(\sqrt{1+3\mathbf{z}^2}-2\mathbf{k}\right)}.
\end{aligned} \tag{5.79}$$

5.2.3 The 5d Truncation on the Spindle

In this section we briefly review the geometric construction used to split the five-dimensional background as the warped product AdS$_3 \times \mathbb{\Sigma}$, where the space $\mathbb{\Sigma}$ is a compact spindle with azimuthal symmetry and conical singularities at the poles. Once introduced the ansatz on the geometry and on the gauge fields, we present the corresponding BPS equations and Maxwell equations of motion.
We refer the reader to [56] for the original derivation and to [42] for a more detailed analysis made using our conventions.

5.2.3.1 The Ansatz and Maxwell Equations

We begin by considering the AdS$_3 \times \mathbb{\Sigma}$ ansatz made in [56][7]:

$$\begin{aligned}
ds^2 &= e^{2V(y)}ds^2_{\text{AdS}_3} + f(y)^2dy^2 + h(y)^2dz^2,\\
A^{(I)} &= a(y)^{(I)}dz,
\end{aligned} \tag{5.80}$$

where $ds^2_{\text{AdS}_3}$ is the metric on unitary AdS$_3$, while (y,z) are the coordinates on $\mathbb{\Sigma}$, which is a compact spindle with an azimuthal symmetry generated by ∂_z. A spindle is a weighted projective space $\mathbb{WCP}^1_{[n_N,n_S]}$ with conical deficit angles at the north (n_N) and at the south (n_S) pole, whose geometry is determined by the two co-prime integers $n_N \neq n_S$ that are associated to the deficit angles $2\pi\left(1-\frac{1}{n_{N,S}}\right)$ at the poles.
The azimuthal coordinate z has periodicity $\Delta z = 2\pi$. The longitudinal coordinate y is compact, bounded by y_N and y_S (with $y_N < y_S$), implying that the function $h(y)$ vanishes at the poles of the spindle.

We assume that the scalars Σ, ϕ, φ depend on the y coordinate only, while the hyperscalar Ξ is linear in z, i.e. $\Xi = \overline{\Xi}z$ (with $\overline{\Xi}$ a constant).
Following [56], we will use an orthonormal frame to simplify the analysis of the Killing spinor equations and of the equations of motion of the gauge fields:

$$e^a = e^V\bar{e}^a, \qquad e^3 = f\,dy, \qquad e^4 = h\,dz, \tag{5.81}$$

---

[7] We are using the mostly plus signature, as in [42].

where $\bar{e}^a$ is an orthonormal frame for AdS$_3$. In this basis, the field strengths read

$$f\,h\,F_{34}^{(I)} = \partial_y a^{(I)}. \tag{5.82}$$

Given that Σ, ϕ, φ are functions of y only and $\Xi = \overline{\Xi} z$, two out of the three gauge equations of motion specified to our ansatz can be integrated, and they can be written in the orthonormal frame as

$$\frac{2e^{3V}}{3\Sigma^2}\Big[(\cosh 2\phi - \mathbf{z}\sinh 2\phi)F_{34}^{(1)} + (\mathbf{z}\cosh 2\phi - \sinh 2\phi)F_{34}^{(2)}\Big] = \mathcal{E}_1, \tag{5.83}$$

$$\frac{2e^{3V}}{3\Sigma^2}\Big[\mathbf{zk}\Sigma^6 F_{34}^{(0)} - (\cosh 2\phi + \mathbf{z}\sinh 2\phi)F_{34}^{(1)} + (\mathbf{z}\cosh 2\phi + \sinh 2\phi)F_{34}^{(2)}\Big] = \mathcal{E}_2, \tag{5.84}$$

$$\partial_y\left(\frac{1}{3}e^{3V}\Sigma^4 F_{34}^{(0)}\right) = \frac{1}{4}e^{4\psi+3V} g\,f\,h^{-1} D_z\Xi, \tag{5.85}$$

where $\mathcal{E}_1$ and $\mathcal{E}_2$ are constants, and we defined $D_z\Xi \equiv \overline{\Xi} + g(a^{(0)} + \mathbf{zk}a^{(1)} - \mathbf{k}a^{(2)})$.

5.2.3.2 The BPS Equations

To derive the BPS equations for the geometry introduced above, we need to factorize the Killing spinor [56]:

$$\epsilon = \psi \otimes \chi, \tag{5.86}$$

where χ is a two-component spinor on the spindle and ϕ is a two-component spinor on AdS$_3$ such that

$$\nabla_m \psi = -\frac{\kappa}{2}\Gamma_m \psi, \tag{5.87}$$

with $\kappa = \pm 1$ depending on the $\mathcal{N} = (2,0)$ or $\mathcal{N} = (0,2)$ supersymmetry chirality of the dual 2d SCFT.

We then decompose the 5d gamma matrices as

$$\gamma^m = \Gamma^m \otimes \sigma^3, \qquad \gamma^3 = \mathbb{I}_2 \otimes \sigma^1, \qquad \gamma^4 = \mathbb{I}_2 \otimes \sigma^2. \tag{5.88}$$

with $\Gamma^m = (-i\sigma^2, \sigma^3, \sigma^1)$.

The analysis of the BPS equations is similar to the one in appendix C of [42] (or to the original of [56]). Here again the spinor χ can be written in terms of an auxiliary function $\xi(y)$ as

$$\chi = e^{V/2} e^{isz} \begin{pmatrix} \sin\frac{\xi}{2} \\ \cos\frac{\xi}{2} \end{pmatrix}, \tag{5.89}$$

with s a constant. Notice that, as expected, the spinor is not constant on the spindle. In the following we summarize the differential relations coming from the BPS equations

$$\begin{aligned}
\xi' - 2f(gW\cos\xi + \kappa e^{-V}) &= 0\\
V' - \frac{2}{3}fgW\sin\xi &= 0\\
\Sigma' + \frac{2}{3}fg\,\Sigma^2\sin\xi\,\partial_\Sigma W &= 0\\
\phi' + 2fg\sin\xi\,\partial_\phi W &= 0\\
\varphi' + \frac{fg}{\sin\xi}\partial_\varphi W &= 0\\
h' - \frac{2fh}{3\sin\xi}(gW(1+2\cos^2\xi) + 3\kappa e^{-V}\cot\xi) &= 0,
\end{aligned} \tag{5.90}$$

where W is the superpotential defined in (5.78). Besides the first-order equations, there are also two algebraic constraints that can be derived from the supersymmetry variations

$$\begin{aligned}
&\sin\xi(s - Q_z) = -h(gW\cos\xi + \kappa e^{-V})\\
&gh\partial_\varphi W\cos\xi = \partial_\varphi Q_z\sin\xi,
\end{aligned} \tag{5.91}$$

where Q_z can be read from the supercovariant derivative $D_\mu\epsilon = \nabla_\mu\epsilon - iQ_\mu\epsilon$ that appears in the gravitino variation and for our model takes the form

$$Q_z = \frac{e^{2\varphi}}{4}D_z\Xi - ga^{(2)}. \tag{5.92}$$

We can also reduce the differential system by observing that

$$h = ke^V\sin\xi \tag{5.93}$$

where k is an arbitrary constant that needs to be determined. Finally, we can take advantage of the BPS equations to express the field strengths in terms of the scalar fields as

$$\begin{aligned}
F^{(0)}_{34} &= \frac{6\kappa e^{-V} + 4gW\cos\xi - 4g\Sigma\,\partial_\Sigma W\cos\xi}{3\Sigma^2},\\
F^{(1)}_{34} &= -\frac{2\Sigma}{3}\Big[\sinh\phi\Big(g\cos\xi\big(2W + \Sigma\,\partial_\Sigma W\big) + 3\kappa e^{-V}\Big) + 3g\partial_\phi W\cos\xi\cosh\phi\Big],\\
F^{(2)}_{34} &= -\frac{2\Sigma}{3}\Big[\cosh\phi\Big(g\cos\xi\big(2W + \Sigma\,\partial_\Sigma W\big) + 3\kappa e^{-V}\Big) + 3g\partial_\phi W\cos\xi\sinh\phi\Big].
\end{aligned} \tag{5.94}$$

5.2.4 Analysis at the Poles

In this section we study the solutions of the BPS equations derived above and we show how to obtain the 2d central charge from the pole analysis. The procedure follows the one originally described in [56] and then applied in [42, 293] for the case of the conifold. We start by summarizing the BPS equations, the constraints and the Maxwell equations. Then we derive the explicit expressions of the conserved charges and the magnetic fluxes. The charge conservation imposes the constraints that allow us to fix the boundary conditions at the poles for the scalars that enter the calculation of the central charge. We then compute the central charge from the Brown-Henneaux formula and discuss its relation with the calculation done on the field theory side.

Before starting our analysis let us stress that, differently from the discussion in [42, 56, 293] we have not found from the pole analysis immediate reasons to exclude the possibility of having solutions in the twist class. We will further comment on this issue in the next section where we provide numeric and analytical solutions of the BPS equations.

5.2.4.1 Conserved Charges and Restriction to the Poles

From the expressions of the field strengths in (5.94) we can study the Maxwell equations using the two conserved charges $\mathcal{E}_{1,2}$ in (5.83) and (5.84). In order to keep the hyperscalar $\varphi(y)$ finite we require that $\partial_\varphi W|_{N,S} = 0$. This constraint gives rise to

$$\mathbf{k}\,\Sigma|^3_{N,S} + \frac{1}{\cosh\phi|_{N,S} - \mathbf{z}\sinh\phi|_{N,S}} = 0\,, \tag{5.95}$$

where W is given in (5.78). Using (5.95) and the fact that $\mathcal{E}_1$ and $\mathcal{E}_2$ are conserved we found simpler expressions by working with the following linear combinations

$$\begin{aligned} Q_1|_{N,S} &= \mathcal{E}_1|_{N,S} = \frac{4}{3}e^{2V|_{N,S}}\left(\frac{\kappa(\sinh(\phi|_{N,S}) - \mathbf{z}\cosh(\phi|_{N,S}))}{\Sigma|_{N,S}} - \mathbf{z}ge^{V|_{N,S}}\cos(\xi|_{N,S})\right), \\ Q_2|_{N,S} &= \mathcal{E}_1|_{N,S} - \mathcal{E}_2|_{N,S} = \frac{4\kappa e^{2V|_{N,S}}}{3\Sigma|_{N,S}}\left(2\sinh(\phi|_{N,S}) - \mathbf{zk}\Sigma|^3_{N,S}\right). \end{aligned} \tag{5.96}$$

At the north and at the south poles we have $k\sin\xi \to 0$. For non-vanishing k this gives $\cos\xi_{N,S} = (-1)^{t_{N,S}}$ with $t_{N,S} = 0$ or $t_{N,S} = 1$. Denoting the poles as $y_{N,S}$ we can work with $y_N \le y \le y_S$. Furthermore,

$$|h'|_{N,S} = |k\sin'\xi|_{N,S} = \frac{1}{n_{N,S}}. \tag{5.97}$$

This relation is due to the metric and to the deficit angles at the poles $2\pi\left(1 - \frac{1}{n_{N,S}}\right)$ where $n_{N,S} > 1$. From the $\mathbb{Z}_2$ symmetry of the BPS equations acting on $h, a^{(I)}, s, Q_z$ and k we can restrict to $h \ge 0$ and $k\sin\xi \ge 0$. We have then $k\sin\xi \ge 0$ and this quantity is vanishing at the poles, with a positive derivative at y_N and a negative one at y_S. Formally we introduce two constants, $l_N = 0$ and $l_S = 1$ such that

$$k\sin'\xi|_{N,S} = \frac{(-1)^{l_{N,S}}}{n_{N,S}}. \tag{5.98}$$

Then the cases $(t_N, t_S) = (0, 0)$ and $(1, 1)$ correspond to the twist while $(t_N, t_S) = (1, 0)$ and $(0, 1)$ correspond to the anti-twist. Plugging the evaluation of $\cos\xi$ at the poles in (5.98), we obtain a relation for ξ' at the poles as well. Furthermore, ξ' following from the first BPS equation in (5.90) in the conformal gauge, can be shown to be proportional to the quantity $(s - Q_z)$ in (5.91), after plugging in this last the relation (5.93). It follows that, the quantity $(s - Q_z)$ at the poles becomes

$$s - Q_Z|_{N,S} = \frac{1}{2n_{N,S}}(-1)^{t_{N,S}+l_{N,S}+1}. \tag{5.99}$$

Furthermore, the relation $\partial_\varphi W|_{N,S} = 0$ imposes from the second relation in (5.91) that $\partial_\varphi Q_z|_{N,S} = 0$. Another assumption (justified a posteriori by the numerical results) is that $\psi|_{N,S} \neq 0$. Such assumption implies also that $D_z\Xi|_{N,S} = 0$.

5.2.4.2 Fluxes

Here we introduce the magnetic fluxes for the reduction of this truncation on the spindle. This will be necessary in order to find the constant k introduced in (5.93) in terms of the data of the spindle. First, from the relations (5.94), we observe that

$$F^{(I)}_{yz} = (a^{(I)})' = \left(\mathcal{I}^{(I)}\right)' \quad \text{with} \quad \mathcal{I}^{(I)} \equiv -ke^V \cos\xi\, h^I\ . \tag{5.100}$$

At this point we need to define the fluxes starting from (5.100). Let's start by defining the integer fluxes p_I from the relations

$$\frac{p_I}{n_N n_S} = \frac{1}{2\pi}\int_\Sigma gF^{(I)} = g\mathcal{I}^{(I)}|_N^S\ . \tag{5.101}$$

The magnetic charge associated to the R-symmetry is

$$-gn_N n_S \mathcal{I}^{(2)}|_N^S = \frac{1}{2}\left(n_S(-1)^{t_N} + n_N(-1)^{t_S}\right). \tag{5.102}$$

This expression is quantized if $n_S(-1)^{t_N} + n_N(-1)^{t_S}$ is even. Observe also that

$$\mathcal{I}^{(0)} + \mathbf{zk}\mathcal{I}^{(1)} - \mathbf{k}\mathcal{I}^{(2)} = 0 \tag{5.103}$$

that implies also that the combination $p_0 + \mathbf{zk}p_1 - \mathbf{k}p_2$ does not give rise to a conserved magnetic flux. The last flux that we need to discuss is the one associated to the flavor symmetry. The integer flavor flux is given by

$$p_F = gn_N n_S \mathcal{I}^{(1)}|_N^S\ . \tag{5.104}$$

It is important to observe that the relation $p_0 = \mathbf{k}(\mathbf{z}p_F + p_2) \in \mathbb{Z}$ requires that for $\mathbf{z} \in \mathbb{Q}\backslash\mathbb{Z}$ we have the further constraint $\mathbf{z}p_F \in \mathbb{Z}$.

Furthermore we also found useful to introduce an auxiliary function δ, in terms of which we can rewrite

$$\tanh(\phi) \equiv \frac{1-\delta}{\mathbf{z}} \tag{5.105}$$

such that the charges evaluated at the poles simplify to

$$\begin{aligned} Q_{1N,S} &= \frac{\mathbf{k}\delta_{N,S}((-1)^{l_{N,S}} - 2\kappa k n_{N,S}(-1)^{t_{N,S}})^2}{6\mathbf{z}g^2k^3n_{N,S}^3} \\ &\times \ (2\kappa k n_{N,S}(\delta_{N,S}-1)\delta_{N,S} - (-1)^{l_{N,S}-t_{N,S}}((\delta_{N,S}-1)^2-\mathbf{z}^2)), \end{aligned} \tag{5.106}$$

$$Q_{2N,S} = \frac{\mathbf{k}\kappa((-1)^{l_{N,S}} - 2\kappa k n_{N,S}(-1)^{t_{N,S}})^2}{3\mathbf{z}g^2k^2n_{N,S}^2}(\mathbf{z}^2 - 1 + \delta_{N,S}(4-3\delta_{N,S})). \tag{5.107}$$

It follows that we have three equations: the first one is (5.104), that after the substitution (5.105) becomes

$$p_F = \frac{(\delta_N - 1)n_S(-1)^{-t_N} + n_N(-1)^{-t_S}(\delta_S - 1) - 2\kappa k n_N n_S(\delta_N - \delta_S)}{2\mathbf{z}} \tag{5.108}$$

while the other two equations correspond to $Q_1|_N = Q_1|_S$, i.e.

$$\frac{(1+2\kappa k n_S(-1)^{t_S})^2}{(1-2\kappa k n_N(-1)^{t_N})^2} \cdot \frac{\delta_S n_N^3}{\delta_N n_S^3} \cdot \frac{2\kappa k n_S(-1)^{t_S}(\delta_S-1)\delta_S + (\delta_S-1)^2 - \mathbf{z}^2}{2\kappa k n_N(-1)^{t_N}(\delta_N-1)\delta_N - (\delta_N-1)^2 + \mathbf{z}^2} = (-1)^{t_S+t_N} \tag{5.109}$$

and $Q_2|_N = Q_2|_S$, i.e.

$$\frac{n_N^2}{n_S^2} \cdot \frac{\mathbf{z}^2 - 1 + \delta_S(4-3\delta_S)}{\mathbf{z}^2 - 1 + \delta_N(4-3\delta_N)} \cdot \frac{(1+2\kappa k n_S(-1)^{t_S})^2}{(1-2\kappa k n_N(-1)^{t_N})^2} = 1 \tag{5.110}$$

for the three variables, k, δ_S and δ_N. By solving these three equations we obtain then the boundary conditions to impose for the scalars V, h, ϕ, Σ in terms of the integers n_S, n_N and p_F of the spindle for generic values of the parameters $\mathbf{z} \in \mathbb{Q}$ and $\mathbf{k} = \pm 1$ in both the twist and the anti-twist class. The requirement of reality for these fields imposes further constraints on the allowed values of the integers $n_{S,N}$ and p_F. The only field that is not involved in this analysis is the hyperscalar φ, that we are assuming as non-vanishing at the poles.

5.2.4.3 Central charge from the Pole Data

Once the boundary data for $\delta_{N,S}$ and the constant k are specified we can read the central charge of the putative 2d CFT from the pole analysis. The central charge is obtained from the Brown–Henneaux formula [120]

$$c_{2d} = \frac{3R_{AdS_3}}{2G_3} = \frac{3}{2G_5}\Delta z \int_{y_N}^{y_S} e^{V(y)}|f(y)h(y)|dy. \tag{5.111}$$

The relation

$$e^{V(y)}f(y)h(y) = -\frac{k}{2\kappa}(e^{3V(y)}\cos\xi(y))' \tag{5.112}$$

implies that the central charge can be computed from the value of the fields at the poles that we have computed above, without specifying the value of the hyperscalar. The consistency of this analysis represents just a necessary condition for the existence of a solution. Nevertheless, when a solution exists, the central charge computed here is the correct one.

In the conformal gauge $f = e^V$ the integrand in (5.111) is $e^{V(y)}|h(y)|$, where we remove the absolute value here and consider $h(y) > 0$ thanks to the symmetries of the BPS equations as discussed above. The central charge becomes $c_{2d} = c_S - c_N$ where

$$c_{N,S} = \frac{3\pi\mathbf{k}\delta_{N,S}}{2\mathbf{z}^2 g^3 G_5\kappa k^2}\left(\kappa k - \frac{(-1)^{l_{N,S}-t_{N,S}}}{2n_{N,S}}\right)^3 ((\delta_{N,S}-1)^2 - \mathbf{z}^2). \tag{5.113}$$

The central charge in the case of the anti-twist, splitting numerator and denominator for ease of readability, is given by

$$\begin{aligned}\text{Numerator} = \; & 3\pi\mathbf{k}\kappa(4p_F^2 - (n_S - n_N)^2)(2\mathbf{z}p_F(-1)^{t_N} - n_N + n_S) \\ & \times (n_S - n_N)\Big(16\mathbf{z}p_F(-1)^{t_N} + (\mathbf{z}^2+3)(n_S - n_N) + 4(3\mathbf{z}^2+1)p_F^2\Big),\end{aligned} \tag{5.114}$$

$$\begin{aligned}\text{Denominator} = \; & 4g^3G_5 n_N n_S\Big(8\mathbf{z}p_F(-1)^{t_N}(n_S - n_N)(3n_N^2 + 2n_N n_S + 3n_S^2 - 4p_F^2) \\ & + 16p_F^2 n_N n_S + 4(n_S - n_N)(n_S^3 - n_N^3) \\ & + \mathbf{z}^2\big(24p_F^2(n_N^2 + n_S^2) - 48p_F^4 + (n_S^2 - n_N^2)^2\big)\Big),\end{aligned} \tag{5.115}$$

while the central charge in the case of the twist is given by

$$\begin{aligned}\text{Numerator} = \; & 3\pi\mathbf{k}\kappa(4p_F^2 - (n_S - n_N)^2)(2\mathbf{z}p_F(-1)^{t_N} - n_N + n_S) \\ & \times \Big((n_N + n_S)(16\mathbf{z}p_F(-1)^{t_N} + (\mathbf{z}^2+3)(n_N + n_S)) + 4(3\mathbf{z}^2+1)p_F^2\Big),\end{aligned} \tag{5.116}$$

$$\begin{aligned}\text{Denominator} = \; & = 4g^3G_5 n_N n_S\Big(8\mathbf{z}p_F(-1)^{t_N}(n_N + n_S)(3n_N^2 - 2n_N n_S + 3n_S^2 - 4p_F^2) \\ & - 16p_F^2 n_N n_S + 4(n_N + n_S)(n_N^3 + n_S^3) \\ & + \mathbf{z}^2\big(24p_F^2(n_N^2 + n_S^2) - 48p_F^4 + (n_S^2 - n_N^2)^2\big)\Big).\end{aligned} \tag{5.117}$$

The five dimensional Newton constant can be read from the holographic dictionary. Indeed from the general relation $a_{4d} = \frac{\pi R^3_{AdS_5}}{8G_5}$ and from the explicit values of the central charge and of the AdS$_5$ radius, given by

$$a_{4d}=\frac{(g-1)\left((1-9\mathbf{z}^2)\mathbf{k}+\left(3\mathbf{z}^2+1\right)^{3/2}\right)}{48\mathbf{k}\mathbf{z}^2},\quad R^3_{AdS_5}=\frac{(1-9\mathbf{z}^2)\mathbf{k}+\left(3\mathbf{z}^2+1\right)^{3/2}}{4\mathbf{z}^2} \tag{5.118}$$

we can extract $G_5=\frac{3\pi\mathbf{k}}{2(g-1)}$. Substituting this expression in the 2d central charge computed above we can then recover the result obtained from the field theory calculation in Section 5.2.1.2.

Some comments are in order. First we have checked in many cases if the various constraints, imposed by the quantization of the fluxes, by the reality condition on the scalars and by the positivity of the central charge, are enough to exclude the existence of some solutions. While in many cases the answer is affirmative, we have not been able to exclude whole families of solutions. In general there are four main families of possible solutions, identified by the value of $\mathbf{k}=\pm1$ and by the fact that they can be in the twist or in the anti-twist class. Anyway, anticipating the results of next section, we have found solutions only in the anti-twist class for $\mathbf{k}=-1$.

5.2.5 The Solution

In this section we obtain the AdS$_3\times\Sigma$ solution for the model discussed above. We separate the analysis in two parts. In the first part we discuss the analytic solution for the universal truncation. This corresponds to a further truncation of the model to the graviton sector. In this case we found the explicit solution corresponding to the general one found in [180, 182]. Similarly to the cases discussed in [42, 56, 293] in presence of hypermultiplets, here we found an analytic solution only in the anti-twist class. Furthermore, we have found such solution only for $\mathbf{k}=-1$. We have also checked that the 2d central charge matches the general expectation [182]

$$c_{2d}=\frac{4}{3}\frac{a_{4d}(n_S-n_N)^3}{n_Nn_S(n_N^2+n_Nn_S+n_S^2)}\,. \tag{5.119}$$

In the second part of this section we study the solution turning on a generic flux p_F. In this case we have obtained the solution numerically. Again we found solutions only in the anti-twist class for $\mathbf{k}=-1$ and for generic values of $\mathbf{z}$.

5.2.5.1 Analytic Solution for the Graviton Sector

Here we study the AdS$_3\times\Sigma$ solution by restricting to the graviton sector. It will turn out that the solution is exactly the same as the one studied in the original Spindle paper [182]. This is consistent with similar results obtained in other 5d truncations in presence of hypermultiplets [42, 56, 293]. This requires to fix $A^{(1)}+\epsilon^*_{4d}A^{(2)}=0$ (with ϵ^*_{4d} defined in (5.19)) and identifying $A^{(R)}=-A^{(2)}$. This further fixes $2p_F=\epsilon^*_{4d}(n_S-n_N)$. We have found a solution in this case for the anti-twist class and $\mathbf{k}=-1$ by fixing the scalars $\Sigma(y)$, $\phi(y)$ and $\varphi(y)$ at their AdS$_5$ value (5.79). Observe that $\phi_{N,S}=\phi_{AdS_5}$ and $\Sigma_{N,S}=\Sigma_{AdS_5}$ when $p_F=\epsilon^*_{4d}(n_S-n_N)/2$.

Before continuing the discussion a comment is in order. The choice of p_F that allows to study the universal twist is, for generic values of $\mathbf{z}$, in contrast with the requirement that $\mathbf{z}p_F$ is an integer. The only cases that are allowed correspond to the ones that give rise to a rational exact R-symmetry. In these cases a solution exists when (the even quantity) $n_S - n_N$ gives rise to an integer $\mathbf{z}p_F$. This analysis restricts the possible truncations to the graviton sector that can be placed on the spindle. This is the counterpart of the field theory argument that we made after formula (5.45). The discussion fits with similar ones appeared in the literature of the spindle (see for example footnote 20 of [226] for an analogous behavior in the case of toric SE$_5$). Having this caveat in mind, the scalar functions $V(y)$, $f(y)$ and $h(y)$ in (5.80) are

$$e^{V(y)} = \frac{\sqrt{y}}{W}, \quad f(y) = \frac{3}{2W}\sqrt{\frac{y}{q(y)}}, \quad h(y) = \frac{c_0\sqrt{q(y)}}{4Wy} \tag{5.120}$$

while the gauge field is

$$A^{(R)} = \left(\frac{c_0\kappa(a-y)}{4y} - s\right) dz\ . \tag{5.121}$$

We also found that

$$\sin\xi(y) = \frac{\sqrt{q(y)}}{2y^{3/2}}, \quad \cos\xi(y) = \frac{\kappa(3y-a)}{2y^{3/2}} \tag{5.122}$$

with

$$q(y) = 4y^3 - 9y^2 + 6ay - a^2\ . \tag{5.123}$$

The constants a and c_0 are obtained from the solutions of the BPS equations at the poles. We found

$$c_0 = \frac{2\left(n_N^2 + n_N n_S + n_S^2\right)}{3n_N n_S\left(n_N + n_S\right)} \tag{5.124}$$

while the constant a is

$$a = \frac{(n_N - n_S)^{\,2}\,(2n_N + n_S)^{\,2}\,(n_N + 2n_S)^{\,2}}{4\,(n_N n_S + n_N^2 + n_S^2)^{\,3}}\ . \tag{5.125}$$

From here it follows that

$$y_N = \frac{\left(-2n_N^2 + n_N n_S + n_S^2\right)^2}{4\left(n_N^2 + n_N n_S + n_S^2\right)^2}, \quad y_S = \frac{\left(n_N - n_S\right)^2\left(n_N + 2n_S\right)^2}{4\left(n_N^2 + n_N n_S + n_S^2\right)^2}\ . \tag{5.126}$$

The central charge becomes

$$c_{2d} = \frac{9\pi\left(n_S - n_N\right)^3}{16G_5 W_{\text{crit}}^3 n_N n_S\left(n_N^2 + n_N n_S + n_S^2\right)}\ . \tag{5.127}$$

Table 5.1: Some numerical solutions found in our analysis for various, consistent, values of deficit angles $n_{N,S}$, flavor flux p_F and geometry $\mathbf{z}$. The boundary values for the field φ are found so to give rise to a finite Spindle in the y direction.

n_S	n_N	p_F	$\mathbf{z}$	φ_S	φ_N	Δy
1	3	0	2	-0.285076	-0.274493	1.83241
1	7	-1	2	-0.172372	-0.170589	2.39707
1	3	0	3	-0.555814	-0.542721	1.82303
1	5	-1	3	-0.300428	-0.300346	2.16012
1	9	3	$\frac{1}{3}$	0.463989	0.363277	2.57446
1	5	0	$\frac{1}{3}$	0.126802	0.124497	2.16392
1	7	2	$\frac{1}{2}$	0.484886	0.347516	2.3322
3	7	0	$\frac{1}{2}$	0.104192	0.103447	1.74866

Using then $a_{4d} = \frac{\pi R^3_{AdS_5}}{8G_5}$ and $R_{AdS_5} = \frac{3}{2W_{\text{crit}}}$ we arrive at the expected universal result (5.119). Observe that W_{crit} can be consistently found from the relation (5.118).

Before turning on a generic value for the flavor flux p_F and studying the solution numerically, a comment is in order. We have so far referred to the solution with p_F set to $\epsilon^*_{4d}(n_S - n_N)/2$ as "universal" solution. Such terminology refers to the fact that the truncation is restricted the "pure" gravity sector, indeed recovering the AdS$_5$ vacuum. On the field theory side the constraint on p_F indeed reflects on the ones on p and q that set the exact R-symmetry to be rational, as discussed at the end of subsection 5.2.1.2.

5.2.5.2 Numerical Solution for Generic p_F

Here we look for more generic solutions of the BPS equations interpolating among the poles of the spindle. From the analysis above we have observed that the analytic solutions with $p_F = \epsilon^*_{4d}(n_S - n_N)/2$ are in the anti-twist class with $\mathbf{k} = -1$. Here we search for numerical solutions for a generic integer $\mathbf{z}p_F$. We have scanned over large regions of parameters and again we have only found solutions with $\mathbf{k} = -1$ in the anti-twist class. The solutions are found along the lines of the analysis of [42, 56, 293]. First we specify the values of $\mathbf{z}$, n_S, n_N and p_F. Then we fix the initial conditions imposed by the analysis at the poles. In this way we are left with one unknown initial condition for the hyperscalar φ. Finding the initial condition of φ corresponds to find the solution for the BPS equations on the spindle. There is just (up to the numerical approximation) a single value φ_S (here we are fixing the south pole at $y_S = 0$) that allows to integrate the BPS equation giving rise to a finite spindle in the y direction. Once this value is found a good sanity check consists of running the numerics until $2\Delta y$, that corresponds to solve the BPS equations from the north to the south pole as well. We have scanned over various values of the parameters and we present some of the solutions that we found in table 5.1.

In each case we have fixed $\mathbf{k} = -1$ and chosen $\kappa = 1$ (corresponding to the choice

$n_N > n_S$). The explicit solutions are plot in Figure 5.3. Observe that the solutions for the cases at $p_F = 0$ do not correspond to the universal twist (at least for $\mathbf{z} \neq 0$). The cases at $p_F = 0$ correspond to a twist along a trial R-symmetry, obtained from a linear combination (with irrational coefficients) of the (irrational) exact R-symmetry and the flavor symmetry.

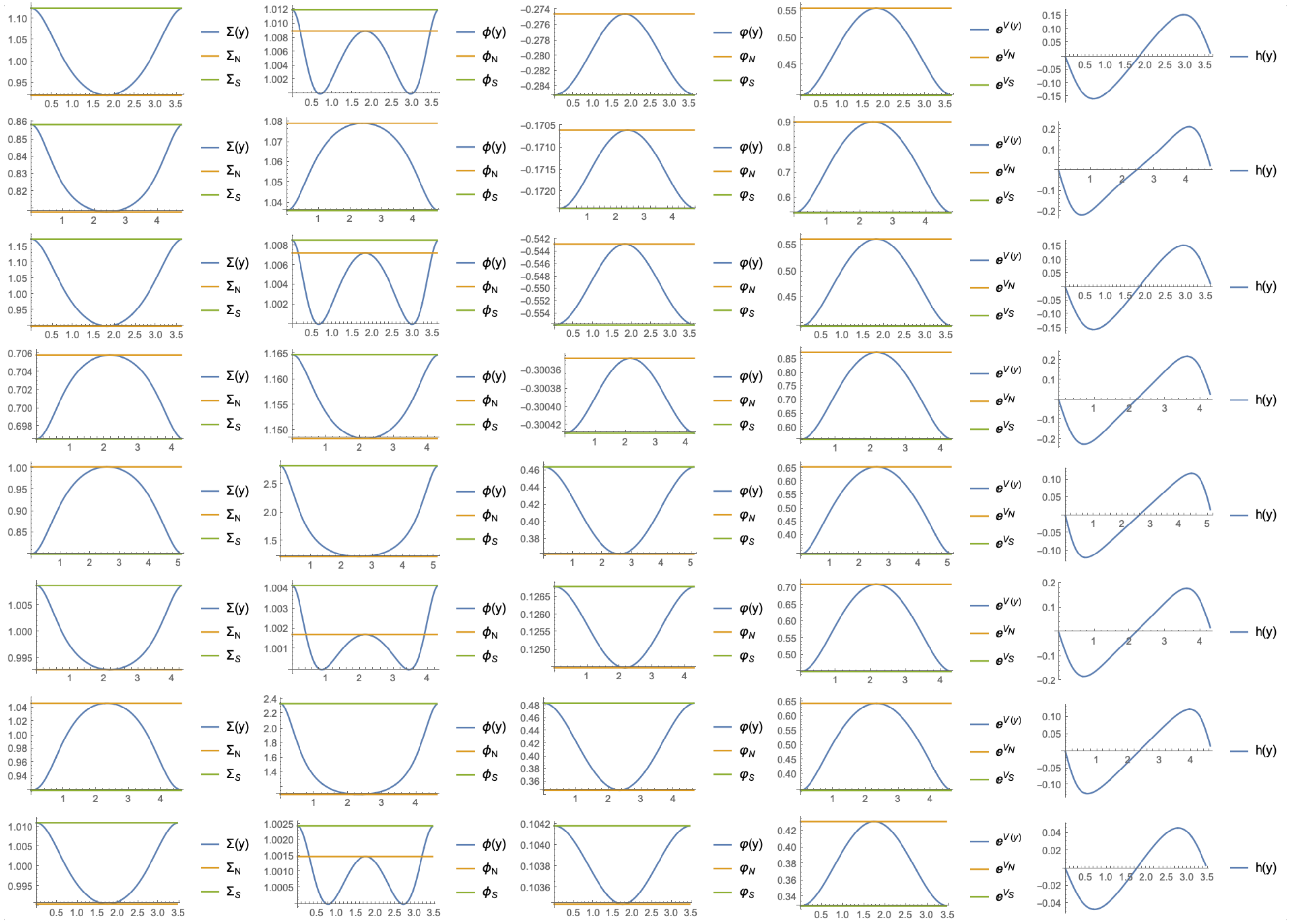

Figure 5.3: Numerical solutions for the scalar fields $\Sigma(y), \phi(y), \varphi(y)$ and the scalar functions $e^V(y)$ and $h(y)$ interpolating between $y = y_S = 0$ and $y = 2(y_N - y_S)$. The values of n_S, n_N, p_F and **z** are the ones fixed in (5.1).The values of the fields at y_S are the green lines and at y_N are the orange ones. From left to right, the plots follow the same order of table 5.1.

5.3 Discussion and Conclusions

In this Chapter we studied the reduction of the consistent truncations found in [132] on the spindle. These truncations are associated to M5 branes wrapping holomorphic curves in a CY_3 and the dual field theories have been obtained in [66, 67]. Using these results we matched the 2d central charge obtained from the field theoretical analysis with the one predicted in gauged supergravity from the analysis at the poles of the spindle. We have also studied the full solution, showing its existence for consistent choices of the parameters, analytically for the universal anti-twist and numerically after including the magnetic charge of the flavor symmetry.

There are many interesting aspects that we did not investigate. A first open question regards the uplift of our solutions to 7d and 11d supergravity. An interesting limit corresponds to set $\mathbf{z} = \pm 1$ and consider $p_F = 2\mathbf{z}\big(q - \frac{1}{4}(n_S - n_N)\big)$. In this case we reproduce the results obtained in [114] for the $\mathcal{N} = 2$ Maldacena-Nuñez theory. Observe that the matching works when p_F and $(n_S - n_N)/2$ have the same parity.

Another open question regards the existence of solutions for $\mathbf{k} = 1$ and $|\mathbf{z}| > 1$ in both the twist and the anti-twist class and for $\mathbf{k} = -1$ in the twist class. Even if we have not been able to exclude these possibilities (for generic values of $\mathbf{z}$) we have not found any solution of this type neither in the analytical nor in the numerical analysis carried out in section 5.2.5. Nevertheless, we observe that by choosing $\mathbf{z} = 0$ we can simplify the problem (for $\mathbf{k} = -1$) and we obtain results similar to the one studied in [42, 56, 293]. This limit corresponds to the $\mathcal{N} = 1$ Maldacena-Nuñez theory and in this case the pole analysis completely excludes the existence of solutions in the twist class. The reason is that in this case we can impose further reality constraints on the conserved charges against the existence of such solutions.

Our analysis has been performed at leading order in N, i.e. the central charge here is scales with N^3. There is a subleading contribution of order N, proportional to the gravitational anomaly of the SCFT, that one could compute from the field theoretical side. It would be interesting to match this contribution from the holographic analysis. A similar calculation was carried out for the case of the topological twist in [65].

It would also be interesting to consider M5 branes wrapping other geometries. For example by considering a disc, an holographic dual of an $\mathcal{N} = 2$ SCFT of AD type was proposed in [69–71] (see also [162]) As then observed in [164, 294] indeed the disc and spindle geometries are different global completions of the same local solution.

Finally, it would be possible to study the models discussed here from the 11d perspective along the lines of the recent discussions of [79, 80, 157, 267] from the theory of equivariant localization.

CHAPTER 6

Probing Black-Holes with Surface Defects from Their Field Theory Duals

Contents

In the recent past important progress in the microstates counting of 5d rotating charged supersymmetric black hole has been possible thanks to the role played by the Superconformal Index (SCI) of [256, 286]. Such a proliferation of results spread after the seminal work of [225] where an entropy function, counting microstates of the dual black hole, was proposed. The microscopic origin of the Bekenstein-Hawking entropy of the holographic dual supersymmetric black hole was provided in [125].

Motivated by these results it became crucial to extract the entropy function of [225] from a pure field theory calculation. The SCI, even if expected to be the natural candidate for this computation, initially failed to provide the $\mathcal{O}(N^2)$ scaling of the microscopic degrees of freedom [256] due to large cancellations among states with opposite statistics. The resolution of the puzzle was found for $\mathrm{SU}(N)$ $\mathcal{N}=4$ SYM in [89, 144] by using two different methodologies. Based on these, one can then distinguish two broad classes of computations of the index for supersymmetric theories in 4d. Either one first compute the integral exactly and then evaluate the leading contribution to the entropy [7, 13, 86, 88–90, 156, 203, 259, 261, 266] or one evaluates the integrand and then extract the entropy from a saddle point analysis [25, 26, 29, 46, 53, 57, 124, 127, 129, 145, 146, 200–202, 220, 241, 254]. The first approach, originally discussed in [88, 89], provides, in principle, an exact answer in any regime of charges. However, the formal exact evaluation turns out

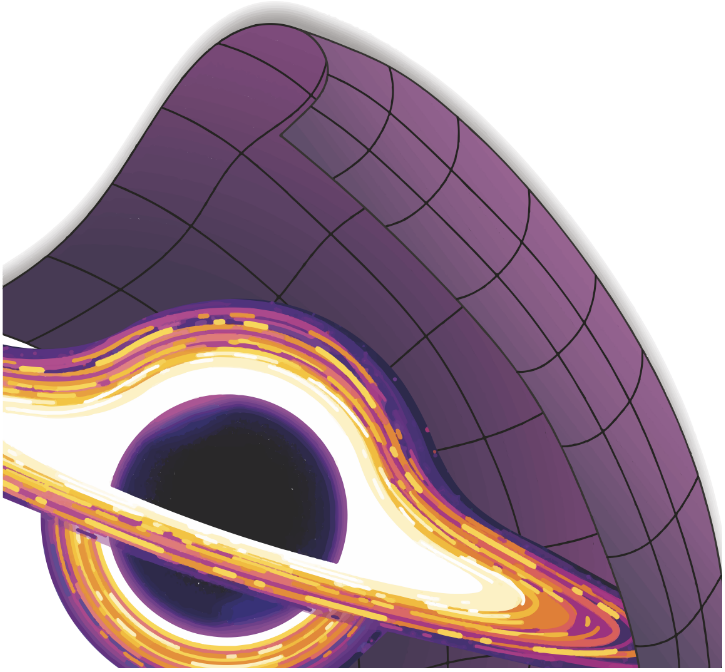

Figure 6.1: Depicion of a D3 brane probing a Black Hole

to be rather complicated and it boils down to solve a set of algebraic equation, referred in the literature as Bethe Ansatz Equations (BAEs). Despite such difficulties in the case of $\mathrm{SU}(N)$ $\mathcal{N} = 4$ SYM the solutions are known at large N and it has allowed to extract the black hole entropy, matching it with the gravitational expectations [208, 209]. A simpler calculation, valid only in a restricted regime of charges, corresponds to the so called Cardy-like limit [144]. In this case one estimates the integral from a saddle point analysis and then the entropy can be obtained also at finite N. Furthermore, a third method consists of a direct saddle point evaluation of the matrix integral at large N in [129]. Observe that the saddle point evaluations of the index can be generalized to other 4d models with different matter content and supersymmetry and such results inspired the EFT calculations of [55, 134] on the high temperature limit on the second sheet of the index.

Despite the power of the results discussed so far it is desirable to go beyond, by perturbing the black hole, and as a consequence the superconformal index in a controlled way. For example, perturbing the system with the addition of a Polyakov loop provides an order parameter to detect the confinement/deconfinement transition, expected to correspond to the dual mechanism of the (first order) Hawking-Page transition from the thermal AdS to the large black hole [14, 297, 309] (see [138, 142, 143, 159, 280] for recent progresses in the understanding of the Hawking-Page transition from the field theory side).

Recently another (supersymmetry preserving) order parameter for the deconfinement phase transition has been proposed in [139] by adding a surface defect corresponding on the gravitational side to a probe D3-brane, extended across the time and a radial direction, and wrapped on one compact direction in AdS$_5$ and one compact direction in S^5. Such a probe D3 is interpreted in the dual field theory as a half BPS Gukov-Witten surface defect placed on $\mathbb{R}^2$ at $x_2 = x_3 = 0$ in $\mathbb{R}^{1,3}$. The defect corresponds to a codimension-2 singularity in $\mathcal{N} = 4$ SYM. A class of Gukov-Witten defects in $\mathcal{N} = 4$ SYM is classified by specifying its Levi subgroup embedding. The defect studied in [139] corresponds to the maximal Levi subgroup embedding. From the 2d SCFT point of view the theory living on the defect is then an $\mathcal{N} = (4, 4)$ $\mathrm{U}(1)$ gauge theory with N fundamental hypermultiplets[1]. The surface operator defined in this way probes a $1/16$ BPS black hole in the generalized thermal ensemble given by the superconformal index. At technical level the effect of the surface defect on the index is obtained by coupling the 4d-2d system along the lines of the discussion of [187]. This coupling is done by gauging the global symmetry of the 2d theory, identifying it to the $\mathrm{SU}(N)$ gauge symmetry of 4d $\mathcal{N} = 4$ SYM. The final expression for the superconformal index of the 4d-2d coupled system is then given by the original index and in addition to the integrand the contribution of opportune insertions of Jacobi θ_0 functions carrying the charges of the 2d fields once expressed in terms of the 4d ones.

From the gravitational side the leading contribution of the probe D3 to the free energy of the Black Hole has been computed in [139] in the case of equal charges and different angular momenta. The final result correspond to a sum of the unperturbed result and to the perturbative contribution from the DBI action. Translating the result to the entropy in turns out that the charge and the entropy of the D3 are complex in this case. The result is apparently contradictory with respect with the one obtained from the field theory side using the superconformal index, where such quantities are real. The way out of the apparent contradiction has been subsequently discussed in [128], where it has been shown that by borrwing the field theory result, where the entropy is obtaind from the laplace transform of the superconformal index, also the gravitational entropy can be shown to be real.

Furthermore while in [139] the evaluation of the defect superconformal index has been pursued using a direct saddle point evaluation at $1/N$ order in a fixed regime of charges, in [128] the evaluation of the superconformal index has been done through a systematic Cardy-like expansion with more generic regimes of charges allowed.

While the two result match in the regime of small angular momenta at fixed charges, this second approach is intriguing because it tells us more informations about the backreaction of the probe D3, predicting a fully backreacted answer at leading order in the Cardy like limit. The result furthermore suggests the structure of the subleading correction in the Cardy-like limit in terms of the angular fugacities. Namely a series expansion in the angular momenta could be derived, going beyond the leading order approximation.

This Chapter is based on [31] and is organized as follows. We firstly introduce the Gukov-Witten operators and their construction as 2d gauge theories in section 6.1. We will then discuss the backreaction effect from the SCI perspective, evaluating the SCI in the Cardy-like limit in section 6.3, and with the Bethe Ansatz (BA) approach in section 6.4. To this end we compute such backreaction in the Cardy-like limit around the

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[1] See also [255] for a similar setup where the actual defect corresponds to the one studied in [273]

holonomy saddles giving rise to the black hole. The result is compatible with [128] and it allows to estimate the 3d partition function with the addition of the effect of the defect. Surprisingly we find that the two regions corresponding to the second and the third sheet from the EFT approach give rise to the same result, symmetrizing the seeming asymmetry of [128].

Then we confirm our result following the Bethe Ansatz approach. We first verify its feasibility in presence of the defect and then we compute the contribution of the basic solution of the BAEs. The computation is done for equal, not necessarily large, angular momenta and for arbitrary flavor charges. Thus, we generalize the result obtained for equal and fixed flavor charges in [139]. The resulting index, once evaluated on the basic solution, recovers its symmetry in agreement with the Cardy-like analysis.

6.1 Gukov-Witten Surface Operators

6.1.1 Field Theory Construction

In this section we give a brief review of the construction of two-dimensional operators by Gukov and Witten (GW) [206, 207]. We then discuss their field theory interpretation as a coupled 4d-2d system and the computation of the SCI in the presence of such defects [187].

Extended operators in a QFT can be broadly divided into two sub-categories: either defined by functionals of local operators on some higher co-dimension manifold or as singularities in the gauge field. From this classification one recognizes Wilson and 't Hooft line operators of four-dimensional pure Yang-Mills theory. The former are classified by representations of the gauge group G, while the latter are, generically, classified by integers labelling the amount of magnetic charge.

Gukov and Witten gave a prescription to generalize this construction to two dimensional surface operators as singularities for the vector field on a surface Σ. Surface defects, in contrast to line defects, are classified not only by the singularity of the vector field along Σ, but also by the subgroup of G under which they are invariant. In this section we will consider such defects in 4d $\mathcal{N} = 4$ SYM even if the construction is more general.

We regard the field content of $\mathcal{N} = 4$ SYM in $\mathcal{N} = 1$ language where we the field content is given by a vector multiplet V and chiral multiplets $\Phi_{i=1,2,3}$. A half-BPS GW surface operator oriented along the (x^0, x^1) direction, is defined as a singularity on the vector field and the scalar component of the chiral multiplets

$$A = a(r)\,\mathrm{d}\theta + \cdots \,, \qquad \phi = b(r)\frac{\mathrm{d}r}{r} - c(r)\,\mathrm{d}\theta + \cdots \,, \tag{6.1}$$

where $A = A_\mu\,\mathrm{d}x^\mu\,,\ \phi = \phi_\mu\,\mathrm{d}x^\mu$ with $\mu = 2, 3$ and $z \equiv r e^{\mathrm{i}\theta} = x^2 + \mathrm{i}x^3$ is the normal direction to Σ. The BPS condition can be casted in the form of Hitchin equations

$$\begin{cases} F_A - \phi \wedge \phi = 0, \\ \mathrm{d}_A\phi = 0, \quad \mathrm{d}_A \star \phi = 0 \end{cases} \tag{6.2}$$

and conformal invariance requires a, b, c in (6.1) to be independent on r. Hitchin equations also require a, b, c to be mutually commuting. The easiest solution to (6.1) is obtained by conjugating the algebra-valued parameters a, b, c to parameters α, β, γ valued in the Lie algebra $\mathfrak{t}$ of the maximal torus $\mathbb{T}$ of the gauge group G. Therefore, the singularity is described by

$$A = \alpha\,\mathrm{d}\theta + \cdots\,, \qquad \phi = \beta\frac{\mathrm{d}r}{r} - \gamma\,\mathrm{d}\theta + \cdots\,. \tag{6.3}$$

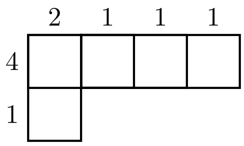
Furthermore, it turns out that one can add a 2d θ-term labelling topologically distinct restrictions of the G-bundle to the defect. The parameter η labelling this choice generically takes value in a subgroup of ${}^{L}\mathbb{T}$, the maximal torus of the Langlands dual of G. This fully defines the insertion of a GW surface defect in the path integral. However, when summing over gauge configurations in the path integral, one divides by the subgroup of G commuting with the parameters $\alpha, \beta, \gamma, \eta$. This condition defines the subgroup that preserves the singularity which is denoted as Levi subgroup $\mathbb{L}$. The choice of $\mathbb{L}$ is regarded as the definition of the defect. Therefore, the insertion of the defect amounts to a choice of

$$(\alpha, \beta, \gamma, \eta) \in (\mathbb{T} \times \mathfrak{t} \times \mathfrak{t} \times {}^{L}\mathbb{T})/\text{Weyl}(\mathbb{L}). \tag{6.4}$$

In the rest of the Chapter we focus on $G = \mathrm{SU}(N)$. The Levi subgroups in this case are classified by partitions of $N = \lambda_1 + \cdots + \lambda_s$. A partition $\lambda = [\lambda_1, \ldots, \lambda_s]$, with $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n < N$, is associated to the Levi subgroup

$$\mathbb{L} = \mathrm{S}\left(\bigotimes_{i=1}^{n} \mathrm{U}(k_i)\right), \qquad N = \sum_{i=1}^{n} k_i\,, \tag{6.5}$$

where k_i are the number of boxes in the i-th column Young tableaux associated to λ. For example, let us consider $G = \mathrm{SU}(5)$ and $\lambda = [4, 1]$. Here the Young tableaux is



(6.6)

and the associated Levi subgroup is

$$\mathbb{L} = \mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(1) \times \mathrm{U}(1) \times \mathrm{U}(1)) \simeq \mathrm{SU}(2) \times \mathrm{U}(1)^3. \tag{6.7}$$

For $\mathrm{SU}(N)$, the Levi subgroup associated to the partition $\lambda = [N-1, 1]$ is dubbed "maximal Levi sub-group". This is going to be the relevant sub-group for our analysis.

The 2d defect is coupled to the 4d theory by imposing the singular behavior on the 4d gauge fields in the path integral. In practice this is quite cumbersome and usually one can add a 2d theory acting as a Lagrange multiplier such to impose the singular behavior on Σ. This 2d gauge theory is built so that, when integrating out the excitation on the defect, one recovers the original 4d theory with the constrained fields.

The 2d theory must satisfy certain properties [187, 206, 207] to prescribe the singularity as described above. Let us consider the scenario where the 2d theory is a Gauge Linear Sigma Model (GLSM) with some target space $\mathcal{M}_{\alpha,\beta,\gamma}$. A half-BPS defect must preserve $\mathcal{N} = (4,4)$ supersymmetry, implying that the target space of the GLSM must be hyper-Kähler. Additionally, it must possess a G action, suggesting that the 2d theory must have a G flavor symmetry that is used to couple it to the 4d bulk. Furthermore, it must be dependent on the choice of $\mathbb{L}$. The simplest target space is the cotangent space of $G/\mathbb{L}$, denoted as $T^*(G/\mathbb{L})$. It is worth noting that $T^*(G/\mathbb{L}) = G_{\mathbb{C}}/\mathbb{L}_{\mathbb{C}}$ is also as the moduli space of solutions with the prescribed singularity of the form (6.3). The action of the 2d system can be obtained straightforwardly: the coupling to the 4d theory induces a singularity in the BPS equations (6.2)

$$F_{23} + \left[\phi_2, \phi_2^\dagger\right] = 2\pi\delta^{(2)}(\vec{x})qq^\dagger, \qquad D_{\bar{z}}\phi_2 = \pi\delta^{(2)}(\vec{x})q\tilde{q}\,, \tag{6.8}$$

where $qq^\dagger$ and $q\tilde{q}$ are moment maps for the G action on $\mathcal{M}_{\alpha,\beta,\gamma}$. By virtue of the BPS equations of the 2d theory, the moment maps are integrated out in favour of the Kähler moduli $\alpha + \mathrm{i}\eta$ and $\beta + \mathrm{i}\gamma$ respectively. This, together with the δ-functions, induces the singular behavior of the solution (6.3).

In order to describe the gauge group G_{2d} of the 2d theory, one further needs to describe $\mathcal{M}_{\alpha,\beta,\gamma}$ as an hyper-Kähler quotient of some vector space by the group G_{2d}. For the case at hand, where the 4d gauge theory is $\mathcal{N} = 4$ SU(N) SYM, this quotient can always be constructed [257] and the resulting gauge theory is a 2d $\mathcal{N} = (4,4)$ theory with flavor symmetry $G = \mathrm{SU}(N)$ and gauge group

$$G_{2d} = \bigotimes_{i=1}^{n-1} \mathrm{U}(p_i), \quad \text{where} \quad p_i = \sum_{j=1}^{i} k_j. \tag{6.9}$$

The matter content is given by bi-fundamental hypermultiplets in the $(\mathbf{p}_i, \mathbf{p}_{i+1})$ representation and N fundamental hypermultiplets for $\mathrm{U}(p_{n-1})$. In $\mathcal{N} = (2,2)$ language the theory is given by the quiver diagram in figure 6.2. The theory also carries a Weyl anomaly [137, 306] c_{2d} which for a defect of the form (6.5) is given by

$$c_{2d} = 3\left(N^2 - \sum_{i=1}^{n} k_i^2\right). \tag{6.10}$$

The construction just described is necessary in order to define the SCI of the 4d-2d coupled system.

6.2 The 2d Elliptic Genus

Here we briefly discuss how the SCI computation changes in the presence of the GW defect. Firstly, to preserve the right supercharges on $S^3 \times S^1$ the GW defect has to be wrapped appropriately on this geometry. This corresponds to wrapping once the defect around the thermal circle and once around the great circle of S^3. By the state-operator

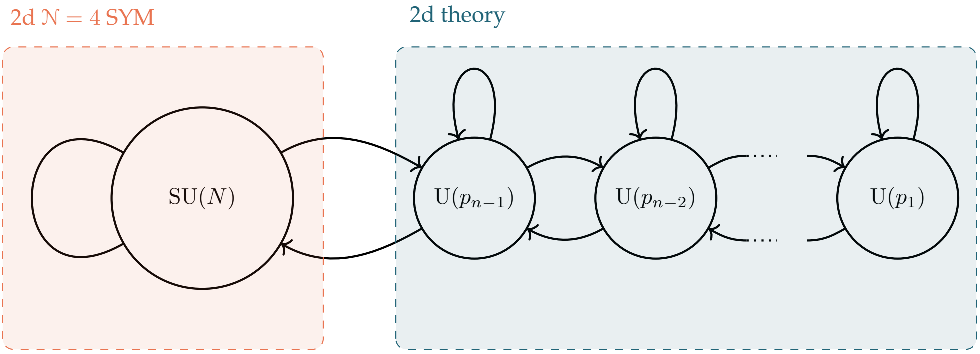


Figure 6.2: Quiver diagram of the 4d-2d coupled system in the $\mathcal{N}=(2,2)$ language in 2d and $\mathcal{N}=2$ language in 4d. Notice that the ranks of the gauge groups of the 2d theory go from right to left.

correspondence, the Hilbert space of the theory is now twisted by the presence of the defect. Therefore, the SCI must be computed on the Hilbert space $\mathcal{H}_{\mathrm{GW}}(S^3)$. However, this approach is quite involved and one usually exploits the fact that the index of the 4d-2d coupled system can be casted in the following form

$$\mathcal{I}(p,q,v_i)=\int_{\mathrm{SU}(N)}\mathrm{d}u\,\mathcal{I}_{4d}(p,q,v_i;u)\mathcal{I}_{2d}(p,q,v_i;u)\,, \tag{6.11}$$

where $\mathcal{I}_{2d}$ is the index of the 2d theory living on the defect, which is wrapped along a temporal T^2 in $S^1\times S^3$.

To fully compute the SCI (6.11) one needs to understand how the superconformal algebra of the 2d $\mathcal{N}=(4,4)$ theory is embedded in the one of 4d $\mathcal{N}=4$. The insertion of an half-BPS GW operator in 4d breaks the superconformal algebra in

$$\mathfrak{u}(1)_A\ltimes(\mathfrak{psu}(1,1|2)\times\mathfrak{psu}(1,1|2))\ltimes\mathfrak{u}(1)_C\subset\mathfrak{psu}(2,2|4) \tag{6.12}$$

which is the usual 2d $\mathcal{N}=(4,4)$ superVirasoro algebra, centrally extended by $\mathfrak{u}(1)_C$. The abelian factor $\mathfrak{u}(1)_A$ acts as the outer-automorphism of the algebra and need not be a realized symmetry on the defect.

The 2d contribution to (6.11) can be found by the following 2d index

$$\mathcal{I}_{\mathrm{NSNS}}=\mathrm{Tr}_{\mathrm{NSNS}}(-1)^F\mathrm{e}^{2\pi\mathrm{i}\tau_{2d}L_0}\mathrm{e}^{-2\pi\mathrm{i}\bar{\tau}_{2d}(\bar{L}_0-\frac{1}{2}\bar{J}_0)}\mathrm{e}^{2\pi\mathrm{i}z_{\mathrm{NS}}J_0}\mathrm{e}^{2\pi\mathrm{i}\chi J_A}\mathrm{e}^{2\pi\mathrm{i}uC}\,, \tag{6.13}$$

where τ_{2d} is the complex structure of the temporal T^2 and the trace is taken over the NSNS-sector. The various bosonic generators of the embedding (6.12) that enter in the definition of this index are the right- and left-moving Hamiltonians $\bar{L}_0, L_0$, the Cartan of $\mathfrak{su}(2)_R$ J_0 and the flavor symmetry J_A of the GLSM. To ease the computation of such index, we will use the fact that the NSNS-sector and RR-sector indices are related, and that the latter is the elliptic genus of the GLSM. In fact, as discussed in Appendix C of

[139], the following relation holds

$$\mathcal{I}_{\text{NSNS}}(\tau_{2d}, z_{\text{NS}}, \chi, u) = \mathrm{e}^{2\pi \mathrm{i} \tau_{2d}\left(-\frac{c_{2d}}{24}\right)} \mathrm{e}^{-\frac{\pi \mathrm{i} c_{2d}}{3}\left(z_{\text{NS}}-\frac{\tau_{2d}}{2}\right)} \mathcal{I}_{\text{RR}}(\tau_{2d}, z_{\text{NS}} - \frac{\tau_{2d}}{2}, \chi, u)\,, \tag{6.14}$$

where $\mathcal{I}_{RR}$ is the RR-sector index (or fully-refined elliptic genus) and one identifies $z_{\text{R}} \equiv z_{\text{NS}} - \frac{\tau_{2d}}{2}$. The 2d contribution in (6.11) is then given by

$$\mathcal{I}_{2d}(\tau_{2d}, z_{\text{NS}}, \chi, u) = \mathrm{e}^{-\frac{\pi \mathrm{i} c_{2d}}{3} z_R} \mathcal{I}_{\text{RR}}(\tau_{2d}, z_{\text{R}}, \chi, u). \tag{6.15}$$

Following the discussion of [87, 139, 187], the elliptic genus of a general $\mathcal{N} = (2, 2)$ gauge theory is given by

$$Z_{T^2} = \frac{1}{|W|} \sum_{u_* \in \mathcal{M}^*_{\text{sing}}} \underset{u=u_*}{\text{JK-Res}}(\mathsf{Q}(u_*), \eta) Z_{\text{1-loop}}\,, \tag{6.16}$$

where the residues are evaluated following the Jeffrey-Kirwan prescription and $\mathsf{Q}(u_*)$ are the chemical potentials for the fields constrained by the poles at u_*. The one-loop contributions for the multiplets are

$$\begin{aligned} Z^{\text{chiral}}_{\mathcal{R}}(\tau, \zeta, u) &= \prod_{\rho \in \mathcal{R}} \frac{\theta_1(y^{\frac{R}{2}-1} x^\rho; q)}{\theta_1(y^{\frac{R}{2}} x^\rho; q)}, \\ Z^{\text{Vector}}_{G}(\tau, \zeta, u) &= \left(\frac{2\pi \eta(q)^3}{\theta_1(y^{-1}; q)}\right)^{\operatorname{rank} G} \prod_{\alpha \in G} \frac{\theta_1(x^\alpha; q)}{\theta_1(y^{-1} x^\alpha; q)} \prod_{a=1}^{\operatorname{rank} G} \mathrm{d}\mu_a\,, \\ Z^{\text{Twisted}}(\tau, \zeta) &= \frac{\theta_1(y^{-\frac{R_A}{2}+1}; q)}{\theta_1(y^{-\frac{R_A}{2}}; q)}, \end{aligned} \tag{6.17}$$

where $\eta(q)$ is the Dedekind eta function and $\theta_1(z; q)$ is the Jacobi theta function. These functions depend on fugacities related to the gauge and global symmetries, which are defined by $q = \mathrm{e}^{2\pi \mathrm{i} \tau_{2d}}$, $y = \mathrm{e}^{2\pi \mathrm{i} \zeta}$, $x_a = \mathrm{e}^{2\pi \mathrm{i} \mu_a}$ and $x^\rho = \mathrm{e}^{2\pi \mathrm{i} \rho(\mu)}$.

For the case of interest, the 2d theory associated to the maximal $\mathrm{SU}(N)$ GW operator $\lambda = [N-1, 1]$ is a $\mathrm{U}(1)$ gauge theory with a $\mathcal{N} = (4, 4)$ hypermultiplet which is just a pair of $\mathcal{N} = (2, 2)$ chiral multiplets in the bi-fundamental of $\mathrm{SU}(N)$. Moreover, we have a $\mathcal{N} = (4, 4)$ $\mathrm{U}(1)$ vector multiplet, corresponding in the $\mathcal{N} = (2, 2)$ language to a chiral multiplet and a vector multiplet[2]. The one-loop determinant is given by

$$Z_{\text{1-loop}} = \frac{2\pi \eta(q)^3}{\theta_1(-\zeta; \tau)} \frac{\theta_1(-2\chi; \tau)}{\theta_1(2\chi - \zeta; \tau)} \prod_{i=1}^{N} \frac{\theta_1(\mu - u_i + \chi - \zeta; \tau)}{\theta_1(\mu - u_i + \chi; \tau)} \frac{\theta_1(-\mu + u_i + \chi - \zeta; \tau)}{\theta_1(-\mu + u_i + \chi; \tau)} \mathrm{d}\mu\,, \tag{6.18}$$

where χ is a fugacity associated to the $\mathrm{U}(1)_A$ symmetry in (6.12). Here the arguments of the Jacobi theta function are the chemical potentials rather than the fugacities to avoid

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[2] In the abelian case, this is equivalent to a twisted chiral multiplet.

clutter. The elliptic genus is computed by integrating over the gauge holonomy μ. Computing the residue around $\mu = u_j - \chi$ one gets cancellations between the vector multiplet determinant and fundamentals with $\beta = \alpha$ ending with the following result

$$Z_{T^2}(\tau, \zeta, \chi) = \sum_{i=1}^{N} \prod_{j \neq i} \frac{\theta_1(u_{ij} + \zeta - 2\chi; \tau)\theta_1(u_{ij} - \zeta; \tau)}{\theta_1(u_{ij}; \tau)\theta_1(u_{ij} - 2\chi; \tau)}, \quad u_{ij} = u_i - u_j. \tag{6.19}$$

The last step is to use this result in (6.15) to get the 2d contribution. The phase factor in (6.15) depends on the 2d central charge, which for the maximal defect is given by $c_{2d} = 6(N-1)$ by using formula (6.10). Here, the phase can be reabsorbed into the Jacobi θ_1-function, using (A.7), to get

$$\mathcal{I}_{2d} = \sum_{i=1}^{N} \prod_{j \neq i}^{N} \frac{\theta_0(u_{ij} + \zeta - 2\chi; \tau)\theta_0(u_{ij} - \zeta; \tau)}{\theta_0(u_{ij}; \tau)\theta_0(u_{ij} - 2\chi; \tau)}, \qquad \zeta \equiv z_{\text{NS}}. \tag{6.20}$$

6.3 Cardy-like Approach

6.3.1 The 4d Superconformal Index

In this section we compute the Cardy-like limit of the index of the 4d-2d coupled system, describing the insertion of a GW defect in $\mathcal{N} = 4$ SU(N) SYM.

The SCI, originally constructed in [256] for 4d $\mathcal{N} = 4$ SYM, can be defined for a generic 4d $\mathcal{N} = 1$ SCFT [286], choosing one supercharge $\mathcal{Q}$, as refined Witten index of the theory in radial quantization. The index counts the difference between bosonic and fermionic states annihilated by $\mathcal{Q}$ in the Hilbert space of the theory on S^3. Explicitly, the SCI, in the notation of Dolan and Osborn [175], is

$$\mathcal{I}_{4d} = \text{Tr}(-1)^F \text{e}^{-\beta\{\mathcal{Q}, \bar{\mathcal{Q}}\}} p^{J_1} q^{J_2} (pq)^{R/2} \prod_k v_k^{Q_k}, \tag{6.21}$$

where the refinement is obtained by including charges in the commutant of $\{\mathcal{Q}, \bar{\mathcal{Q}}\}$. In (6.21) J_1 and J_2 correspond to the angular momenta of the S^3, R is the U(1) R-charge and the fugacities v_k parametrize the Cartan subalgebra, with charges Q_k, of other generic symmetries the theory may have.

We are interested in the case of $\mathcal{N} = 4$ SYM with SU(N) gauge group, for which the index (6.21) takes the form

$$\mathcal{I}_{4d} = \text{Tr}_{\text{gauge}}(-1)^F \text{e}^{-\beta\{\mathcal{Q}, \bar{\mathcal{Q}}\}} p^{J_1} q^{J_2} (pq)^{R/2} v_1^{Q_1} v_2^{Q_2}, \tag{6.22}$$

where the trace is taken over gauge singlets and Q_1, Q_2 parametrize the Cartan of the $\mathfrak{su}(3) \subset \mathfrak{su}(4)_R$ commuting with $\{\mathcal{Q}, \bar{\mathcal{Q}}\}$. In order to discuss the Cardy-like limit of the index we define

$$p = \text{e}^{2\pi \text{i}\tau}, \quad q = \text{e}^{2\pi \text{i}\sigma}, \quad v = \text{e}^{2\pi \text{i}\xi} \tag{6.23}$$

and introduce the chemical potentials Δ_a associated to the matter fields, given by

$$\Delta_a = \rho_a(\xi) + \frac{\tau+\sigma}{2} R_a, \tag{6.24}$$

together with the constraint

$$\sum_{a=1}^{3} \Delta_a = \sigma + \tau \mod 1. \tag{6.25}$$

The trace over gauge-invariant states can be achieved by integrating over the holonomies of the gauge group. Defining the elliptic gamma functions

$$\Gamma(z;p,q) \coloneqq \prod_{m=0}^{\infty}\prod_{n=0}^{\infty} \frac{1-p^{m+1}q^{n+1}/z}{1-p^m q^n z}\,, \qquad \widetilde{\Gamma}(u) \coloneqq \Gamma(\mathrm{e}^{2\pi \mathrm{i} u};\mathrm{e}^{2\pi\mathrm{i}\tau},\mathrm{e}^{2\pi\mathrm{i}\sigma}) \tag{6.26}$$

and the q-Pochhammer symbol

$$(z;q)_\infty \coloneqq \prod_{k=0}^{\infty}\left(1-zq^k\right), \tag{6.27}$$

the index can be expressed as an elliptic hypergeometric integral [175]

$$\mathcal{I}(\Delta,\tau,\sigma) = \frac{(p;p)_\infty^{N-1}(q;q)_\infty^{N-1}}{N!}\prod_{a=1}^{3}\widetilde{\Gamma}(\Delta_a)^{N-1}\int\prod_{i=1}^{N-1}\mathrm{d}u_i \frac{\prod_{a=1}^{3}\prod_{i\neq j}^{N}\widetilde{\Gamma}(u_{ij}+\Delta_a)}{\prod_{i\neq j}^{N}\widetilde{\Gamma}(u_{ij})}, \tag{6.28}$$

where $u_{ij} \coloneqq u_i - u_j$. The integral is taken over the gauge holonomies, subjected to the $\mathrm{SU}(N)$ constraint

$$\sum_{i=1}^{N} u_i \in \mathbb{Z}. \tag{6.29}$$

Expression (6.28), although originally defined for purely imaginary modular parameters τ and σ, can be extended to the upper-half complex plane $\tau,\sigma \in \mathbb{H}$, and complex Δ. Analytic continuation of chemical potentials introduces phases in (6.21), which in principle allow for obstructions to Bose/Fermi cancellations. The Cardy-like limit is defined as a generalization of the standard hyperbolic limit of the index [284, 304], or high-temperature Cardy limit, where an appropriate scaling behavior is assigned to the analytically continued chemical potentials of the theory, so to preserve the aforementioned cancellations [53, 144]. Typically, one defines a complexified inverse temperature parameter β, such that

$$\tau = \frac{\mathrm{i}\beta b}{2\pi}, \qquad \sigma = \frac{\mathrm{i}\beta b^{-1}}{2\pi}, \tag{6.30}$$

where b can be identified with the squashing parameter of the S^3 in the underlying $S^1 \times S^3$ geometry. While usually b is a real positive number, also complex values are allowed. The non-collinearity of τ and σ in this last case is understood as a twisting of the S^3 on the S^1 as discussed in [134, 149, 159]. Then, the Cardy-like limit is defined first by choosing a scaling behavior for the flavor chemical potentials. Such scaling is constituted of a constant fixed term, crucial for preventing Bose/Fermi cancellations, and a linear part in β. Then, the limit sends the complexified inverse temperature β to zero. Within this framework the index (6.21) can be interpreted as the supersymmetric partition function of the theory[3] on $S^1 \times S^3$ background, with appropriate twisted boundary conditions for the fields, corresponding to the fugacities refinements. We make the connection with the underlying geometry explicit by parametrizing the fugacities as

$$\tau := r\omega_1, \qquad \sigma := r\omega_2, \qquad r \in \mathbb{R},\ \omega_1, \omega_2 \in \mathbb{H}. \tag{6.31}$$

The definition (6.31) explicitly singles out the r parameter, identified with the radius of the S^1 in the $S^1 \times S^3$ background on which the theory is defined, and the ω_1, ω_2 squashing parameters for the (possibly squashed) S^3. Then, the Cardy-like limit is understood geometrically as a dimensional reduction of the Euclidean theory along the thermal circle.

6.3.2 The Cardy-like Limit of the 4d Index

In order to compute Cardy-like limit of the 4d $\mathcal{N} = 4$ SYM index we parametrize the flavor chemical potentials as

$$\Delta_a = \tilde{\Delta}_a + r(\hat{\Delta}_a\omega_1 + \check{\Delta}_a\omega_2) \equiv \tilde{\Delta}_a + r\bar{\Delta}_a \qquad \tilde{\Delta}_a \in \mathbb{R}/\mathbb{Z},\ \hat{\Delta}_a,\ \check{\Delta}_a \in \mathbb{R}. \tag{6.32}$$

Then, the index can be expanded asymptotically as r is sent to zero and expression (6.28) can be evaluated through a saddle point approach at fixed N. In doing so, the large N limit behavior can be inferred extracting a dominant saddle configuration, associated to the black hole solution in the dual gravitational theory [144].

In order to evaluate (6.28) in the Cardy-like limit, we rewrite the index in terms of an effective action for the matrix model

$$\mathcal{I} = \frac{1}{N!}\int \prod_{i=1}^{N-1} \mathrm{d}u_i\, \mathrm{e}^{S_{4d}(u,\Delta,\tau,\sigma)}, \tag{6.33}$$

where

$$\begin{aligned} S_{4d}(u,\Delta,\tau,\sigma) =& (N-1)\sum_{a=1}^{3} \log\widetilde{\Gamma}(\Delta_a) + \sum_{a=1}^{3}\sum_{i\neq j}^{N} \log\widetilde{\Gamma}(u_{ij}+\Delta_a) + \\ &+ \sum_{i<j}^{N} \log\theta_0(u_{ij}) + \sum_{i<j}^{N} \log\theta_0(-u_{ij}) + \end{aligned} \tag{6.34}$$

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[3] The difference between the two corresponds to a supersymmetric Casimir energy term [62, 63, 78]. Being proportional to the radius of the S^1, such contribution can be neglected in a Cardy-like limit evaluation, where a small radius limit is taken from a geometric perspective, dimensionally reducing the theory on the three-sphere.

$$+(N-1)\Bigg(\log(p;p)_\infty + \log(q;q)_\infty\Bigg),$$

where the special function θ_0 is given by

$$\theta_0(u;\omega) := (\mathrm{e}^{2\pi\mathrm{i}u};\mathrm{e}^{2\pi\mathrm{i}\omega})_\infty\,(\mathrm{e}^{2\pi\mathrm{i}\omega}\mathrm{e}^{-2\pi\mathrm{i}u};\mathrm{e}^{2\pi\mathrm{i}\omega})_\infty\,. \tag{6.35}$$

and we employed (A.11) in appendix 7.1. The index reduces to a sum of contributions from multiple saddles

$$\mathcal{I} = \sum_{u^*} I_{\mathrm{lead}}(u^*,\Delta,\tau,\sigma)Z_{\mathrm{sub}}, \quad Z_{\mathrm{sub}} := \frac{1}{N!}\int\prod_{i=1}^{N-1}\mathrm{d}\delta u_i I_{\mathrm{sub}}(\delta u) \tag{6.36}$$

in which we can isolate a leading part, at $\mathcal{O}(r^{-2})$, dependent only on Δ and the details of the holonomy saddle u^*, and a subleading term, $\sim\mathcal{O}(r^0)$, $I_{\mathrm{sub}}(\delta u)$ emerging as an effective potential for the matrix model perturbed near u^*. As discussed above, the Cardy-like limit reproduces the dimensional reduction of the theory as the thermal S^1 is sent to zero. For generic non-zero $\tilde{\Delta}$ the whole KK tower of modes for the matter fields becomes massive and gets lifted as $r\to 0$, while a zero-mode survives in the vector multiplet and a gapped 3d pure CS gauge theory emerges. When the size of the S^3 is much larger than the size of the S^1, correlators at two different points on S^3 are exponentially suppressed and in the limit $r\to 0$ the leading order description of the theory is captured only by CS contact terms [134]. In the language of localization the contribution from such contact terms are encoded in I_{lead} in (6.36). The sub-leading contribution Z_{sub} encodes the three-sphere partition function for the topological $\mathrm{SU}(N)_{\pm N}$ theory emerging in the reduction.

The saddle points configurations satisfy the saddle point equations

$$\frac{\partial S_{4d}}{\partial u_i} \overset{!}{=} 0 \qquad i=1,\dots,N-1. \tag{6.37}$$

While in principle one needs to solve the saddle point equations derived from (6.34), it is more convenient to compute the leading order saddle-point equations from the leading-order effective action, as for $r\to 0$ the saddles will converge to the leading order ones as discussed in [201].

Employing modular properties of the elliptic gamma functions, Jacobi functions and q-Pochhammer symbols presented in appendix 7.1, we can derive the complete expansion of the effective action in r from (6.34) up to exponentially suppressed terms. The leading order term $\sim\mathcal{O}(r^{-2})$ receives contributions only from matter fields

$$S_{\mathrm{lead}} = -\frac{\pi\mathrm{i}}{3\tau\sigma}\left((N-1)\sum_{a=1}^{3}B_3\left(\{\Delta_a\}\right) + \sum_{a=1}^{3}\sum_{i\neq j}^{N}B_3\left(\{u_{ij}+\Delta_a\}\right)\right), \tag{6.38}$$

where we define $\{x\} = \{\tilde{x}\} + r\bar{x} \equiv \tilde{x} - \lfloor\tilde{x}\rfloor + r\bar{x}$ for any x with $\tilde{x}\neq 0$. A set of solutions to the saddle point equations

$$\frac{\partial S_{4d}}{\partial u_i} = \sum_{a=1}^{3}\sum_{j=1}^{N}\Big(B_2(\{u_{ij}+\Delta_a\}) \,-\, B_2(\{-u_{ij}+\Delta_a\}) \\ + B_2(\{-u_{Nj}+\Delta_a\}) \,-\, B_2(\{u_{Nj}+\Delta_a\})\Big) = 0 \tag{6.39}$$

is the so-called family of C-center solutions [54, 126], organized accordingly to the presence of a discrete one-form symmetry, namely the center symmetry $\mathbb{Z}_N$, and its subgroups [54, 134]

$$u_j = \frac{m}{N} + \frac{\left\lfloor \frac{j-1}{N/C} \right\rfloor - \frac{C-1}{2}}{C} \qquad j = 1, \ldots, N, \tag{6.40}$$

where $m = 0, \ldots \frac{N}{C} - 1$ and C a divisor of N.
In the following we will be interested only in the $C = 1$ case, corresponding to the saddle point reproducing the black hole entropy. For $C = 1$, we have N holonomy configurations with all the holonomies packed at $u_i = \frac{m}{N}$ for fixed $m = 0, \ldots, N-1$, contributing the same to the index in (6.28). The $\log N$ degeneracy arising from these saddles in the entropy function ($\sim \log \mathcal{I}$) is due to the index being insensitive to the presence of global properties of the gauge group, namely it is unable to detect the action of the $\mathbb{Z}_N$ center symmetry, mapping different saddles into each other. Expanding the effective action near vanishing holonomies with ansatz

$$u_i = r\lambda_i, \qquad u_i \in \left[-\frac{1}{2}, \frac{1}{2}\right) \tag{6.41}$$

the gauge terms and Pochhammer symbols combines to produce the measure for a three-sphere partition function in the reduction

$$\mathcal{I} = \frac{(p;p)_\infty^{N-1}(q;q)_\infty^{N-1}}{N!}\int \prod_{i=1}^{N-1} \mathrm{d}u_i \prod_{i<j}\theta_0(u_{ij})\theta_0(-u_{ij})\ldots \underset{r\to 0}{\sim} \\ \sim \frac{\mathrm{e}^{-\frac{\pi \mathrm{i}(\omega_1+\omega_2)(N^2-1)}{12 r\omega_1\omega_2}}}{N!}\int \prod_{i=1}^{N-1}\frac{\mathrm{d}\lambda_i}{\sqrt{-\omega_1\omega_2}}\frac{1}{\prod_{i<j}\Gamma_h(\lambda_{ij})\Gamma_h(-\lambda_{ij})}\cdots \tag{6.42}$$

where the dots stand for the matter content, whose contribution depends on the details of the reduction. Parameterizing the complex chemical potentials Δ_a as in equation (6.32) the matter terms contribute as

$$\exp\left(2\pi\mathrm{i}\left((N-1)\sum_{a=1}^{3} Q\left(\{\Delta_a\}\right) + \sum_{a=1}^{3}\sum_{i\neq j}^{N} Q\left(\{r\lambda_{ij}+\Delta_a\}\right)\right) + \mathcal{O}\left(\mathrm{e}^{-\frac{1}{r}}\right)\right) \tag{6.43}$$

which can be expanded for small r, generating a leading term $\sim r^{-2}$ related to central charges a, c of the theory and a quadratic Chern-Simons term in the holonomies of order

$\mathcal{O}(1)$ in r

$$\exp\left(2\pi\mathrm{i}\left((N^2-1)\sum_{a=1}^{3}Q\left(\{\Delta_a\}\right)+r^2\left(\sum_{a=1}^{3}Q''\left(\{\Delta_a\}\right)\right)\sum_{i<j}^{N}(\lambda_i-\lambda_j)^2\right)+\ldots\right), \tag{6.44}$$

where $Q''(x)$ denotes the second derivative of $Q(x)$ with respects to its argument and the dots stands for negligible terms in the Cardy-like limit at most of linear order in r. By virtue of the reality of the adjoint representation, the generation of an FI term linear in the gauge holonomies is prevented.

Upon employing the constraint

$$\sum_{a=1}^{3}\{\Delta_a\}=\tau+\sigma+\frac{3+n_0}{2},\quad n_0=\pm 1, \tag{6.45}$$

which follows from the constraint (6.25), and the definition of $Q(\{\Delta_a\})$

$$\begin{aligned} Q(\{\Delta_a\}) = & -\frac{B_3(\{\Delta_a\})}{6\,\sigma\tau}+B_2(\{\Delta_a\})\frac{(\sigma+\tau)}{4\,\sigma\tau}-B_1(\{\Delta_a\})\frac{\left((\sigma+\tau)^2+\sigma\tau\right)}{12\,\sigma\tau} \\ & +\frac{\sigma}{24}+\frac{\tau}{24}, \end{aligned} \tag{6.46}$$

we get

$$\begin{aligned} \exp\Bigg(& -\frac{\pi\mathrm{i}(N^2-1)}{r^2\omega_1\omega_2}\prod_{a=1}^{3}\left(\{\Delta_a\}-\frac{n_0+1}{2}\right)-\frac{\pi\mathrm{i}n_0N}{\omega_1\omega_2}\sum_{i=1}^{N}\lambda_i^2 \\ & +\frac{\pi\mathrm{i}(\omega_1+\omega_2)(N^2-1)}{12r\omega_1\omega_2}-\frac{\pi\mathrm{i}n_0(N^2-1)(\omega_1^2+\omega_2^2+3\omega_1\omega_2)}{12\omega_1\omega_2}\ldots\Bigg), \end{aligned} \tag{6.47}$$

All in all, the index becomes

$$\begin{aligned} \mathcal{I}=\exp & \left(-\frac{\pi\mathrm{i}(N^2-1)}{\sigma\tau}\prod_{a=1}^{3}\left(\{\Delta_a\}-\frac{n_0+1}{2}\right)-\frac{\pi\mathrm{i}n_0(N^2-1)(\omega_1^2+\omega_2^2+3\omega_1\omega_2)}{12\omega_1\omega_2}\right) \\ & \times\frac{1}{N!}\int\prod_{i=1}^{N-1}\frac{\mathrm{d}\lambda_i}{\sqrt{-\omega_1\omega_2}}\frac{\mathrm{e}^{-\frac{\pi\mathrm{i}n_0N}{\omega_1\omega_2}\sum_{i=1}^{N}\lambda_i^2}}{\prod_{i<j}\Gamma_h(\lambda_{ij})\Gamma_h(-\lambda_{ij})}, \end{aligned} \tag{6.48}$$

consistently with [57], where the domain of integration of each $\lambda_i=\frac{u_i}{r}$ runs over $(-\infty,+\infty)$ as S^1 shrinks to 0.

The partition function for a pure CS theory on a squashed three-sphere background can be evaluated exactly in terms of the constrained $\mathrm{U}(N)$ CS partition function

$$\frac{1}{N!}\int\mathrm{d}\Lambda\int\prod_{i=1}^{N}\frac{\mathrm{d}\lambda_i}{\sqrt{-\omega_1\omega_2}}\frac{\mathrm{e}^{-\frac{\pi\mathrm{i}n_0N}{\omega_1\omega_2}\sum_{i=1}^{N}\lambda_i^2+2\pi\mathrm{i}\Lambda\sum_{j=1}^{N}\lambda_j}}{\prod_{i<j}\Gamma_h(\lambda_{ij})\Gamma_h(-\lambda_{ij})} \tag{6.49}$$

giving

$$\mathcal{Z}^{S^3}_{\mathrm{SU}(N)_{n_0 N}} = \mathcal{Z}^{S^3}_{\mathrm{U}(N)_{n_0 N}} \sqrt{-\mathrm{i}n_0} = \mathrm{e}^{\frac{\pi \mathrm{i} n_0 \left(N^2-1\right)\left(\omega_1^2+\omega_2^2+3\omega_1\omega_2\right)}{12\omega_1\omega_2}} . \tag{6.50}$$

Therefore, taking into account the N degeneracy of the 1-center saddles due to the action of the $\mathbb{Z}_N$-center symmetry, the final contribution to the index is

$$\mathcal{I} = N \exp\left(-\frac{\pi \mathrm{i}(N^2-1)}{\sigma\tau} \prod_{a=1}^{3} \left(\{\Delta_a\} - \frac{n_0+1}{2} \right) + \mathcal{O}(r) \right) . \tag{6.51}$$

6.3.3 Adding the Defect

The insertion of a maximal Gukov-Witten defect amounts, in the SCI, to couple the 2d theory with the 2d model (6.20) as described in Section 6.1. The Cartan generators of the half-BPS algebra can be identified with the ones in 2d and by comparing (6.13) with (6.21). The dictionary between the fugacities is the following [139]

$$\sigma = \tau_{2d}, \quad \tau = \frac{\tau_{2d}}{2} - u_C, \quad \Delta_1 = \frac{\tau_{2d}}{2} + 2\chi - u_C, \quad \Delta_2 = \frac{\tau_{2d}}{2} + z - 2\chi, \tag{6.52}$$

where $\sigma, \tau, \Delta_1, \Delta_2$ are the usual fugacities for $\mathcal{N} = 4$. In a Cardy-like limit approach for the evaluation of the index we notice that the insertion of such a defect modifies the original effective action with an order $\mathcal{O}\left(\frac{1}{r}\right)$ term

$$\mathcal{I} = \frac{1}{N!} \sum_{i=1}^{N} \int \prod_{j=1}^{N-1} \mathrm{d}u_j \mathrm{e}^{S_{4d}(u,\Delta,\tau,\sigma) + S_{2\mathrm{d},i}(u,\Delta,\tau,\sigma)}, \tag{6.53}$$

where

$$S_{2d,i}(u,\Delta,\tau,\sigma) = \sum_{j\neq i}^{N} \frac{\log\theta_0(-u_{ij} - \Delta_2 + \sigma;\sigma)}{\log\theta_0(-u_{ij} + \Delta_1 - \tau;\sigma)} + \frac{\log\theta_0(u_{ij} + \Delta_3;\sigma)}{\log\theta_0(u_{ij};\sigma)}, \tag{6.54}$$

giving rise to subleading corrections to the contribution of the 4d theory and crucially leaving the saddle-point equations (6.39) unaffected. Therefore, we can still identify the combined black hole/probe D3-brane system in the gravitational theory with the holonomy saddle, associated to the sole black hole solution in the unperturbed 4d theory. This is perfectly consistent with the holographic dual picture, in which a probe regime for the backreaction of the D3-brane on the black hole background is considered. In this regime, the backreaction effects are negligible and the insertion of a probe D3-brane does not spoil the underlying black hole geometry. More generally, this can be extended to regime of charges beyond the black hole one, where other gravitational saddles are expected to provide a dominant contribution to the gravitational path integral. When the unperturbed dual 4d theory is considered, their behavior is described by the other $C-$center solutions to the saddle points equations or equivalently, in a Bethe-Ansatz approach for the evaluation of the index, by the contributions arising from the Hong-Liu solutions to the Bethe-Ansatz equations [221, 227]. The insertion of a probe D3-brane to such gravitational solutions does not spoil the background geometry and the identification between C-center saddle points, in the dual theory, and the combined gravitational system still holds. In section 6.4 we will discuss how the probe regime manifests in the Bethe-Ansatz approach.

Let us evaluate the effective action (6.54) in the Cardy-like limit near the vanishing holonomies configuration defined in (6.41). By employing the asymptotic expansion for $\theta_0(u;\sigma)$, listed in Appendix 7.1, we get

$$\begin{aligned} S_{2d,i} \sim &+ \frac{\pi \mathrm{i}(N-1)\left(-2(\tau+\sigma)\{\Delta_1\} + \sigma\sum_{a=1}^{3}\{\Delta_a\} + \tau\right)}{\sigma} \\ &+ \frac{\pi \mathrm{i}(N-1)\left(2\{\Delta_1\}(\{\Delta_1\}-1) - \sum_{a=1}^{3}\{\Delta_a\}(\{\Delta_a\}-1)\right)}{\sigma} \\ &+ \frac{\pi \mathrm{i}(N-1)\tau(\tau+\sigma)}{\sigma} - \frac{2\pi \mathrm{i}(N-1)\left(\sum_{a=1}^{3}\{\Delta_a\} - 1 - \tau - \sigma\right)}{\sigma}\sum_{j\neq i}^{N}\frac{u_{ij}}{N-1} \\ &- \sum_{j\neq i}\Bigg(\log\left(1-e^{-\frac{2\pi \mathrm{i}}{\sigma}u_{ij}}\right)\left(1-e^{-\frac{2\pi \mathrm{i}}{\sigma}(1-u_{ij})}\right) \\ &+ \log\left(1-\mathrm{e}^{-\frac{2\pi \mathrm{i}}{\sigma}(\sigma-u_{ij}+1-\{\Delta_2\})}\right)\left(1-\mathrm{e}^{-\frac{2\pi \mathrm{i}}{\sigma}(\sigma-u_{ij}+\{\Delta_2\})}\right) \\ &- \log\left(1-\mathrm{e}^{-\frac{2\pi \mathrm{i}}{\sigma}(\{\Delta_1\}-u_{ij}-\tau)}\right)\left(1-\mathrm{e}^{-\frac{2\pi \mathrm{i}}{\sigma}(1-\{\Delta_1\}-u_{ij}-\tau)}\right) \\ &+ \log\left(1-\mathrm{e}^{-\frac{2\pi \mathrm{i}}{\sigma}(u_{ij}\{\Delta_3\})}\right)\left(1-\mathrm{e}^{-\frac{2\pi \mathrm{i}}{\sigma}(1-u_{ij}-\{\Delta_3\})}\right)\Bigg). \end{aligned} \tag{6.55}$$

For general values of $\Delta_a = \tilde{\Delta}_a + r\bar{\Delta}_a$ all the logarithmic terms but one are suppressed in the Cardy-like limit, assuming $\tilde{\Delta}_a \neq 0$. Upon constraining the chemical potentials according to eq. (6.25) and employing the SU(N) constraint (6.29) we can rewrite the effective action as

$$\begin{aligned} S_{2d,i} =& \frac{2\pi \mathrm{i}(N-1)}{\sigma}\prod_{a=2}^{3}(\{\Delta_a\}-n) + \pi \mathrm{i}n(N-1) - 2\pi \mathrm{i}nN\frac{\lambda_i}{\omega_2} \\ &- \sum_{j\neq i}\log\left(1-\mathrm{e}^{-2\pi \mathrm{i}\frac{\lambda_{ij}}{\omega_2}}\right), \end{aligned} \tag{6.56}$$

where we defined $n = \frac{1+n_0}{2}$.

Before moving on, we would like to compare our result with the one in the recent paper [128]. In that paper the authors derived the Cardy-like expansion of the 2d system by introducing a regulator in the asymptotic expansion of $\theta_0(u_{ij};\sigma)$ in terms of polylogarithms

$$\log\theta_0(u_{ij};\sigma) = \frac{1}{2\pi \mathrm{i}\sigma}\sum_{r=0}^{\infty}(-1)^r\frac{(2\pi \mathrm{i}\sigma)^r}{r!}\left(B_r(1-z_{ij})\mathrm{Li}_{2-r}(\mathrm{e}^{2\pi \mathrm{i}\varepsilon}) + B_r(z_{ij})\mathrm{Li}_{2-r}(\mathrm{e}^{-2\pi \mathrm{i}\varepsilon})\right), \tag{6.57}$$

so to have a well-defined expression during the manipulations. As far as the leading behavior of the 2d system is concerned, this is a perfectly fine choice, as the regulator does not spoil the leading $\mathcal{O}\left(\frac{1}{r}\right)$ terms in the effective action and indeed we find perfect agreement between their and our result, up to this order. The effect of the regulator only shows up at finite order in r, altering subleading corrections arising from the 2d model and thus, possibly preventing a clear understanding of corrections to the large N limit and an EFT interpretation of the Cardy-like limit approach. The net effect of the regulator is to suppress the logarithmic finite contribution arising from the asymptotic expansion of $\log \theta_0(u_{ij};\sigma)$. To properly retrieve subleading effects in this approach, one would need to properly implement the regulator also in the 2d theory and include extra $\mathcal{O}(\sigma^0)$ effects arising from the vector multiplet of the 2d matrix model, before sending it to zero.

As long as a probe regime for the backreaction of the D3-brane is considered in the dual theory, we expect the contribution arising from the reduction of the 2d defect not being able to alter the effective 3d theory arising from the reduction of the sole 2d $\mathcal{N}=4$ SYM. In the language of localization this translates into obtaining a matrix model for the gauge holonomies, expanded near the saddle point, associated to a pure CS partition function as in (6.48). As a consequence, subleading corrections to the index in the probe limit from the 2d system can arise only from the expansion $\Delta = \tilde{\Delta} + r\bar{\Delta}$ in

$$\frac{2\pi\mathrm{i}(N-1)}{\sigma}\prod_{a=2}^{3}(\{\Delta_a\}-n), \tag{6.58}$$

as we will explicitly show below. Plugging (6.56) into (6.53) we get

$$\begin{aligned}\mathcal{I} =& N\exp\Bigg(-\frac{\pi\mathrm{i}(N^2-1)}{\sigma\tau}\prod_{a=1}^{3}(\{\Delta_a\}-n)+\frac{2\pi\mathrm{i}(N-1)}{\sigma}\prod_{a=2}^{3}(\{\Delta_a\}-n)\\ &-\frac{\pi\mathrm{i}n_0(N^2-1)(\omega_1^2+\omega_2^2+3\omega_1\omega_2)}{12\omega_1\omega_2}\Bigg)\\ &\times\frac{1}{N!}\int \mathrm{d}\Lambda\int\prod_{i=1}^{N}\frac{\mathrm{d}\lambda_i}{\sqrt{-\omega_1\omega_2}}\frac{\mathrm{e}^{-\frac{\pi\mathrm{i}n_0N}{\omega_1\omega_2}\sum_{i=1}^{N}\lambda_i^2+2\pi\mathrm{i}\Lambda\sum_{j=1}^{N}\lambda_j}}{\prod_{i<j}\Gamma_h(\lambda_{ij})\Gamma_h(-\lambda_{ij})}\sum_{i=1}^{N}\frac{\mathrm{e}^{-2\pi\mathrm{i}nN\frac{\lambda_i}{\omega_2}+\pi\mathrm{i}n(N-1)}}{\prod_{j\neq i}^{N}\left(1-\mathrm{e}^{-2\pi\mathrm{i}\frac{\lambda_{ij}}{\omega_2}}\right)}.\end{aligned} \tag{6.59}$$

We see that the defect deforms the original CS partition function with an extra term. However, this deformation is only apparent. In fact, let us consider

$$\sum_{i=1}^{N}\frac{\mathrm{e}^{-2\pi\mathrm{i}nN\frac{\lambda_i}{\omega_2}+\pi\mathrm{i}n(N-1)}}{\prod_{j\neq i}^{N}\left(1-\mathrm{e}^{-2\pi\mathrm{i}\frac{\lambda_{ij}}{\omega_2}}\right)} \tag{6.60}$$

and define $z_j = \mathrm{e}^{-2\pi\mathrm{i}\frac{\lambda_j}{\omega_2}}$. Focusing first on the case $n = 0$, eq. (6.60) can be rewritten as

$$\sum_{i=1}^{N}\prod_{j\neq i}^{N}\frac{z_j}{(z_j - z_i)} = \left(\sum_{i=1}^{N}(-1)^{N-i}\prod_{j\neq i}^{N} z_j \prod_{\substack{1\leq k<l\leq N\\ k,l\neq i}}(z_k - z_l)\right)\left(\prod_{1\leq i<j\leq N}(z_i - z_j)\right)^{-1}. \tag{6.61}$$

The last product is simply the Vandermonde determinant up to a sign due to the reordering of all the columns, while the first parenthesis can be rewritten in terms of a sum of monomials of degree $N(N-1)/2$ of the form

$$\sum_{i=1}^{N}(-1)^{N-i}\prod_{j\neq i}^{N} z_j \prod_{\substack{1\leq k<l\leq N\\ k,l\neq i}}(z_k - z_l) = \sum_{i=1}^{N}(-1)^{N-i}\sum_{\sigma\in S_{N-1}}\mathrm{sign}(\sigma)\prod_{k=1}^{N-1} z^{N-i}_{\sigma(\mathfrak{I}^i_k)}, \tag{6.62}$$

where $\mathfrak{I}^i = \{1, 2, \ldots, i-1, i+1, \ldots, N\}$ and $\mathfrak{I}^i_k$ is the k-th element of such set. Written in this way, expression (6.62) is simply the Laplace expansion of the Vandermonde determinant with an extra sign due to the very same reordering of columns already mentioned in the denominator. Similarly, for $n = 1$ eq. (6.60) can be rewritten as, constraining $\prod_{j=1}^{N} z_j = 1$,

$$\mathrm{e}^{\pi\mathrm{i}(N-1)}\sum_{i=1}^{N}\frac{z_i^{N-1}}{\prod_{j\neq i}^{N}(z_j - z_i)} = 1. \tag{6.63}$$

where we used the $\mathrm{SU}(N)$ constraint $\prod_{j=1}^{N} z_j = 1$. Then, the final expression for the index of the 4d-2d combined system in the Cardy-like limit is

$$\mathcal{I} = N\exp\left(-\frac{\pi\mathrm{i}(N^2-1)}{\sigma\tau}\prod_{a=1}^{3}(\{\Delta_a\} - n) + \frac{2\pi\mathrm{i}(N-1)}{\sigma}\prod_{a=2}^{3}(\{\Delta_a\} - n)\right). \tag{6.64}$$

In section 6.4 we will see that this result perfectly agrees with the Bethe-Ansatz evaluation of the index in the case of collinear angular momenta $\tau = \sigma$.

6.3.4 EFT Interpretation

In [139] an EFT interpretation, along the lines of [55, 134], was given for the case of equal charges at leading order in N. Such an interpretation uses the $\mathrm{U}(1)$ gauge theory formulation of the defect to reconstruct its contribution to the entropy function in terms of 2d anomalies. At sub-leading order, in the absence of the defect, a purely topological gapped CS gauge theory emerges for the massless modes in the Kaluza-Klein reduction [55, 134]. This contirbution arises in the Cardy-like limit as a 3d supersymmetric partition function on the squashed S^3 [25, 26, 201]. In order to complete this EFT interpretation in presence of the surface defect, it is necessary to include its effect in the 3d topological theory. Here we have proved that the CS partition function (6.49) is left unchanged by the addition of the surface defect. In the following we will give an EFT interpratiotion of this result.

The reduction of a GW defect wrapping the temporal S^1 produces a line defect in the effective 3d pure CS theory. Let us discuss this feature by temporarily neglecting the denominator $\prod_{j\neq i}^{N}(z_j - z_i)$ in (6.63), coming from the contribution $-\log\theta_0(u_{ij};\sigma)$ in (6.54). We see that, when $n = 1$, the index receives the following sub-leading contribution

$$\frac{\mathrm{e}^{\pi\mathrm{i}(N-1)}}{N!}\int \mathrm{d}\Lambda \int \prod_{i=1}^{N} \frac{\mathrm{d}\lambda_i}{\sqrt{-\omega_1\omega_2}} \frac{\mathrm{e}^{-\frac{\pi\mathrm{i}n_0 N}{\omega_1\omega_2}\sum_{i=1}^{N}\lambda_i^2 + 2\pi\mathrm{i}\Lambda\sum_{j=1}^{N}\lambda_j}}{\prod_{i<j}\Gamma_h(\lambda_{ij})\Gamma_h(-\lambda_{ij})} \left(\sum_{i=1}^{N}\mathrm{e}^{-2\pi\mathrm{i}N\frac{\lambda_i}{\omega_2}}\right) \tag{6.65}$$

which defines an N-wounded anti-fundamental Wilson loop insertion in the partition function of a pure CS theory on a squashed three-sphere with (analytically-continued) squashing parameters ω_1, ω_2. More precisely, for purely imaginary squashing parameters $\omega_1 = \mathrm{i}b$ and $\omega_2 = \mathrm{i}b^{-1}$, we have

$$\left(\sum_{i=1}^{N}\mathrm{e}^{-2\pi N b^{-1}\lambda_i}\right), \tag{6.66}$$

which defines the insertion of a the Wilson loop

$$W_\gamma(\lambda) = \mathrm{Tr}_R \exp\left(\lambda \oint |\dot{x}|\mathrm{d}s\right), \tag{6.67}$$

with lenght

$$\oint \mathrm{d}s = 2\pi N b^{-1}. \tag{6.68}$$

Defining the ellipsoid metric on S^3 as

$$ds^2 = b^2(dx_0^2 + dx_1^2) + \tilde{b}^{-2}(dx_2^2 + dx_3^2), \tag{6.69}$$

with

$$x_0 = \cos\theta\cos\phi, \quad x_1 = \cos\theta\sin\phi, \quad x_2 = \sin\theta\cos\chi, \quad x_3 = \sin\theta\sin\chi, \tag{6.70}$$

equation (6.66) describes a $1/2$ BPS N-wounded Wilson loop wrapping the 1-cycle at fixed χ on the T^2 in S^3, parametrized by χ, ϕ coordinates at $\theta = \pi/2$, as discussed in [302]. The appearance of an exactly N-wounded Wilson loop from the reduction of a Gukov-Witten surface is crucial for a couple of reasons. Firstly, in a pure CS theory at level-N only the expectation values of pN-wounded (anti-)fundamental Wilson loops, with $p \in \mathbb{Z}$, are non-vanishing. This can be seen in the case of collinear angular momenta, for which $\tau = \sigma$ and $b = 1$. In this case the insertion of a pN-wounded Wilson loop in the partition function of a pure CS theory gives

$$\mathrm{e}^{\mathrm{i}\pi(N-1)} Z^{\mathrm{SU}(N)_{\pm N}}_{W_{-N}} = N Z^{\mathrm{SU}(N)_{\pm N}}_{CS}. \tag{6.71}$$

A derivation of this result is presented in Appendix 7.2. Secondly, we expect a symmetry between the $n = 0$ case, where the Wilson loop is not present, and the $n = 1$ case. In fact, the 4d index cannot detect global properties of the gauge groups and thus it is insensitive with respect to the action of the $\mathbb{Z}_N$ center symmetry. In addition, the insertion of a maximal Gukov-Witten defect introduces sub-leading corrections and thus it cannot alter this property, as discussed before for the saddle-point equations. Therefore, only an $N-$wounded Wilson loop is consistent with the case $n = 0$, being uncharged under the $\mathbb{Z}_N$ symmetry. Let us now reintroduce and discuss the term $\prod_{j\neq i}(z_j - z_i)$. As discussed before, the holographic counterpart of a maximal Gukov-Witten defect is described by a D3-brane in the probe limit. For this reason we expect the contribution of the defect not being able to alter the EFT emerging from the reduction of the theory along the thermal S^1. This manifests in the presence of

$$-\log\theta_0(u_{ij};\sigma) \tag{6.72}$$

in the 2d model describing the defect, which can be interpreted as a counter-term suppressing the effects of the Wilson loop emerging in the effective 3d pure CS theory. It would be interesting to study the fate of the counter-term for other GW defects in regimes where backreaction effects of the probe D3-brane are not necessarily negligible.

6.4 Bethe Ansatz Approach

Motivated by the results just obtained, in this section we provide a derivation of the index in presence of the maximal GW defect using the BA approach. This technique was originally used in [89], following the derivation of [152], in order to provide a derivation of the black-hole entropy at large N beyond the Cardy-like regime. The result has been shown to be in perfect agreement with the one found by saddle-point approximation in the Cardy-like limit [54, 201]. We show that the agreement survives also in the presence of the maximal GW surface defect.

6.4.1 The Bethe-Ansatz Formula

Here we review the BA formula [88] in the context of 4d $\mathcal{N} = 4$ SYM theory [89], for equal angular momenta

$$\tau = \sigma \equiv \omega. \tag{6.73}$$

We start by rewriting the SCI (6.28) in a more convenient way for the forthcoming discussion

$$\mathcal{I} = \kappa_N \oint_{\mathcal{R}} \mathrm{d}^{N-1}u\, \mathcal{Z}_{4d}(u;\omega,\Delta)\,, \tag{6.74}$$

where the integral is taken over the region

$$\mathcal{R} = \left\{(u_1,\ldots,u_{N-1}) \in \mathbb{C}^{N-1} \mid \mathrm{Re}\, u_i \in [0,1]\ , \mathrm{Im}\, u_i = 0\,, \forall\, i = 1,\ldots,N-1\right\} \tag{6.75}$$

and the gauge holonomies are constrained by (6.29). Then the prefactor κ_N is given by

$$\kappa_N = \frac{1}{N!}\left(\frac{(\mathrm{e}^{2\pi\mathrm{i}\omega};\mathrm{e}^{2\pi\mathrm{i}\omega})_\infty^2\,\widetilde{\Gamma}(\Delta_1;\omega,\omega)\,\widetilde{\Gamma}(\Delta_2;\omega,\omega)}{\widetilde{\Gamma}(\Delta_1+\Delta_2;\omega,\omega)}\right)^{N-1}\,, \tag{6.76}$$

with the usual definitions of the q-Pochhammer symbol (6.27) and of the elliptic gamma function (6.26). The integrand in (6.74)

$$\mathcal{Z}_{4d}(u;\omega,\Delta) = \prod_{i=1}^{N} \prod_{i\neq j=1}^{N} \frac{\widetilde{\Gamma}(u_{ij}+\Delta_1;\omega,\omega)\,\widetilde{\Gamma}(u_{ij}+\Delta_2;\omega,\omega)}{\widetilde{\Gamma}(u_{ij}+\Delta_1+\Delta_2;\omega,\omega)\,\widetilde{\Gamma}(u_{ij};\omega,\omega)}\,. \tag{6.77}$$

It is important to stress that, in order to have a plethystic expansion of the elliptic functions, we need to restrict to a certain region of chemical potentials [88]

$$\mathcal{B} = \{\omega, \Delta \in \mathbb{C}\,|\, 0 < \operatorname{Im}\Delta < 2\operatorname{Im}\omega\}\ , \tag{6.78}$$

with $\Delta = \Delta_1,\ \Delta_2,\ \Delta_1+\Delta_2$. Then, once we have computed the index, we can eventually analytically continue the result outside $\mathcal{B}$.
The BA operators Q_i are defined as

$$Q_i(u;\omega,\Delta) := \mathrm{e}^{2\pi\mathrm{i}(\lambda+3\sum_j u_{ij})} \prod_{j=1}^{N} \frac{\theta_0(u_{ji}+\Delta_1;\omega)\,\theta_0(u_{ji}+\Delta_2;\omega)\,\theta_0(u_{ji}-\Delta_1-\Delta_2;\omega)}{\theta_0(u_{ij}+\Delta_1;\omega)\,\theta_0(u_{ij}+\Delta_2;\omega)\,\theta_0(u_{ij}-\Delta_1-\Delta_2;\omega)}\,, \tag{6.79}$$

where $i = 1,\ldots,N$ and the function θ_0 is defined in (6.35). These operators, written for $\mathrm{U}(N)$ gauge symmetry, are restricted to the case of $\mathrm{SU}(N)$ by the action of the "Lagrange multiplier" λ. A crucial property of Q_i is that they shift the integrand in (6.74) as

$$\mathcal{Z}_{4d}(u-\delta_i\omega) = \frac{Q_i}{Q_N}\,\mathcal{Z}_{4d}(u)\,, \qquad \forall\, i = 1,\ldots,N-1\,, \tag{6.80}$$

where

$$u-\delta_i\omega = (u_1,\ldots,u_i-\omega,\ldots,u_{N-1},u_N+\omega)\,. \tag{6.81}$$

Moreover, these operators are doubly periodic, i.e. they are invariant under the shifts

$$u_i \mapsto u_i + m + n\omega\,, \quad \forall\, m,n\in\mathbb{Z}\,, \quad \forall\, i=1,\ldots,N-1\,. \tag{6.82}$$

As we mentioned above, we can use (6.80) to rewrite the integral representation (6.74) as

$$\mathcal{I} = \kappa_N \oint_{\mathcal{C}} \mathrm{d}^{N-1}u\, \frac{\mathcal{Z}_{4d}(u;\omega,\Delta)}{\prod_{i=1}^{N}\left(1-Q_i(u;\omega,\Delta)\right)}\,, \tag{6.83}$$

where now we are integrating over the contour $\mathcal{C}$ encircling the region

$$\mathcal{A} = \left\{u\in\mathbb{C}^{N-1} |\, \operatorname{Re}u_i \in [0,1]\ , -\operatorname{Im}\omega < \operatorname{Im}u_i < 0, \forall\, i{=}1,\ldots,N{-}1\right\}. \tag{6.84}$$

At this point we can apply the residue theorem and recognize in the zeros of the denominator the only poles that really contribute. In fact, a priori we should also consider those poles that come from $\mathcal{Z}_{4d}$. However, it turns out that, for each pole coming from the gamma functions inside $\mathcal{Z}_{4d}$, either there is a zero of the denominator of some Q_i with higher multiplicity (thus canceling the pole), or such pole is outside $\mathcal{A}$ in (6.84) and thus cannot contribute. Then the poles are obtained by the solutions of the set of trascendental equations

$$Q_i(u;\omega,\Delta)=1\,,\quad \forall\, i=1,\dots,N\,, \tag{6.85}$$

the so called Bethe Ansatz Equations (BAEs).
We are almost ready to write the BA formula but first we need to clarify two aspects. Firstly, due to the double periodicity of the operators Q_i, we can solve the BAEs on $N-1$ copies of the complex torus with modular parameter ω. This means that the solutions can be grouped into a finite number of equivalence classes $[\hat{u}_i]$ such that

$$\hat{u}_i\sim\hat{u}_i+1\sim\hat{u}_i+\omega\,,\quad \forall\, i=1,\dots,N\,,\qquad \sum_{i=1}^{N}\hat{u}_i=0\mod(\mathbb{Z}+\omega\mathbb{Z})\,. \tag{6.86}$$

Secondly, as discussed in the Appendix C of [88], among all the solutions of the BAEs, there is the subset of all those solutions that are fixed by a non-trivial element of the Weyl group of $\mathrm{SU}(N)$. It turns out that the integrand function $\mathcal{Z}_{4d}$ is such that the contributions from this subset sum up to zero and thus can be discarded. These two clarifications bring us to define the set

$$\mathcal{M}_{\text{BAE}}\coloneqq\{[\hat{u}]\in\mathcal{A}\mid Q_i([\hat{u}];\omega,\Delta)=1\,,w\cdot[\hat{u}]\neq[\hat{u}]\,,\ \forall\, i=1,\dots,N\,,\ \forall\, w\in S_N\} \tag{6.87}$$

and finally the BA formula is given by

$$\mathcal{I}=\kappa_N\sum_{\hat{u}\in\mathcal{M}_{\text{BAE}}}\mathcal{Z}_{4d}(\hat{u};\omega,\Delta)\,H(\hat{u};\omega,\Delta)^{-1}\,, \tag{6.88}$$

where H^{-1} is the inverse of a Jacobian due to the change of variables in the integral

$$H(\hat{u};\omega,\Delta)=\det\left(\frac{1}{2\pi\mathrm{i}}\,\frac{\partial(Q_1,\dots,Q_N)}{\partial(u_1,\dots,u_{N-1},\lambda)}\bigg|_{\hat{u}}\right)\,. \tag{6.89}$$

Unfortunately the full set of solutions of the BAEs (6.85) has not been found yet. However, a subset of solutions is known [89, 221, 227]. Within this subset, one solution, also known as basic solution,

$$\hat{u}_i=\bar{u}-\frac{\omega}{N}\,i\,,\quad \text{with }\bar{u}\text{ such that}\quad \sum_{i=1}^{N}\hat{u}_i=0\mod(\mathbb{Z}+\omega\mathbb{Z})\,, \tag{6.90}$$

reproduces the leading contribution to the index whose logarithm matches with the entropy function of the dual 5d rotating black hole solution in holography. This result was first obtained in [89] and then improved in [201]

$$\begin{aligned}\mathcal{I}\big|_{\text{basic}} &= N N!\, \kappa_N\, \mathcal{Z}_{4d}\, H^{-1}\big|_{\text{basic}} = \\ &= \exp\left(-\frac{\pi \mathrm{i}}{\omega^2}\, N^2 \prod_{a=1}^{3} \left(\{\Delta_a\}_\omega - n \right) + \log N + O(N^0)\right) ,\end{aligned} \tag{6.91}$$

where $n = \frac{1+n_0}{2}$, $n_0 = \pm 1$, the function $\{\,\cdot\,\}_\omega$ is defined as

$$\{\Delta\}_\omega \coloneqq \Delta + m \quad \text{such that} \quad m \in \mathbb{Z} \;\text{ and }\; 0 > \operatorname{Im} \frac{\Delta + m}{\omega} > \operatorname{Im} \frac{1}{\omega} \tag{6.92}$$

and the auxiliary chemical potential Δ_3 in (6.91) such that

$$\{\Delta_1\}_\omega + \{\Delta_2\}_\omega + \{\Delta_3\}_\omega = 2\omega + \frac{3 + n_0}{2}\,. \tag{6.93}$$

We added an extra pre-factor $N \cdot N!$ representing the multiplicity of the basic solution, that can be justified as follows. As we mentioned, we consider only those solutions that are not fixed by any non-trivial element of S_N, but this implies that there is a multiplicity factor $N!$ related to the Weyl group action on each solution. Moreover we observe that, since the BAEs and the index depend only on the differences u_{ij}, we can shift $\bar{u} \mapsto \bar{u}{+}i/N$ into (6.90), with $i = 0, \ldots, N-1$, to obtain a set of N inequivalent solutions giving the same contribution to the index. The shift is chosen in such a way that the constraint (6.86) always holds. However, this implies that there is another multiplicity factor N related to these shifts. This is the reason why the total multiplicity for the basic solution is $N \cdot N!$.

Finally we can analytically continue (6.91) outside the region (6.78), so to extend the result to any $\Delta \in \mathbb{C}$ such that

$$\operatorname{Im} \frac{\Delta}{\omega} \notin \mathbb{Z} \times \operatorname{Im} \frac{1}{\omega}\,, \tag{6.94}$$

because $\{\,\cdot\,\}_\omega$ is not defined on these lines, denoted as Stokes lines in [89].

Before continuing to the next sub-section, we make a further comment on the matching between the entropy function obtained from the Cardy-like limit of the 4d SCI and the BA approach. Such matching extends beyond the functional agreement and it relates the saddle holonomies of (6.40) with the Hong-Liu solutions [221]. A complete discussion of such matching has been done in [54]. We have seen from the saddle-point analysis in sub-section 6.3.3, that the holonomy saddles giving rise to the 5d black hole are not modified in presence of the defect. Here we wonder if a counterpart of this behavior is realized in the BA approach. In the next sub-section we will give an affirmative answer to this question by studying the pole of the BA formula in presence of the defect.

6.4.2 The Bethe-Ansatz Formula in Presence of the Defect

The addition of the GW defect modifies the SCI as follows

$$\mathcal{I} = \int_{\mathcal{R}} \mathrm{d}^{N-1}u\; \mathcal{Z}_{4d}(u;\omega,\Delta)\, \mathcal{Z}_{2d}(u;\omega,\Delta)\,, \tag{6.95}$$

where the 2d contribution is given by

$$\mathcal{Z}_{2d} = \sum_{i=1}^{N} \exp\left(\sum_{i\neq j=1}^{N}\left(\log\frac{\theta_0(-u_{ij}-\Delta_2+\omega;\omega)}{\theta_0(-u_{ij}+\Delta_1-\omega;\omega)} + \log\frac{\theta_0(u_{ij}-(\Delta_1+\Delta_2)+2\omega;\omega)}{\theta_0(u_{ij};\omega)}\right)\right) . \tag{6.96}$$

In light of the discussion at the end of sub-section 6.4.1, we study the feasibility of the BA approach in this new situation by inserting $\mathcal{Z}_{2d}$ in (6.88) as

$$\mathcal{I} \overset{?}{=} \kappa_N \sum_{\hat{u}\in\mathcal{M}_{\text{BAE}}} \mathcal{Z}_{4d}(\hat{u};\omega,\Delta)\, \mathcal{Z}_{2d}(\hat{u};\omega,\Delta)\, H(\hat{u};\omega,\Delta)^{-1} . \tag{6.97}$$

The answer will turn out to be affirmative even if one has to be cautious. The proof of the BA formula guarantees that there is no contributing pole from $\mathcal{Z}_{4d}$ but a priori we cannot be sure that this still holds when we have $\mathcal{Z}_{4d}\,\mathcal{Z}_{2d}$. In fact, the presence of poles in $\mathcal{Z}_{2d}$ within the region $\mathcal{A}$ previously defined in (6.84), could potentially spoil the result. We rewrite the region in (6.84) as

$$\mathcal{A} = \left\{u\in\mathbb{C}^{N-1} \middle| \operatorname{Re} u_i \in [0,1]\ , -\operatorname{Im}\omega < \operatorname{Im} u_{ij} < \operatorname{Im}\omega, \forall\, i,j=1,\dots,N-1\right\} , \tag{6.98}$$

in order to make the next discussion more intuitive.
Recall that, according to [252], the zeros of the θ_0 functions are given by

$$\theta_0(u;\omega) = 0 \qquad \Longleftrightarrow \qquad u = m + n\omega\,, \quad \forall\, m,n\in\mathbb{Z}\,. \tag{6.99}$$

Therefore, from the definition (6.97), there is a pole whenever

$$\begin{array}{lll} \text{(A)} & \theta_0(u_{ij};\omega) = 0 & \\ \text{(B)} & \theta_0(-u_{ij}+\Delta_1;\omega) = 0 & \end{array} \quad \Longleftrightarrow \quad \begin{array}{l} u_{ij} = m\omega\,, \\ u_{ij} = \Delta_1 + m\omega \end{array} \quad \forall\, m\in\mathbb{Z}\,. \tag{6.100}$$

By rewriting

$$\prod_{i=1}^{N}\prod_{i\neq j=1}^{N} \frac{1}{\widetilde{\Gamma}(u_{ij};\omega,\omega)} = \prod_{i=1}^{N}\prod_{i\neq j=1}^{N} \theta_0(u_{ij};\omega) \tag{6.101}$$

in $\mathcal{Z}_{4d}$, we see that each pole of type (A) is cancelled by the corresponding zero in $\mathcal{Z}_{4d}$. For poles of type (B), we start by noticing that, since $\Delta_1\in\mathcal{B}$ (6.78), only those poles with $m = 0,-1,-2$ can lie inside the region $\mathcal{A}$. Secondly, we recall that inside Q_j there is a term of the form

$$\frac{1}{\theta_0(u_{ji}+\Delta_1;\omega)}\,, \tag{6.102}$$

hence its denominator is zero whenever

$$u_{ij} = \Delta_1 + n\omega\,, \quad \forall\, n\in\mathbb{Z}\,. \tag{6.103}$$

However, some of these zeros were formerly used to take care of the corresponding poles coming from the gamma function in $\mathcal{Z}_{4d}$. In fact,

$$\widetilde{\Gamma}(u_{ji}+\Delta_1;\omega,\omega)=\prod_{\ell=0}^{\infty}\left(\frac{1-\mathrm{e}^{2\pi\mathrm{i}(\ell+2)\omega}\mathrm{e}^{2\pi\mathrm{i}(u_{ij}-\Delta_1)}}{1-\mathrm{e}^{2\pi\mathrm{i}\ell\omega}\mathrm{e}^{-2\pi\mathrm{i}(u_{ij}-\Delta_1)}}\right)^{\ell+1} \tag{6.104}$$

has a pole of multiplicity $\ell+1$ whenever

$$u_{ij}=\Delta_1+\ell\omega\,,\quad \forall\,\ell\in\mathbb{N}\,. \tag{6.105}$$

Poles with $\ell\neq 0$ are outside $\mathcal{A}$ because $\Delta_1\in\mathcal{B}$ of (6.78). On the other hand, the pole $u_{ij}=\Delta_1$ lies inside $\mathcal{A}$, thus we need the corresponding zero of the denominator of (6.102) with $n=0$ to cancel it. Therefore, on one hand we can use the zeros of the denominator of Q_j with $n=-1,-2$ to cancel the corresponding poles in $\mathcal{Z}_{2d}$ with $m=-1,-2$. On the other, we lack a further zero to cancel the pole with $m=0$.

This is not the end of the story yet. In the proof of the BA formula in absence of the defect, we chose to restrict the integral to the domain $\mathcal{B}$ in order to have a plethystic expansion of the elliptic functions, and only at the end of computation we extended the resulting index outside this domain by analytic continuation. In the same fashion, here we can first restrict the integration on a smaller domain for Δ_1,

$$\mathcal{B}'=\{\Delta_1\in\mathbb{C}\,|\,\operatorname{Im}\omega<\operatorname{Im}\Delta_1<2\operatorname{Im}\omega\}\ , \tag{6.106}$$

such that the pole of $\mathcal{Z}_{2d}$ with $m=0$ pops out of $\mathcal{A}$. This allow us to apply the BA formula because $\mathcal{Z}_{2d}$ will not bring new poles contributing to the integral. Finally, we will extend the result outside $\mathcal{B}'$ by analytic continuation, when possible.

6.4.3 Contribution of the Basic Solution

We are now ready to evaluate the index (6.96) on the basic solution of the BAEs (6.90) in presence of the GW defect. Such computation aims to generalize the result of [139] which is restricted to the case of equal chemical potentials

$$\Delta_1=\Delta_2=\Delta_3=\frac{2\omega-1}{3}. \tag{6.107}$$

Here we will consider the case of arbitrary ω and $\Delta_{1,2,3}$, with the constraint (6.93). As a starting point, we compute the following sum

$$\sum_{i\neq j=1}^{N}\log\theta_0(\pm\hat{u}_{ij}+v;\omega) \tag{6.108}$$

Firstly, by using the modular transformations

$$\theta_0(u;\omega)=\mathrm{e}^{\pi\mathrm{i}B(u;\omega)}\theta_0\left(\frac{u}{\omega};-\frac{1}{\omega}\right)\ ,\quad B(u;\omega)=-\frac{u^2}{\omega}-\frac{u}{\omega}+u-\frac{\omega}{6}-\frac{1}{6\omega}+\frac{1}{2} \tag{6.109}$$

we obtain

$$(6.108) = \pi\mathrm{i} \sum_{i\neq j=1}^{N} B\left(\pm\hat{u}_{ij}+v;\omega\right) + \sum_{i\neq j=1}^{N} \log\theta_0\left(\frac{\pm\hat{u}_{ij}+v}{\omega};-\frac{1}{\omega}\right) \equiv \phi\left(v,\omega\right)+\varphi\left(v,\omega\right)\ . \tag{6.110}$$

Secondly, focusing on $\varphi(v,\omega)$ of the sum above and recalling the definitions (6.27), (6.35), we get the following expression

$$\varphi(v,\omega) = \sum_{i\neq j=1}^{N}\sum_{m=0}^{\infty} \log\left(1-\tilde{w}\,\tilde{h}^{m}\,\left(\frac{\tilde{z}_i}{\tilde{z}_j}\right)^{\pm1}\right)\left(1-\tilde{w}^{-1}\,h^{m+1}\,\left(\frac{\tilde{z}_j}{\tilde{z}_i}\right)^{\pm1}\right)\,, \tag{6.111}$$

where, for compactness, we have defined

$$\tilde{z}_j \equiv \mathrm{e}^{\frac{2\pi\mathrm{i}}{\omega}\hat{u}_j}\,, \qquad \tilde{w}_a \equiv \mathrm{e}^{\frac{2\pi\mathrm{i}}{\omega}v}\,, \qquad \tilde{h} \equiv \mathrm{e}^{-\frac{2\pi\mathrm{i}}{\omega}}\ . \tag{6.112}$$

By the Taylor expansion of the logarithm, we obtain

$$\varphi(v,\omega) = -\sum_{i\neq j=1}^{N}\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\frac{1}{n}\left(\tilde{w}^{n}\,\tilde{h}^{mn}\,\left(\frac{\tilde{z}_i}{\tilde{z}_j}\right)^{\pm n} + \tilde{w}^{-n}\,\tilde{h}^{(m+1)n}\,\left(\frac{\tilde{z}_j}{\tilde{z}_i}\right)^{\pm n}\right)\ . \tag{6.113}$$

Now we evaluate explicitly the following sums on the basic solution (6.90)

$$A_n \equiv \sum_{i\neq j=1}^{N}\left(\frac{\tilde{z}_i}{\tilde{z}_j}\right)^{n}\,, \qquad B_n \equiv \sum_{i\neq j=1}^{N}\left(\frac{\tilde{z}_j}{\tilde{z}_i}\right)^{n}\,, \tag{6.114}$$

which amount to

$$A_n = B_n = \begin{cases} -1 & n\neq 0 \quad \mathrm{mod}\ N \\ N-1 & n=0 \quad \mathrm{mod}\ N\,. \end{cases} \tag{6.115}$$

By summing over m, we can reorganize $\varphi(v,\omega)$ in a simpler form

$$\varphi(v,\omega) = \sum_{n=1}^{\infty}\frac{1}{n}\,\frac{\tilde{w}^{n}+\tilde{w}^{-n}\,\tilde{h}^{n}}{1-\tilde{h}^{n}} - \sum_{n=1}^{\infty}\frac{1}{n}\,\frac{\tilde{w}^{Nn}+\tilde{w}^{-Nn}\,\tilde{h}^{Nn}}{1-\tilde{h}^{Nn}} \tag{6.116}$$

where the first term is convergent if we take

$$|\tilde{h}| < |\tilde{w}| < 1 \qquad \Longleftrightarrow \qquad \mathrm{Im}\,-\frac{1}{\omega} > \mathrm{Im}\,\frac{v}{\omega} > 0\,, \tag{6.117}$$

while the second term, within the region (6.117), vanishes for large N.
The first three terms of $\mathcal{Z}_{2d}$, depending on $\Delta_1, \Delta_2, \Delta_1+\Delta_2$ respectively, have the same form of (6.108) and thus we use the result (6.116) to compute them. More precisely, we first apply the modular transformations (6.109) and then, by the quasi-periodicity,

$$\theta_0\left(\frac{u}{\omega};-\frac{1}{\omega}\right) = \mathrm{e}^{\pi\mathrm{i}S(u;\,\omega)}\theta_0\left(\frac{u}{\omega}-\frac{1}{\omega};-\frac{1}{\omega}\right)\,, \quad S(u;\omega) = 1+\frac{2u}{\omega}\,, \tag{6.118}$$

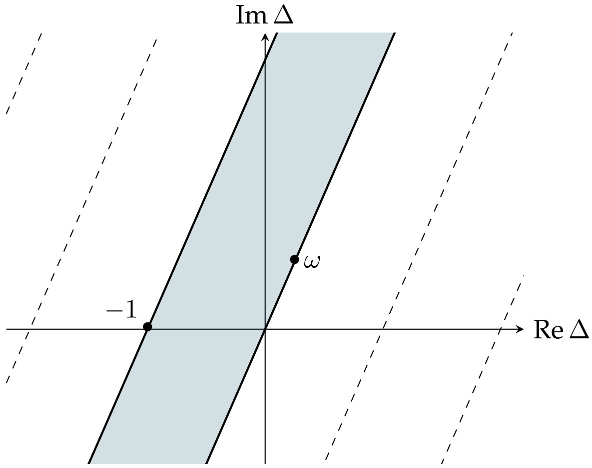

Figure 6.3: The colored region represents the portion of the Δ-complex plane where $\Delta_1, \Delta_2, \Delta_1 + \Delta_2$ live in order to have a convergent plethystic expansion. The dashed lines are defined in (6.94).

we shift by $-1/\omega$ the arguments of the two θ_0 in the numerator of $\mathcal{Z}_{2d}$, that depend on $\Delta_2, \Delta_1 + \Delta_2$ respectively. This further step guarantees the same domain of convergence for each of these three terms. Hence these terms of $\mathcal{Z}_{2d}$ become

$$\begin{aligned}
1^{\text{st}} \quad &: \quad +\pi\mathrm{i} \sum_{i\neq j=1}^{N} \left(B(-\hat{u}_{ij} - \Delta_2 + \omega;\omega) + S(-\hat{u}_{ij} - \Delta_2;\omega)\right) + \mathcal{O}\big(N^0\big)\,, \\
2^{\text{nd}} \quad &: \quad +\pi\mathrm{i} \sum_{i\neq j=1}^{N} \left(B(\hat{u}_{ij} - (\Delta_1 + \Delta_2) + 2\omega;\omega) + S(\hat{u}_{ij} - (\Delta_1 + \Delta_2);\omega)\right) + \mathcal{O}\big(N^0\big)\,, \\
3^{\text{rd}} \quad &: \quad -\pi\mathrm{i} \sum_{i\neq j=1}^{N} B(-\hat{u}_{ij} + \Delta_1 - \omega;\omega) + \mathcal{O}\big(N^0\big)\,,
\end{aligned} \tag{6.119}$$

provided that the chemical potentials are taken in the domain of convergence

$$\Delta_1,\, \Delta_2,\, \Delta_1 + \Delta_2 \in \mathcal{D} \equiv \left\{\Delta \in \mathbb{C} \;:\; \mathrm{Im}\,-\frac{1}{\omega} > \mathrm{Im}\,\frac{\Delta}{\omega} > 0\right\}\,, \tag{6.120}$$

which is shown in Figure 6.3. However, because of the quasi-periodicity of θ_0, the 2d defect integrand is invariant under the shifts $\Delta_a \mapsto \Delta_a + n$, for any integer n. This means that, if we introduce the function $[\,\cdot\,]_\omega$, defined as

$$[\Delta]_\omega := \Delta + n \quad \text{such that} \quad n \in \mathbb{Z} \quad \text{and} \quad \mathrm{Im}\,-\frac{1}{\omega} > \mathrm{Im}\,\frac{\Delta + n}{\omega} > 0\,, \tag{6.121}$$

for any $\Delta \in \mathbb{C}$ such that

$$\mathrm{Im}\,\frac{\Delta}{\omega} \notin \mathbb{Z} \times \mathrm{Im}\,\frac{1}{\omega}\,, \tag{6.122}$$

we can extend the result to the whole complex plane by analytic continuation, with the exception of the lines (6.122), and get

$$\begin{aligned} 1^{\text{st}} \quad &: \quad +\pi\mathrm{i} \sum_{i\neq j=1}^{N} \left(B(-\hat{u}_{ij} - [\Delta_2]_\omega + \omega;\omega) + S(-\hat{u}_{ij} - [\Delta_2]_\omega;\omega)\right) + \mathcal{O}\big(N^0\big)\,, \\ 2^{\text{nd}} \quad &: \quad +\pi\mathrm{i} \sum_{i\neq j=1}^{N} \left(B(\hat{u}_{ij} - [\Delta_1+\Delta_2]_\omega + 2\omega;\omega) + S(\hat{u}_{ij} - [\Delta_1+\Delta_2]_\omega;\omega)\right) + \mathcal{O}\big(N^0\big)\,, \\ 3^{\text{rd}} \quad &: \quad -\pi\mathrm{i} \sum_{i\neq j=1}^{N} B(-\hat{u}_{ij} + [\Delta_1]_\omega - \omega;\omega) + \mathcal{O}\big(N^0\big)\,, \end{aligned} \tag{6.123}$$

We still have to compute the fourth term of $\mathcal{Z}_{2d}$ which requires a different approach

$$\begin{aligned} 4^{\text{th}} \,:\quad & -\sum_{i\neq j=1}^{N}\sum_{m=0}^{\infty} \log\left(1-\tilde{h}^m\left(\frac{\tilde{z}_i}{\tilde{z}_j}\right)\right)\left(1-\tilde{h}^{m+1}\left(\frac{\tilde{z}_j}{\tilde{z}_i}\right)\right) = \\ & = -\sum_{i\neq j=1}^{N} \log\left(1-\left(\frac{\tilde{z}_i}{\tilde{z}_j}\right)\right) + 2\sum_{i\neq j=1}^{N}\sum_{m=1}^{\infty} \log\left(1-\tilde{h}^m\left(\frac{\tilde{z}_i}{\tilde{z}_j}\right)\right)\,. \end{aligned} \tag{6.124}$$

The second term in (6.124), in analogy with the previous computation, once evaluated on the basic solution, is given by

$$2\sum_{i\neq j=1}^{N}\sum_{m=1}^{\infty} \log\left(1-\tilde{h}^m\left(\frac{\tilde{z}_i}{\tilde{z}_j}\right)\right) = \sum_{n=1}^{\infty}\frac{1}{n}\frac{\tilde{h}^n}{1-\tilde{h}^n} - \sum_{n=1}^{\infty}\frac{1}{n}\frac{\tilde{h}^{Nn}}{1-\tilde{h}^{Nn}} = \mathcal{O}\big(N^0\big)\,, \tag{6.125}$$

at large N and $|\tilde{h}| < 1$. For the first term in (6.124), notice that, since

$$x^N - 1 = \prod_{k=1}^{N}\left(x - \mathrm{e}^{2\pi i\frac{k}{N}}\right)\,, \tag{6.126}$$

by factorizing $x-1$, we have

$$x^{N-1} + \cdots + x + 1 = \prod_{k=1}^{N-1}\left(x - \mathrm{e}^{2\pi i\frac{k}{N}}\right)\,, \tag{6.127}$$

that, for $x=1$, becomes

$$N = \prod_{k=1}^{N-1}\left(1 - \mathrm{e}^{2\pi i\frac{k}{N}}\right) \tag{6.128}$$

and finally we conclude that the contribution we are looking for amounts to

$$4^{\text{th}} \;:\quad -\log N + O(N^0)\,. \tag{6.129}$$

Collecting our results (6.123) and (6.129), we obtain

$$\mathcal{Z}_{2d}|_{\text{basic}} = \sum_{i=1}^{N} \exp\left(\Psi_i([\Delta_1]_\omega, [\Delta_2]_\omega, [\Delta_1+\Delta_2]_\omega) - \log N + \mathcal{O}(N^0)\right) , \tag{6.130}$$

where we introduced the function

$$\begin{aligned} &\Psi_i([\Delta_1]_\omega, [\Delta_2]_\omega, [\Delta_1+\Delta_2]_\omega) = \\ &= \pi \mathrm{i} N\left(-3[\Delta_1]_\omega + [\Delta_2]_\omega + 3[\Delta_1+\Delta_2]_\omega\right) - \frac{2\pi\mathrm{i}}{\omega} N\left([\Delta_1]_\omega + [\Delta_2]_\omega - [\Delta_1+\Delta_2]_\omega\right) \sum_{i\neq j=1}^{N} \hat{u}_{ij} + \\ &+ \frac{\pi\mathrm{i}}{\omega} N\left([\Delta_1]_\omega - [\Delta_2]_\omega - [\Delta_1+\Delta_2]_\omega\right) + \frac{\pi\mathrm{i}}{\omega} N\left([\Delta_1]_\omega^2 - [\Delta_2]_\omega^2 - [\Delta_1+\Delta_2]_\omega^2\right) . \end{aligned} \tag{6.131}$$

Here we can distinguish two cases:

$$[\Delta_1+\Delta_2]_\omega = \begin{cases} [\Delta_1]_\omega + [\Delta_2]_\omega & \text{for} \quad \operatorname{Im} -\frac{1}{\omega} > \operatorname{Im} \frac{[\Delta_1]_\omega + [\Delta_2]_\omega}{\omega} > 0 \\ [\Delta_1]_\omega + [\Delta_2]_\omega + 1 & \text{for} \quad \operatorname{Im} -\frac{2}{\omega} > \operatorname{Im} \frac{[\Delta_1]_\omega + [\Delta_2]_\omega}{\omega} > \operatorname{Im} -\frac{1}{\omega} \end{cases} \tag{6.132}$$

or equivalently, in a more compact form,

$$[\Delta_1+\Delta_2]_\omega = [\Delta_1]_\omega + [\Delta_2]_\omega + \frac{1-n_0}{2} \qquad \text{where} \qquad n_0 := \begin{cases} +1 & \mathrm{I} \\ -1 & \mathrm{II}\, . \end{cases} \tag{6.133}$$

We introduce the auxiliary chemical potential Δ_3 constrained as in (6.25) with the further assumption (6.73) and, because of the properties of $[\,\cdot\,]_\omega$,

$$[\Delta+n]_\omega = [\Delta]_\omega\, , \qquad [\Delta+\omega]_\omega = [\Delta]_\omega + \omega\, , \qquad [-\Delta]_\omega = -[\Delta]_\omega - 1\, , \tag{6.134}$$

the constraint becomes

$$[\Delta_1]_\omega + [\Delta_2]_\omega + [\Delta_3]_\omega = 2\omega - \frac{3-n_0}{2} . \tag{6.135}$$

In terms of these new chemical potentials the function Ψ_i becomes

$$\Psi_i = \frac{2\pi\mathrm{i}}{\omega} N\left([\Delta_2]_\omega + m\right)\left([\Delta_3]_\omega + m\right) + \frac{2\pi\mathrm{i}}{\omega} N m \left(\sum_{i\neq j=1}^{N} \frac{\hat{u}_{ij}}{N-1} - \frac{\omega}{2}\right) , \quad m = \frac{1-n_0}{2} . \tag{6.136}$$

Here one could think that, in case II there is an extra term, suggesting a different behavior of the D3-brane backreaction. However, once we evaluate such extra term on the basic solution (6.90), and we sum over j, it is irrelevant. This makes the symmetry between the two cases become manifest. This result is the BA counterpart of what we obtained by computing the subleading term in the Cardy-like limit.

We conclude that

$$\mathcal{Z}_{2d}|_{\text{basic}} = \exp\left(\frac{2\pi\mathrm{i}}{\omega} N \prod_{a=2}^{3} \left([\Delta_a]_\omega + m \right) + \mathcal{O}(N^0)\right) . \tag{6.137}$$

In order to compare the result with the 2d term (6.91), we rewrite $[\,\cdot\,]_\omega$ in terms of $\{\,\cdot\,\}_\omega$ as

$$[\Delta]_\omega = \{\Delta\}_\omega - 1 \,, \tag{6.138}$$

the constraint (6.136) reduces again to (6.93) and the 2d index contribution becomes

$$\mathcal{Z}_{2d}|_{\text{basic}} = \exp\left(\frac{2\pi\mathrm{i}}{\omega} N \prod_{a=2}^{3} \left(\{\Delta_a\}_\omega - n \right) + \mathcal{O}(N^0)\right) , \quad n = \frac{1+n_0}{2} . \tag{6.139}$$

By combining this result with the 2d term (6.91), we write our final result for the defect superconformal index

$$\log \mathcal{I}|_{\text{basic}} = -\frac{\pi\mathrm{i}}{\omega^2} N^2 \prod_{a=1}^{3} \left(\{\Delta_a\}_\omega - n \right) + \frac{2\pi\mathrm{i}}{\omega} N \prod_{a=2}^{3} \left(\{\Delta_a\}_\omega - n \right) + \log N + \mathcal{O}(N^0) \,. \tag{6.140}$$

This result is in agreement with the one obtained in (6.64) in the limit of small ω.

6.5 Discussion and Conclusions

In this Chapter we have studied a setup corresponding to a 5d rotating BH in presence of surface defects with maximal supersymmetry. From the holographic perspective the system corresponds to a stack of D3-branes in AdS$_5 \times S^5$ type IIB supergravity with the addition of a probe D3, extending across both time and the radial direction, while being wrapped around one compact direction in AdS$_5$ and another on the five-sphere. We have evaluated the SCI of the corresponding dual field theory, consisting in a 4d-2d system, namely SU(N) $\mathcal{N} = 4$ SYM coupled to a maximal Gukov-Witten surface defect. We have used two distinct methodologies to evaluated the index: firstly, by considering the Cardy-like limit, and secondly by applying the BA approach matching the two results at large-N for equal angualar momenta. Such regimes correspond to the large N limit for equal and small angular velocities. In this case we have extracted the sub-leading logarithmic corrections to the index, that are expected to capture the leading order effect of the backreaction of the probe D3-brane in the dual gravitational picture. Furthermore, a three-dimensional EFT emerges from the calculation: the effective picture corresponds to a sum over the anomalies of the 4d and 2d system in addition to a pure SU$(N)_{\pm N}$ topological theory.

There are many open questions and lines of research left. It would be interesting to study the coupling with other GW operators. So far only the coupling of the four-dimensional theory with a maximal GW operator has been extensively studied. Nevertheless, the possibility of coupling SU(N) $\mathcal{N} = 4$ SYM to GW surface defects corresponding to other Levi sub-gropus has been discussed in [139]. It would be interesting to have an explicit analysis of such Levi subgroups for the GW defects and to understand how this is realized at the level of the SCI by working out the localization procedure of the coupled 4d-2d system along the lines of what we have done here.

Another possible extension consists in working out the maximally supersymmetric case for real gauge groups and their connection with the S-duality orbits of SYM. Although the defect is generally defined by prescribing boundary conditions for the vector field, a useful approach to include such in the SCI is to consider it as a coupling of a 2d theory to the four-dimensional one. The standard prescription used here for the $\mathrm{SU}(N)$ case could be extended by considering other Lie algebras corresponding to $\mathrm{USp}(2N)$ and $\mathrm{SO}(N)$ gauge groups. From the Cardy-like limit of the 4d-2d system SCI it would require to study then the saddles, similarly to what was done in [46] for the pure 4d system. It would also be interesting to study the fate of S-duality for the $\mathrm{USp}(2N)$ and $\mathrm{SO}(2N+1)$ gauge groups in the coupled system.

Another generalization of the analysis consists in understanding the behavior of the coupled system around different holonomy saddles than the one treated here. Such saddles admit a holographic interpretation in terms of wrapped D3-brane solutions [7]. Futhermore, an EFT interpretation in terms of orbifold partition functions has been discussed in [55]. One may wonder the fate of the 4d-2d coupled system and the role played by the circle reduction of the defect in this case.

While on the Cardy side we have obtained the result for different angular momenta, on the Bethe side we restricted to the case of $\sigma = \tau$. It would be important to extend the BA analysis to the case of $\sigma \neq \tau$. Indeed, the BA approach is well-defined also in this more general setup [88] and, when we add a defect, the analysis of sub-section 6.4.2 can be generalized, providing a BA formula for the defect SCI. However, the case of different angular momenta is still an open problem even in absence of defects [13, 86]. In fact, the contribution of the basic solution itself requires the evaluation of extra terms that are quite hard to compute. In [86] some of these terms are obtained, and they do not alter the result at leading order. The recent analysis of [13] has revealed that some other terms are $\mathcal{O}(N^2)$, thus they cannot be discarded in a large-N limit. Moreover, the cancellation among the extra terms is argued by focusing on the $\mathrm{SU}(2)$ case. However, even if these quantities become relevant for large N, they are always negligible for large angular momenta. Therefore, restricting to the comparison with the Cardy-like approach, one could estimate these terms in a double limit of large N and large angular momenta, instead of computing them explicitly. Such terms are then discarded, upon ensuring that their recombination is negligible at leading order in the evaluation of the index. Following this strategy one could generalize, at least in this double limit, the result obtained here for the defect SCI to the case of different angular momenta.

A last line of research consists in expanding on the three-dimensional EFT interpretation arising from the circle reduction of the parent four-dimensional theory. Our analysis suggests the emergence of an N-wound anti-fundamental Wilson loop from the defect. We expect that a complete analysis of the backreaction of the probe D3 will require also an explicit construction of an effective 3d-$1d$ system.

Appendices

CHAPTER 7

Appendix to Chapter 6

7.1 Special Functions and Asymptotics Expansions

In this appendix we list general properties and the asymptotic expansions of the special functions used in this work.
The q-Pochhammer symbol is defined for complex z, q with $|q| < 1$ by

$$(z;q)_\infty := \prod_{j=0}^{\infty} \left(1 - zq^j\right). \tag{A.1}$$

We can derive an asymptotic expansion for the q-Pochhammer symbol $(q;q)_\infty$ by rewriting it in terms of the Dedekind Eta function

$$\eta(\tau) := \mathrm{e}^{\frac{\pi \mathrm{i}\tau}{12}} \prod_{n=1}^{\infty} (1 - \mathrm{e}^{2n\mathrm{i}\pi\tau}), \tag{A.2}$$

and employing its modular properties

$$(q;q)_\infty = \mathrm{e}^{-\frac{\pi \mathrm{i}\tau}{12}} \eta(\tau) \underset{r\to 0}{\sim} -\frac{\pi \mathrm{i}}{12}\left(\tau + \frac{1}{\tau}\right) - \frac{1}{2}\log(-\mathrm{i}\tau). \tag{A.3}$$

Similarly $\theta_0(u;\tau)$ is defined as

$$\theta_0(u;\tau) := (\mathrm{e}^{2\pi \mathrm{i}u}; \mathrm{e}^{2\pi \mathrm{i}\tau})_\infty\, (\mathrm{e}^{2\pi \mathrm{i}\tau}\mathrm{e}^{-2\pi \mathrm{i}u}; \mathrm{e}^{2\pi \mathrm{i}\omega})_\infty. \tag{A.4}$$

It satisfies the quasi-double periodicity property

$$\theta_0(u + m + n\tau;\tau) = (-1)^n \mathrm{e}^{-2\pi \mathrm{i}nu} \mathrm{e}^{-\pi \mathrm{i}n(n-1)\tau} \theta_0(u;\tau), \quad m, n \in \mathbb{Z} \tag{A.5}$$

and the inversion formula

$$\theta_0(-u;\tau) = -\mathrm{e}^{2\pi \mathrm{i}u} \theta_0(u;\tau). \tag{A.6}$$

In addition, we remind the relation between $\theta_0(u;\tau)$ and the Jacobi theta function $\theta_1(u;\tau)$:

$$\theta_1(u;\tau) = \mathrm{i}\mathrm{e}^{\pi\mathrm{i}\tau/4-\pi\mathrm{i}u}(q;q)_\infty\theta_0(u;\tau). \tag{A.7}$$

The elliptic gamma function is defined as

$$\Gamma(z;p,q) \coloneqq \prod_{m=0}^{\infty}\prod_{n=0}^{\infty}\frac{1-p^{m+1}q^{n+1}/z}{1-p^m q^n z}, \qquad \widetilde{\Gamma}(u) \coloneqq \Gamma(\mathrm{e}^{2\pi\mathrm{i}u};\mathrm{e}^{2\pi\mathrm{i}\tau},\mathrm{e}^{2\pi\mathrm{i}\sigma}). \tag{A.8}$$

Similarly to $\theta_0(u,\tau)$, also elliptic gamma function satisfies an inversion formula

$$\widetilde{\Gamma}(u;\tau,\sigma) = \widetilde{\Gamma}(\sigma+\tau-u;\tau,\sigma)^{-1}, \tag{A.9}$$

and a quasi double-periodicity relation

$$\widetilde{\Gamma}(u;\tau,\sigma) = \theta_0(u;\tau)^{-1}\widetilde{\Gamma}(u+\sigma;\tau,\sigma) = \theta_0(u;\sigma)^{-1}\widetilde{\Gamma}(u+\tau;\tau,\sigma). \tag{A.10}$$

Using (A.9) and (A.10) together with (A.5) and (A.6) one obtains

$$\sum_{i\neq j}\log\widetilde{\Gamma}(u_{ij};\tau,\sigma) = -\sum_{i<j}\left(\log\theta_0(u_{ij};\tau) + \log\theta_0(-u_{ij};\sigma)\right). \tag{A.11}$$

Exploiting the modular properties of $\theta_0(u;\tau)$ one derives the asymptotic expansion for small τ

$$\begin{aligned}\log\theta_0(u;\tau) = \quad & \frac{\mathrm{i}\pi}{\tau}\{u\}_\tau(1-\{u\}_\tau) + \mathrm{i}\pi\{u\}_\tau - \frac{\mathrm{i}\pi}{6\tau}\left(1+3\tau+\tau^2\right) + \\ & + \log\left(\left(1-\mathrm{e}^{-\frac{2\pi\mathrm{i}}{\tau}\{u\}_\tau}\right)\left(1-\mathrm{e}^{-\frac{2\pi\mathrm{i}}{\tau}(1-\{u\}_\tau)}\right)\right) + \mathcal{O}\left(\mathrm{e}^{-\frac{2\pi\sin\arg(\tau)}{|\tau|}}\right),\end{aligned} \tag{A.12}$$

where

$$\{u\}_\tau \equiv \{\tilde{u}\} + \tau\bar{u}, \quad u \equiv \tilde{u} + \tau\bar{u}, \quad \tilde{u},\bar{u} \in \mathbb{R} \tag{A.13}$$

and $\{\tilde{u}\} = \tilde{u} - \lfloor u \rfloor$. For small $\tau \neq \sigma$, such definition is generalized to

$$\{x\} = \{\tilde{x}\} + r\bar{x} \equiv \tilde{x} - \lfloor\tilde{x}\rfloor + r\bar{x} \tag{A.14}$$

for any x with $\tilde{x} \neq 0$. To recover an asymptotic expansion for the elliptic gamma function one can start from the infinite product formula

$$\widetilde{\Gamma}(u;\tau,\sigma) = \mathrm{e}^{2\pi\mathrm{i}Q(u;\tau,\sigma)}\prod_{n=-\infty}^{\infty}\mathrm{e}^{-\mathrm{sign}(n)\frac{\pi\mathrm{i}}{2\tau\sigma}\left(\left(\frac{u+n}{r}-\frac{\tau+\sigma}{2}\right)^2-\frac{\tau^2+\sigma^2}{12}\right)}\Gamma_h\left(\frac{u+n}{r};\omega_1,\omega_2\right), \tag{A.15}$$

where $\Gamma_h(u;\omega_1,\omega_2)$ is the hyperbolic gamma function. As r approaches zero the infinite tower of KK modes associated with the hyperbolic gamma functions gets lifted, when

$u \notin \mathbb{Z}$, and we get

$$\log \widetilde{\Gamma}(u;\tau,\sigma) = 2\pi \mathrm{i} Q(\{u\}) + \mathcal{O}(\mathrm{e}^{-1/r}) \tag{A.16}$$

with

$$\begin{aligned} Q(\{u\}) = \quad & -\frac{B_3(\{u\})}{6\,\sigma\tau} + B_2(\{u\})\frac{(\sigma+\tau)}{4\,\sigma\tau} - B_1(\{u\})\frac{\big((\sigma+\tau)^2+\sigma\tau\big)}{12\,\sigma\tau} \\ & + \frac{\sigma}{24} + \frac{\tau}{24}, \end{aligned} \tag{A.17}$$

and the Bernoulli polynomials

$$B_3(u) = u^3 - \frac{3}{2}u^2 + \frac{u}{2}, \quad B_2(u) = u^2 - u + \frac{1}{6}, \quad B_1(u) = u - \frac{1}{2}. \tag{A.18}$$

7.2 Wilson Loop in Pure CS

In this appendix we want to evaluate the partition function of a $SU(N)_{n_0 N}$ pure CS theory with an (anti-)fundamental N-wounded Wilson loop insertion, where $n_0 = \pm 1$. We start by first recalling the result for the three-sphere partition function of a pure $3d$ CS theory. The squashed three-sphere partition function of a pure $\mathrm{U}(N)_{n_0 N}$ CS theory is

$$\frac{1}{N!}\int \prod_{i=1}^{N} \frac{\mathrm{d}\lambda_i}{\sqrt{-\omega_1\omega_2}} \frac{\mathrm{e}^{-\frac{\pi i n_0 N}{\omega_1\omega_2}\sum_{i=1}^{N}\lambda_i^2}}{\prod_{i<j}\Gamma_h(\lambda_{ij})\Gamma_h(-\lambda_{ij})}. \tag{A.19}$$

We can constrain the holonomies with a Lagrange multiplier to derive the partition function for the case of $\mathrm{SU}(N)$ gauge group:

$$\frac{1}{N!}\int \mathrm{d}\Lambda \int \prod_{i=1}^{N} \frac{\mathrm{d}\lambda_i}{\sqrt{-\omega_1\omega_2}} \frac{\mathrm{e}^{-\frac{\pi i n_0 N}{\omega_1\omega_2}\sum_{i=1}^{N}\lambda_i^2 + 2\pi \mathrm{i}\Lambda\sum_{j=1}^{N}\lambda_j}}{\prod_{i<j}\Gamma_h(\lambda_{ij})\Gamma_h(-\lambda_{ij})}. \tag{A.20}$$

Employing the identity

$$\frac{1}{\Gamma_h(x)\Gamma_h(-x)} = -4\sin\left(\frac{\pi x}{\omega_1}\right)\sin\left(\frac{\pi x}{\omega_2}\right) \tag{A.21}$$

and specializing to the case $\omega_1 = \omega_2$ we get

$$Z^{S^3}_{\mathrm{SU}(N)_{n_0 N}} = \mathrm{e}^{\frac{5\pi i n_0 (N^2-1)}{12}}. \tag{A.22}$$

Let us introduce a Wilson loop operator in the CS theory. The supersymmetric Wilson loop is defined as

$$W_\gamma(\sigma) = \mathrm{Tr}_R\, P\exp\left\{\oint_\gamma iA_\mu \dot{x}^\mu + \sigma|\dot{x}|\mathrm{d}\tau\right\}. \tag{A.23}$$

The localization locus for a gauge theory on S^3 is defined by the equations

$$\begin{cases} F_{\mu\nu} = 0 \\ D = -\sigma \equiv -\sigma_0. \end{cases} \tag{A.24}$$

Thus, an n-wounded Wilson loop insertion in the functional integral modifies the matrix model arising from localization with a term

$$W_\gamma(\sigma_0) = \mathrm{Tr}_R \exp\left\{\sigma_0 \oint \mathrm{d}s\right\} = \mathrm{Tr}_R \exp\{2\pi n \sigma_0\}. \tag{A.25}$$

Let us compute the partition function of a k-level CS-theory with an n-wounded Wilson loop in the (anti-)fundamental representation of $\mathrm{U}(N)$. Starting from (A.19) and using the Weyl denominator formula

$$\prod_{1\leq i<j\leq N} 2\sinh\left(\frac{x_i - x_j}{2}\right) = \sum_{\sigma}(-1)^\sigma \prod_j \mathrm{e}^{((N+1)/2-\sigma(j))x_j}, \tag{A.26}$$

where the sum runs over the permutations S_N, we get

$$\begin{aligned} \mathcal{Z}_W = \frac{1}{N!}\int \prod_{j=1}^{N} \mathrm{d}\lambda_j e^{-i\pi k \lambda_j^2} &\sum_{\sigma_1,\sigma_2} (-1)^{\varepsilon(\sigma_1)+\varepsilon(\sigma_2)} \\ \prod_{j=1}^{N} \mathrm{e}^{2\pi(N+1-j-\sigma_1(j)-\sigma_2(j))\lambda_j} &\left(\sum_{i=1}^{N} \mathrm{e}^{2\pi n \lambda_i}\right) \end{aligned} \tag{A.27}$$

Since $\left(\sum_{i=1}^{N} \mathrm{e}^{2\pi n\lambda_i}\right)$ is symmetric under exchange of λ_i with λ_j we can freely relabel variables as before to get rid of one sum over permutations, without spoiling the result. We get

$$\begin{aligned} \mathcal{Z}_W^{\mathrm{U}(N)} =& \sum_{i=1}^{N} \sum_{\sigma\in S_N} (-1)^{\varepsilon(\sigma)} \int \left(\mathrm{d}\lambda_i\, \mathrm{e}^{-ik\pi\lambda_i^2} \mathrm{e}^{2\pi(N+1-i-\sigma(i))\lambda_i}\right) \\ & \int \left(\prod_{j\neq i}^{N} \mathrm{d}\lambda_j\, \mathrm{e}^{-ik\pi\lambda_j^2} \mathrm{e}^{2\pi(N+1-j-\sigma(j))\lambda_j}\right) = \\ =& (ik)^{-N/2} \sum_{i=1}^{N} \sum_{\sigma\in S_N} (-1)^{\varepsilon(\sigma)} \prod_{j=1}^{N} \mathrm{e}^{-\frac{i\pi}{k}(N+1+n\delta_{i,j}-j-\sigma(j))^2} \end{aligned} \tag{A.28}$$

Expanding the square

$$\sum_{j=1}^{N} (N+1-j+n\delta_i, j-\sigma(j))^2 = \sum (x_j^2 - 2x_j\sigma(j) + j^2), \tag{A.29}$$

where $x_j = N+1-j+n\delta_{i,j}$, we isolate a term independent on $\sigma(j)$ and a term $x_j\sigma(j)$,

which can be rearranged with the Weyl denominator formula after being combined with the sum over σ.
We get

$$\begin{aligned}
\mathcal{Z}_W^{\mathrm{U}(N)} =&(ik)^{-N/2}\sum_{i=1}^{N}\mathrm{e}^{-\frac{i\pi}{k}\sum_j(x_j^2+j^2)}\sum_{\sigma\in S_N}(-1)^{\varepsilon(\sigma)}\mathrm{e}^{\frac{2\pi i}{k}\sum_j x_j\sigma(j)}=\\
=&(ik)^{-N/2}\sum_{i=1}^{N}\mathrm{e}^{-\frac{\pi i}{k}\sum_j(x_j^2+j^2)-(N+1)x_j}\prod_{i<j}2\sinh\left(\frac{x_i-x_j}{2}\right)=\\
=&(ik)^{-N/2}\mathrm{e}^{-\frac{\pi i}{6k}\left(N(N^2-1)+6n(N+1+n)\right)}(-i)^{N(N-1)/2}\\
&\sum_{j=1}^{N}\mathrm{e}^{\frac{2\pi i}{k}jn}\prod_{m<l}2\sin\left(\frac{\pi}{k}(l-m+n(\delta_{j,m}-\delta_{j,l}))\right)
\end{aligned} \tag{A.30}$$

$$\begin{aligned}
&\prod_{m<l}2\sin\left(\frac{\pi}{k}(l-m+n(\delta_{j,m}-\delta_{j,l}))\right)=\\
&=\prod_{m<l,\,m,l\neq j}2\sin\left(\frac{\pi}{k}(l-m)\right)\prod_{m=1}^{j-1}2\sin\left(\frac{\pi}{k}(j-m-n)\right)\prod_{l=j+1}^{N}2\sin\left(\frac{\pi}{k}(l-j+n)\right)=\\
&=\prod_{m<l}2\sin\left(\frac{\pi}{k}(l-m)\right)\prod_{m=1}^{j-1}\frac{\sin\left(\frac{\pi}{k}(j-m-n)\right)}{\sin\left(\frac{\pi}{k}(j-m)\right)}\prod_{m=j+1}^{N}\frac{\sin\left(\frac{\pi}{k}(m-j+n)\right)}{\sin\left(\frac{\pi}{k}(m-j)\right)}=\\
&=\prod_{m<l}2\sin\left(\frac{\pi}{k}(l-m)\right)\prod_{m\neq j}^{N}\frac{\sin\left(\frac{\pi}{k}(m-j+n)\right)}{\sin\left(\frac{\pi}{k}(m-j)\right)}
\end{aligned} \tag{A.31}$$

Then, the final result is

$$\begin{aligned}
\mathcal{Z}_{W_n}^{\mathrm{U}(N)_k} =&(ik)^{-N/2}\mathrm{e}^{-\frac{\pi i}{6k}\left(N(N^2-1)+6n(N+1+n)\right)}(-i)^{N(N-1)/2}\\
&\prod_{m<l}2\sin\left(\frac{\pi}{k}(l-m)\right)\sum_{j=1}^{N}\mathrm{e}^{\frac{2\pi i}{k}jn}\prod_{m\neq j}^{N}\frac{\sin\left(\frac{\pi}{k}(m-j+n)\right)}{\sin\left(\frac{\pi}{k}(m-j)\right)}
\end{aligned} \tag{A.32}$$

Let us consider the gapped case for the pure CS theory $k=N$ (the case $k=-N$ goes along the same line). Then, (A.32) is zero unless the wounding of the Wilson loop is $n=pN$,

$$\begin{aligned}
\mathcal{Z}_{W_{pN}}^{\mathrm{U}(N)_N} =&(N)^{-N/2}\mathrm{e}^{-\frac{\pi i}{6}\left((N^2-1)+6p(N(p+1)+1)\right)}\mathrm{e}^{-i\pi N^2/4}\\
&(-1)^{N-1}N\prod_{m<l}2\sin\left(\frac{\pi}{N}(l-m)\right)
\end{aligned} \tag{A.33}$$

Notice that $\prod_{m<l}2\sin\left(\frac{\pi}{N}(l-m)\right)=N^{N/2}$ and we get

$$\mathcal{Z}_{W_{pN}}^{\mathrm{U}(N)_N}=(-1)^{N-1+p(N(p+1)+1)}N\mathcal{Z}_{CS}^{\mathrm{U}(N)}. \tag{A.34}$$

When $p = \pm 1$ we have

$$\mathcal{Z}_{W_{\pm N}}^{\mathrm{U}(N)_N} = (-1)^N N \mathcal{Z}_{CS}^{\mathrm{U}(N)_N}. \tag{A.35}$$

Ultimately we specialize to $\mathrm{SU}(N)$ by introducing the Lagrange multiplier Λ as in (A.20), and we obtain

$$\mathcal{Z}_{W_n}^{\mathrm{SU}(N)_k} = \mathcal{Z}_{W_n}^{\mathrm{U}(N)_k} \mathrm{e}^{-\frac{i\pi n^2}{kN}} \sqrt{\frac{ik}{N}}. \tag{A.36}$$

We are interested in the case $n = -N$ and $k = N$:

$$\mathcal{Z}_{W_{-N}}^{\mathrm{SU}(N)_N} = (-1)^{N+1} N \mathcal{Z}_{CS}^{\mathrm{SU}(N)_N}. \tag{A.37}$$

List of Publications

[31] Antonio Amariti, Pietro Glorioso, Davide Morgante, and Andrea Zanetti. "Cardy matches Bethe on the Surface: a Tale of a Brane and a Black Hole". In: (Mar. 2024). arXiv: 2403.17190 [hep-th].

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[37] Antonio Amariti and Davide Morgante. "Chiral dualities for $SQCD_3$ with D-type superpotential". In: *JHEP* 02 (2023), p. 032. DOI: 10.1007/JHEP02(2023)032. arXiv: 2209.12673 [hep-th].

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