Electroweak interactions

Notes from the EW course held by prof. Martinelli A.A. 2019/2020

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Preface

I'll begin with a big disclaimer: this notes are not meant to be a full on course on EW theory, they're just a collection, and a summary, of what I've gathered during the semester. The idea behind this notes is to improve the notes given by last years students, making them more fluid and organized. Many of the subjects will be taken from their notes as well as mine and from various books that I'll highlight through the notes.
The electroweak interaction course is a though one and I'm not in the position of saying that I've understood very well the subject, but by writing this notes I hope to lay down my understanding and hopefully study in a better way.

I'll leave out some sections that prof. Martinelli will do at the start of the course since there are his well made notes and I feel that on the subject there's enough coverage.

BE AWARE: there may be many errors that I didn't catch yet, I urge you to be a critical reader and to second guess any of the things you read in this notes. If you find that there could be some error at some point, write me an e-mail and I'll make sure to rectify it! Many thanks.

PART

I

Introduction

Standard Model is a physics theory considered one of the most important results of the last 70 years concerning the comprehension of the particles and the fundamental interactions of the Universe. Despite this theory would find huge success and be deemed totally correct, currently Standard Model is not satisfactory yet because of some problems:

1. Gravitational interaction is not included in the Standard Model.
2. Neutrinos have no mass, even though it was discovered in 2015.
3. There is no explanation for the presence of the dark matter.
4. There is no explanation for the baryon asymmetry (we don't have data to explain that).

There is another problem related to the mass of the Higgs boson whose value is $m_H = (125,18 \pm 0,16)\ GeV/c^2$. Since the Higgs field permeates all the space like the gravitational one and that the relative boson gives mass to the gauge bosons (and also to itself), it certainly interacts with the gravitational field. Consequently, the mass value of the Higgs boson is modified by quantum corrections that bring it to an order of magnitude of 10^{15} - $10^{16}\ GeV$, comparable with the Planck energy scale ($10^{19}\ GeV$). The problem arises when we realize that, experimentally, the value of the mass always remains the same as the starting one but there is no protection mechanism that cancels the correction terms: this is the so called hierarchy problem.

1 Fundamental interactions and its constituents

It is possible to summarize all the possible ways in which particles can interact through 4 fundamental interactions: gravitational, electromagnetic, weak, strong.

Interaction	**Mediator**	**Range (m)**	**Intensity**	**Symmetry**
Gravitational	Graviton	∞	O(10^{-42})	U(1)
Electromagnetic	Photon	∞	O(10^{-3})	U(1)
Weak	$W^{\pm}$, Z^0	O(10^{-18})	O(10^{-5})	SU(2)
Strong	Gluon	O(10^{-15})	O(10^{+1})	SU(3)

1.1 Gravitational

The gravitational interaction involves all existing particles and extends throughout space. The gravitational potential is:

$$V(r) = -G_N \frac{m_1 m_2}{r} \qquad G_N = (6,67408 \pm 0,00031) \cdot 10^{-11}\ \frac{Nm^2}{kg^2} \tag{1.1}$$

The mediating particle of this interaction is called graviton and it is assumed to be a spin 2 particle. Unfortunately, it has not yet been revealed, due to the very low intensity with which it would act on the particles; the coupling constant g is

$$g = \frac{G_N m_1 m_2}{\hbar c} = O(10^{-42}) \tag{1.2}$$

Moreover, it is possible to obtain the value of the Planck mass, a useful parameter for the Planck energy scale

$$M_{Planck} = \sqrt{\frac{\hbar c}{G_N}} = 1,2209 \cdot 10^{19}\ GeV \tag{1.3}$$

1.2 Electromagnetic

The electromagnetic interaction, with which it is possible to make parallels with the gravitational interaction, involves all those particles that carry electric charge or magnetic moment and extends throughout the space.
The electromagnetic four-potential A is defined as $A = (\phi, \mathbf{A})$, where the time component ϕ represents the electric potential, while $\mathbf{A}$ the magnetic one, and is built such for which it is worth

$$\begin{aligned} E &= -\nabla\phi - \frac{\partial A}{\partial t} \\ B &= \nabla \times A \end{aligned} \tag{1.4}$$

The particle mediating the interaction is the photon, which has zero mass and charge and spin equal to 1. The coupling constant is the fine structure constant $\alpha \simeq \frac{1}{137}$.

1.3 Weak

The weak interaction involves only the leptons, quarks and related gauge bosons. The potential that describes this interaction is:

$$V(r) = \frac{g_w^2}{2r}\ \exp\left(-\frac{M_w\ c}{\hbar}\ r\right) \tag{1.5}$$

where g_w is the dimensionless coupling constant and M_w the mass of the mediating particle. Since it describes the decays of atomic nuclei, this interaction acts over very short distances, so the mass of the mediating particle can be assumed to be very large, approximately $M_w \simeq 80 - 100\ GeV/c^2$. In this way, the distance over which the weak interaction acts turns out to be:

$$r_0 = \frac{\hbar}{M_w c} \simeq 2 \cdot 10^{-18}\ m = 2 \cdot 10^{-3}\ fm \tag{1.6}$$

which is compatible with experimental data.
The coupling constant for the weak interaction is not the dimensionless constant g_w, but the Fermi constant G_F, introduced in 1934 within the Fermi theory to try to describe the interaction in the decays of the atomic nucleus, which turns out to have the dimension of a quadratic inverse energy, i.e.

$$G_F = \frac{\sqrt{2}\ g_w^2}{8\ M_w^2} = 1,1663787 \cdot 10^{-5}\ GeV^{-2} \cdot c^4 \tag{1.7}$$

This result is obtained if we use the value of the $W^{\pm}$ bosons as the mass value, that is $(80,385 \pm 0,015)\ GeV/c^2$.

1.4 Strong

The strong interaction involves quarks and gluons and explains why the protons inside the atomic nucleus do not repel each other due to Coulomb repulsion. In 1935 the Japanese physicist Hideki Yukawa hypothesized a mechanism based on such a potential that the relative force was of an attractive type and more intense than the electromagnetic one; this potential is defined of Yukawa and is

$$V(r) = \frac{g_s^2}{r} \exp\left(-\frac{M_s c}{\hbar} r\right) \tag{1.8}$$

with g_s dimensionless coupling constant of the interaction and M_s mass of the mediating particle, renamed mesotron, since Yukawa's calculations show that the value of this mass should be more or less halfway between that of the electron and that of the proton, that is $100 - 200\ MeV$. Moreover, since this particle is exchanged between nucleons (protons and neutrons), considering all possible combinations, Yukawa assumed that such a particle exists with positive, negative and neutral charge. With the discovery in 1936 of the particles redefined later muons, Yukawa hypothesized they were suitable candidates to mediate the strong interaction, but Conversi-Pancini-Piccioni experiment refuted this idea, since these particles do not interact with the elements of the nucleus. It was necessary to wait until 1947, with the discovery of pions from cosmic ray (whose mass is around $135-140\ MeV$), to find the suitable candidate to be the mediating particle. Only with the introduction of the quark model, in 1964, the theory of strong interaction was completed, introducing gluons as massless particles mediating the interaction that perform the action of glue between quarks within nucleons, that is protons and neutrons; these particles generate an attractive force that overcomes the repulsive Coulomb one and manifests itself through the exchange of pions.
The distance over which the strong interaction acts turns out to be

$$r = \frac{\hbar}{M_\pi c} \simeq 1,5 \cdot 10^{-15}\ m = 1,5\ fm \tag{1.9}$$

and it is compatible with the radius of a generic atomic nucleus, while the intensity of the coupling constant is

$$\frac{g_s^2}{\hbar c} = O(10^{+1}) \tag{1.10}$$

2 Particle instability

It is possible to distinguish under which interactions a given decay can occur not only by evaluating the transition amplitude, but also by estimating the life time of the particles, since there is a parameter that allows to make this connection: the decay width Γ. In detail

$$\Gamma = \frac{2\pi}{\hbar} \sum_f \left| \langle f | \hat{V} | i \rangle \right|^2 \rho_f(E_i) \tag{2.1}$$

representing Fermi's golden rule, where the matrix element of $\hat{V}$ expresses the amplitude of transition from a state i to a f, while the other formula is

$$\Gamma = \frac{\hbar}{\tau} \tag{2.2}$$

with τ average life time of the considered particle. Thanks to this we can say that

$$|A|^2 \sim \frac{1}{\tau} \tag{2.3}$$

When studying the interactions between particles, it is generally useful to consider an external perturbation. For example, using the interaction of a particle with the electromagnetic field, we will have the Hamiltonian of the type $\hat{H} = \hat{H}_0 + \hat{V}$, with $\hat{V} = J \cdot A$, while the wave function will be

$$\psi(x,t) = \psi_0(x) \exp\left(-i \frac{(E_n - \frac{i}{2}\Gamma_n)}{\hbar} t\right) \tag{2.4}$$

where Γ_n represents the resonance width and is related to its average life through the formula $\tau_n = \hbar/\Gamma_n$. The 2^{nd} order eigenvalues will be

$$E_n = E_n^{(0)} + \langle n|\hat{V}|n\rangle + \sum_k \frac{|\langle k|\hat{V}|n\rangle|^2}{E_n^{(0)} - E_k^{(0)}} \tag{2.5}$$

The probability of being at a certain energy level n will decrease exponentially

$$P_n = |A|^2 = \psi_n^* \psi_n = \exp\left(-\frac{\Gamma_n}{\hbar} t\right) = \exp\left(-\frac{t}{\tau_n}\right) \tag{2.6}$$

The probability amplitude that allows to define the probability P_n is of the form

$$A(E) \sim \frac{1}{E - E_n + i\frac{\Gamma}{2}} \tag{2.7}$$

so

$$P(E) = |A(E)|^2 \sim \frac{1}{(E - E_n)^2 + \frac{\Gamma^2}{4}} \tag{2.8}$$

$P(E)$ is what is called **Breit-Wigner function** and is used in High Energy Physics to model resonances, i.e. unstable particles.

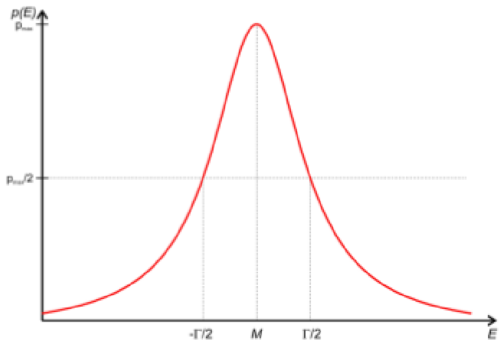


Figure 1. Breit-Wigner distribution

The form that this formula takes arises from the propagator of an unstable particle, which has the denominator of the form

$$\frac{1}{p^2 - M^2 + i\epsilon} = \frac{1}{p_0^2 - E^2 + i\epsilon} \tag{2.9}$$

If we call the interaction propagator $S(x,t)$, we will have

$$S(x - x_0, t - t_0) = \int \frac{dk_0}{2\pi} \int \frac{d^3k}{(2\pi)^3} \frac{i\, e^{ik(x-x_0)}}{k_0^2 - k^2 - M^2 + i\epsilon} \tag{2.10}$$

Setting $x_0 = 0$, we write

$$\begin{aligned} S(p,t) &= \int d^3x\, e^{ipx} S(x,t) \\ &= \int \frac{dk_0}{2\pi} \int \frac{d^3k}{(2\pi)^3} \frac{i}{k_0^2 - k^2 - M^2 + i\epsilon} \underbrace{\int d^3x\, e^{ipx} e^{i(p-k)x}}_{(2\pi)^3\delta(p-k)} \end{aligned} \tag{2.11}$$

Therefore

$$\begin{aligned} S(p,t) &= \int \frac{dk_0}{2\pi} \frac{i}{k_0^2 - (p^2 + M^2) + i\epsilon} \\ &= \int \frac{dk_0}{2\pi} \frac{i}{k_0^2 - E^2(p) + i\epsilon} \end{aligned} \tag{2.12}$$

Considering two different paths, namely the one in the upper half plane and in the lower one, using the residuals method I get

$$S(p,t) = \begin{cases} \dfrac{e^{-iE(p)t}}{2E(p)}, & x_0 > 0 \\ \\ \dfrac{e^{iE(p)t}}{2E(p)}, & x_0 < 0 \end{cases}$$

In the event that the particle is unstable, the resonance width should also be considered and the denominator of the propagator has to be in the following form

$$\frac{1}{p_0^2 - (\sqrt{E^2(p) - iM\Gamma})^2 + i\epsilon} \tag{2.13}$$

and the poles will be $E_{1,2} = \mp\sqrt{E^2(p) - iM\Gamma}$. When the particle has low instability, it is possible to develop the propagator in series, obtaining the denominator

$$E(p)\sqrt{1 - \frac{iM\Gamma}{E^2(p)}} \simeq E(p)\left(1 - \frac{iM\Gamma}{2E^2(p)}\right) \tag{2.14}$$

3 Potential description

By observing the potentials that describe the fundamental interactions, it can be seen that all of them depend on the distance between the particles in the form $1/r$; it is possible to mathematically describe this evidence.

3.1 3-dimensional case

Let's consider the electromagnetic field. The free field wave equation is described in this way

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right) \phi(r,t) = 4\pi\rho(r,t) = \delta^{(4)}(r,t) \tag{3.1}$$

In order to do simple accounts, we consider the stationary case; the previous equation transforms in the **Poisson equation**

$$-\nabla^2\phi(r) = \delta^{(3)}(r) \tag{3.2}$$

We first solve this equation in Fourier space, where

$$\phi(r) = \int \frac{d^3q}{(2\pi)^3} e^{iqr}\phi(q) \qquad \delta^{(3)}(r) = \int \frac{d^3q}{(2\pi)^3} e^{iqr} \tag{3.3}$$

Recalling that the ∇^2 operator acts only on the r coordinates, we obtain

$$\int \frac{d^3q}{(2\pi)^3}(-\nabla^2 e^{iqr})\phi(q) = \int \frac{d^3q}{(2\pi)^3} e^{iqr}$$

$$\int \frac{d^3q}{(2\pi)^3} q^2\phi(q)e^{iqr} = \int \frac{d^3q}{(2\pi)^3} e^{iqr} \tag{3.4}$$

By comparison

$$q^2\phi(q) = 1 \implies \phi(q) = \frac{1}{q^2} \tag{3.5}$$

Now that the potential in Fourier space has been obtained, the potential in space is calculated r

$$\phi(r) = \int \frac{d^3q}{(2\pi)^3}\frac{e^{iqr}}{q^2} \tag{3.6}$$

Now the spherical coordinates are used

$$\begin{aligned}\phi(r) &= \frac{1}{(2\pi)^3}\int_0^{2\pi} d\varphi \int_0^{\pi} d\cos\theta \int_0^{\infty} dq\, \cancel{q^2}\frac{e^{iqr\cos\theta}}{\cancel{q^2}} \\ &= \frac{1}{(2\pi)^2}\int_0^{\infty} dq \int_0^{\pi} d\cos\theta\; e^{iqr\cos\theta} = \frac{1}{2\pi^2}\int_0^{\infty} dq \frac{e^{iqr} - e^{-iqr}}{2iqr}\end{aligned} \tag{3.7}$$

The change of variable is performed ($z = qr,\ dz = r\ dq$) and in this way we obtain an integral with the known result

$$\phi(r) = \frac{1}{2\pi^2}\int_0^{\infty}\frac{dz}{r}\frac{\sin z}{z} = \frac{1}{2\pi^2 r}\frac{\pi}{2} = \frac{1}{4\pi r} \tag{3.8}$$

3.2 D-dimensional case

Now we perform the calculations by extending the equation (3.2) in the general case to D dimensions

$$-\nabla^2\phi(r) = \delta^{(D)}(r) \tag{3.9}$$

The accounts in this case are to be performed identical from equation (3.3) to equation (3.4), remembering to generalize to D dimensions. Here too we get the same result of equation (3.5).
To be able to run the accounts more easily, use the Schrödinger parameter to rewrite better

$$\frac{1}{q^2} = \int_0^{\infty} d\beta\; e^{-\beta q^2} \tag{3.10}$$

You can rewrite the potential as

$$\phi(r) = \int_0^{\infty} d\beta \int \frac{d^Dq}{(2\pi)^D}\; e^{iqr-\beta q^2} = \int_0^{\infty} d\beta \int \frac{dq_1}{2\pi}\; e^{iq_1x_1-\beta q_1^2} \ldots \int \frac{dq_D}{2\pi}\; e^{iq_Dx_D-\beta q_D^2} \tag{3.11}$$

Every single integral dq_i can be solved using a Quantum Mechanics formula

$$\int dx\; e^{-(Ax^2+Bx+C)} = \exp\left(\frac{B^2}{4A} - C\right)\sqrt{\frac{\pi}{A}} \tag{3.12}$$

In the end we get

$$\begin{aligned}\phi(r) &= \frac{\pi^{D/2}}{(2\pi)^D}\int_0^\infty d\beta\beta^{-D/2}\prod_{i=1}^{D}\exp\left(-\frac{x_i^2}{4\beta}\right)\\ &= \frac{\pi^{D/2}}{(2\pi)^D}\int_0^\infty d\beta\;\beta^{-D/2}\exp\left(-\frac{\sum_i x_i^2}{4\beta}\right)\\ &= \frac{\pi^{D/2}}{(2\pi)^D}\int_0^\infty d\beta\beta^{-D/2}\;\exp\left(-\frac{r^2}{4\beta}\right)\end{aligned}\tag{3.13}$$

Now we change the variable $\beta = 1/t$, hence

$$\begin{aligned}\phi(r) &= \frac{\pi^{D/2}}{(2\pi)^D}\int_\infty^0 \left(-\frac{dt}{t^2}\right)t^{D/2}\exp\left(-\frac{r^2}{4}t\right)\\ &= \frac{\pi^{D/2}}{(2\pi)^D}\int_0^\infty dt\;t^{\frac{D}{2}-2}\exp\left(-\frac{r^2}{4}t\right)\end{aligned}\tag{3.14}$$

Using the formula for the Euler's gamma

$$\int dt\;t^\alpha\;e^{-\beta t} = \frac{1}{\beta^{\alpha+1}}\Gamma(\alpha+1)\tag{3.15}$$

we'll have

$$\phi(r) = \frac{\pi^{D/2}}{(2\pi)^D}\left(\frac{4}{r^2}\right)^{\frac{D}{2}-1}\Gamma\left(\frac{D}{2}-1\right) = \frac{\Gamma\left(\frac{D}{2}-1\right)}{4\;\pi^{D/2}\;r^{D-2}}\tag{3.16}$$

It is possible to rewrite this equation knowing the property $x\Gamma(x) = \Gamma(x+1)$

$$\begin{aligned}\phi(r) &= \frac{\Gamma\left(\frac{D}{2}-1\right)}{4\;\pi^{D/2}\;r^{D-2}}\;\frac{\left(\frac{D}{2}-1\right)}{\left(\frac{D}{2}-1\right)}\\ &= \frac{\Gamma\left(\frac{D}{2}\right)}{2\;\pi^{D/2}\;r^{D-2}\;(D-2)}\\ &= \frac{1}{\Omega(D)\;r^{D-2}\;(D-2)}\end{aligned}\tag{3.17}$$

where the solid angle variable has been defined $\Omega(D) = \dfrac{\Gamma\left(\frac{D}{2}\right)}{2\;\pi^{D/2}}$.

Using the equation (3.17) we analyze the two-dimensional case. We can see the presence of a divergence for D = 2. To tackle the problem, we introduce a parameter $\varepsilon > 0$ as small as you like in order to redefine $D \simeq 2+\varepsilon$ and perform a development in series. It is possible to observe how some parameters are developed

$$\begin{aligned}&r^{D-2} = r^\varepsilon = e^{\,\varepsilon\ln r} \simeq 1+\varepsilon\ln(r)+\frac{\varepsilon^2\ln^2(r)}{2}+O\Big((\varepsilon\ln(r))^3\Big)\\ &\Gamma\left(\frac{D}{2}\right) = \Gamma\Big(1+\frac{\varepsilon}{2}\Big) = 1-\gamma_E\frac{\varepsilon}{2}-\frac{\varepsilon}{2}\ln\pi-\varepsilon\ln r+\cdots\end{aligned}\tag{3.18}$$

with $\gamma_E = 0,544$. So you can rewrite the potential as

$$\phi(r) \simeq \frac{1}{2\pi\varepsilon}\Big(1-\gamma_E\frac{\varepsilon}{2}-\frac{\varepsilon}{2}\ln\pi-\varepsilon\ln r\Big) = \frac{1}{2\pi\varepsilon}-\frac{1}{4\pi}(\gamma_E+\ln\pi+2\ln r)\tag{3.19}$$

The term $1/(2\pi\varepsilon)$ has a singularity point for which there may be divergence, but the potential is always defined up to a constant, therefore choosing as constant

$$-\frac{\ln(r/r_0)}{2\pi}$$

it is possible to delete this singularity.

For D = 1 we will have a potential written in the form $\phi(r) = a + b \cdot r$; the potential no longer follows the characteristic $1/r$ trend. In other words, the concept of potential has changed according to size; it will no longer be like the inverse of the distance, but it will be linear. To better understand this concept, an example can be made by looking at figure 2. The mouse, which has a size equal to the diameter of the tube, will only be able to move in one direction (back and forth) while the movements up and down and left and right will be prevented; it will have a one-dimensional conception of space. The flea, small enough, will not have the same difficulties as the mouse and will be able to move in all directions; it will have a three-dimensional conception of space. Depending on the size of the observer, the space will be perceived differently and some conceptions will change. This idea could be innovative for the purpose of discovering the graviton; until a few years ago no one had thought of studying the effects of gravity for orders of magnitude lower than *cm*, so modern Physics is setting itself the goal, in the coming years, to study gravity under these conditions and to observe hypothetical changes to the potential.
It is possible to develop a model to describe a linear potential by creating a new magnetic monopole using two different magnetic monopoles, as can be seen in figure 3. We obtain a new magnetic monopole at the local level for which the magnetic field flux will not be zero due to the presence of the two magnetic monopoles. If we now place ourselves in a superconductor, at temperatures close to absolute zero, we can observe how the flux lines of the local magnetic field follow a linear trend, in order to minimize the volume of the field itself, as can be seen in figure 4.

This model is the basis for explaining the potential between quarks and gluons. Unlike all other particles, quarks and gluons exert a reciprocal force such that, as the distance between the two particles increases, the intensity of this force increases, as if there were a linear potential in distance. This would also explain why quarks and gluons do not exist isolated in nature but lived in a state of confinement.

Figure 2. Mouse and flea inside a sewer pipe. The two animals will conceive the surrounding space differently

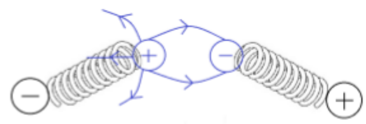


Figure 3. Local magnetic monopole where $\nabla \cdot B = \rho_M \neq 0$

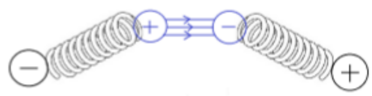


Figure 4. In a superconductor, at $T \sim 0\ K$, a linear magnetic flux is observed

PART

II

Elements of Group Theory

Group theory is one of the most important subject for the study of elementary particles and their properties. They come up since in physics we're very interested in symmetries and group theory serves us a very strong mathematical tool to study them.

In particle physics we're interested in a particular type of groups, the so called Lie Groups. The mathematical foundations upon which groups and Lie groups grow on, won't be treated since it's a very deep subject and, for the time being, it's not so relevant for our analysis. For the inclined reader I'll leave some snippets on the deeper meaning of some stuff.

4 Some definitions

4.1 An explicit example

Before mentioning the definition of a group, we'll see a well known example from which the idea of group can be easily understood. Take a point $(x, y) \in \mathbb{R}^2$ and apply a rotation to the well known transformation rule

$$\begin{cases} x' = x\cos\phi + y\sin\phi \\ y' = -x\sin\phi + y\cos\phi \end{cases} \tag{4.1}$$

that can be put in matrix form

$$R(\phi) = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \implies \mathbf{r}' = R(\phi)\mathbf{r}. \tag{4.2}$$

I could take two successive rotations and it would be clear that

$$\mathbf{r}'' = R(\phi_2)\mathbf{r}' = R(\phi_2)R(\phi_1)\mathbf{r} \tag{4.3}$$

or a could take a single rotation with an angle $\phi_1 + \phi_2$ and get to the same point. This means that

$$R(\phi_2)R(\phi_1) = R(\phi_1 + \phi_2) \tag{4.4}$$

This can be easily calculated using row by column matrix multiplication.

By much the same argument we can conclude that

$$R(\phi)R(-\phi) = 1 \implies R^{-1}(\phi) = R(-\phi) \tag{4.5}$$

Moreover, in two dimensions we have that

$$R(\phi_1)R(\phi_2) = R(\phi_2)R(\phi_1) \tag{4.6}$$

In three dimensions things change. First thing first we have two angles of rotation around an axis ω, we call them $\phi_\omega, \theta_\omega$. Every rotation is therefore defined by three parameters $\omega = (\cos\phi_\omega \sin\theta_\omega, \sin\phi_\omega \sin\theta_\omega, \cos\theta_\omega)\omega$. In vector form, a rotation is given by

$$\mathbf{r}' = R(\omega)\mathbf{r} \implies x_i' = R_{ij}(\omega)x_j. \tag{4.7}$$

The characteristic property of rotations is that they leave invariant the scalar product, given $v, w \in \mathbb{R}^3$ and $(v, w) = v_i w_i$, where Einstein summation convention is understood

$$(v, w) = (v', w') \implies v_j R_{ij} R_{ik} w_k = v_j \delta_{jk} w_k \implies R_{ij} R_{ik} = \delta_{ik} \tag{4.8}$$

So a rotation matrix is an orthogonal matrix since $R^{-1}(\omega) = R^T(\omega)$. In three dimensions relation 4.6 doesn't hold anymore.

4.2 Groups and representations

All the properties that rotations have are exactly the ones that define a group! Mathematically a group is defined as follows

Definition 4.1. A group $(G, \bullet)$ is a set with an operation

$$\bullet : G \times G \to G$$

with the following rules

- $\forall\, f, g \in G \implies f \bullet g \in G$
- $\forall\, f, g, h \in G \implies f \bullet (g \bullet h) = (f \bullet g) \bullet h$
- $\exists\, e \in G$ s.t. $f \bullet e = e \bullet f \qquad \forall f \in G$
- $\forall\, f \in G \qquad \exists\, f^{-1} \in G$ s.t. $f \bullet f^{-1} = f^{-1} \bullet f = e$

Whenever $\bullet$ is commutative, G is said to be abelian. A group can either be continuous or discrete depending on the number of elements of the group.
Most of the time the operation $\bullet$ on G won't be explicitly written and will be understood.

As said before, rotations belong to what is called **orthogonal** group. This is its definition

Definition 4.2. Given an n-dimensional vector space V equipped with a pseudo-inner product $(\cdot, \cdot)$ with Sylvester signature (p, q), we define the set

$$O(p, q) := \{\phi : V \xrightarrow{\sim} V \,|\, \forall v, w \in V : (\phi(v), \phi(w)) = (v, w)\} \tag{4.9}$$

together with function composition operation as the orthogonal group.

If this definition seems quite abstract it's because it is. A group is an abstract object. To make the group practically useful we use a **representation**. As we've seen before, rotations are elements of the orthogonal group and we wrote them down explicitly using matrices: we used a matrix representation of the orthogonal group. Representations are not unique and there are many of them depending on the space upon which they act.
The theory of representations is quite important in physics, so we'll give now a proper definition

Definition 4.3. A representation of a group G is a mapping D of the elements of G onto a set of linear operators

$$D : G \xrightarrow{\sim} \text{End}(V) \tag{4.10}$$

where V is some finite dimensional vector space, with the following properties

- $D(e) = 1$, where 1 is the identity operator in V
- $D(g_1)D(g_2) = D(g_1 g_2)\ \forall\ g_1, g_2 \in G$, in other words the group multiplication law is mapped onto the natural multiplication law in the space V. This is understood since we defined D as an isomorphism.

Let's make an example considering a quantum state ψ described by the angular momentum L. For every possible value that the angular momentum can take, the state ψ will be described by a linear combination of possible states; for example, for $L = 1$ the state ψ will be a linear combination of 3 possible states, those for which $l_z = +1, 0, -1$. If I perform the unitary transformation $e^{i\omega L}$, considering all the possible values of the angular momentum, I will have

$$|\psi\rangle' = e^{i\omega L}|\psi\rangle = e^{i\omega L}|0,0\rangle \oplus\ e^{i\omega L}(|1,1\rangle \oplus |1,0\rangle \oplus |1,-1\rangle) \oplus ..... \tag{4.11}$$

where each single term is the representation of an element of the angular momentum. In matrix terms I will instead have a block representation

$$|\psi\rangle' = \begin{pmatrix} 1\times 1 & & & \\ & 3\times 3 & & \\ & & 5\times 5 & \\ & & & \ddots \end{pmatrix} |\psi\rangle \tag{4.12}$$

We will therefore say that a representation is **reducible** when it is possible to decompose it into a direct sum of non-decomposable subgroup representations (equivalently a representation in matrix terms is reducible if it is possible to describe it by means of block matrices). A representation will instead be **irreducible** when the decomposition into representations of subgroups is not possible.
We'll see many representations of the various groups that are of some importance in particle physics like $SU(2), SU(3), U(1), SO(3,1)$ etc.

Using the matrix representation we can give an easier definition of the orthogonal group

Remark. The orthogonal group is the set of all matrices

$$O(p,q) = \{O \in GL(p+q, K) \mid OO^T = 1\} \tag{4.13}$$

where $GL(p+q, K)$ is the general linear group over a generic field K.

Remaining on the rotation group example. We see that another property of the orthogonal matrices is that

$$\det\big(O^T O\big) = \det\big(O^T\big)\det(O) = \det(O)^2 = \det(1) = 1 \implies \det(O) = \pm 1 \tag{4.14}$$

and so we can subdivide the set in two sets: one with the matrices with unitary determinant and one with determinant -1. Notice that i said that we divide into two **sets** and not into two **groups**. This is because only one of those sets is a group, mainly because the set with determinant -1 has no identity element. Therefore we define the following

Definition 4.4. A subgroup $H \leqslant G$ is a group closed under the product $\bullet$ on G restricted on H

$$\bullet|_H : H \times H \to H \tag{4.15}$$

The rotations for which $\det(O) = 1$ are called **proper rotations** and are indicated by

$SO(p,q) \leqslant O(p,q)$, the others are called **improper rotations** and are given by the proper ones times a parity inversion

$$P = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad \forall\, O \in SO(p,q) \implies \det(PO) = \det(P)\det(O) = -1 \tag{4.16}$$

4.3 Orthogonal and Unitary group

We take a short look now at some properties of tho orthogonal group and then introduce another very important group in physics, the **unitary** group.

From the definition of the orthogonal group, we would like to find how many parameters one need to fully describe the elements of $O(N)$. Surely, which can be easily seen by the matrix representation, we'll need at least N^2 parameters. Some of them are redundant. From the condition $OO^T = 1$ we have that, using again the matrix representation,

Just as a notation $O(N) = O(p+q,0)$.

$$(v_i, v_j) = 0 \qquad (v_i, v_i) = 1 \qquad \forall\, v_i, v_j \in V \subset \mathbb{R}^N \tag{4.17}$$

The first condition of 4.17 constrain $N(N-1)/2$ parameters, while the second constrain N parameters. So we can describe an element of the orthogonal group using

$$N^2 - \frac{N(N-1)}{2} - N = \frac{N(N-1)}{2} \tag{4.18}$$

elements.

Another very important group in particle physics is the unitary group. Let's speak of the unitary transformations U, which leave the system unchanged and act only on the internal indices; we will therefore speak of *internal symmetry*. In detail

$$|\phi^i\rangle' = U^i_j\, |\phi^j\rangle \tag{4.19}$$

The characteristic of these transformations is that $(U^\dagger)^i_j (U)^j_k = \delta^i_k$, or simply $U^\dagger U = 1$. As for the orthogonal group, we determine the number of free parameters for the unit group. The number of total parameters that define a unitary matrix are $2N^2$, since they are defined for both the real and the complex part. To this the unitarity constraint $U^\dagger U = 1$ must be removed which translates, in terms of matrix lines, as $(v_i, v_j) = \delta_{ij}$. For $i = j$ must be 1 and you will get a result similar to the orthogonal case, except that you must also consider the complex part, therefore $N(N-1)$. For $i \neq j$ must be 0; the complex part is not considered, so we will have N. The free parameters will be

$$2N^2 - N(N-1) - N = N^2 \tag{4.20}$$

Starting from the determinant of the matrices of the unitary group, it is possible to express the sign as if it were a phase, that is

$$U(N) = (e^{i\delta})SU(N) \tag{4.21}$$

SU(N) indicates the group of special unitary matrices, which have the property of having the determinant equal to +1. The number of free parameters for the matrices belonging to this group will be the same as that of the unit matrices to which, however, another constraint must be considered, namely that of selecting only determinants with a positive sign, therefore

$$2N^2 - N(N-1) - N - 1 = N^2 - 1 \tag{4.22}$$

The main consequence of having only positive determinants is that the special unit matrices are all **traceless**. In fact, by renaming the generator of this group $\hat{G}_i$, that is

$$SU(N) = I + i\hat{G}_i\alpha_i = e^{(i\sum_{i=1}^{N^2-1} G_i\alpha_i)} \tag{4.23}$$

we'll have

$$\det(SU) = \prod_i e^{(iG_i\alpha_i)} = e^{(i\,\mathrm{Tr}(G_i\alpha_i))} = 1 \tag{4.24}$$

For the equation 4.24 to be verified, it must be $Tr(G_i\alpha_i) = 0$, that is, zero trace.

5 Representation theory

A representation will be defined **adjoint** when the number of elements of a representation, that is the generators, coincides with the number of free parameters of that symmetry group.

5.1 Lie algebra

5.2 Cartan Subalgebra

5.3 Schur's lemma

6 Composition of various representations

6.1 $\mathbf{3} \otimes \mathbf{3}$ and $\mathbf{3} \otimes \bar{\mathbf{3}}$

6.2 $\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3}$

6.3 $\mathbf{8} \otimes \mathbf{8}$

PART

III

The Quark Model

7 The Heisenberg model of nuclei

In the early days of particle physics where proposed many ways of explaining the whole set of particle that were found by scattering experiments. The particles where so many that everybody thought that it would be impossible for them to be elementary. Another problem that they had to explain was that certain particles were produced always together. One of the first way in which symmetry was used to describe an experimental evidence was by Heisenberg: it was evident at the time that proton and neutrons inside a nuclei behaved in much the same way if it wasn't for the electric charge. Isotopes had the same energy. Now we know that, by looking at the scale of EM interactions with respect to the the strong interaction which binds the nucleus together, inside nuclei protons and neutrons do not experience EM interactions or at least they are much suppressed.
Heisenberg proposed to introduce a new quantum number, the **isospin**. Much like the Dirac index for spin ψ_α, we could define a nucleon with a Dirac index

$$\underset{\text{Dirac index}}{\underset{\downarrow}{N_\alpha(x)}} = \begin{pmatrix} p_\alpha(x) \\ n_\alpha(x) \end{pmatrix} \begin{matrix} \leftarrow \text{ proton} \\ \leftarrow \text{ neutron} \end{matrix} \tag{7.1}$$

where the operator for the isotopic spin τ has eigenvalues for the projection on a given "axis"

$$\tau_3^p = \frac{1}{2} \qquad \tau_3^n = -\frac{1}{2} \tag{7.2}$$

The isotopic spin is the quantum number associated with the symmetry which changes protons with neutrons because we know that, inside a nuclei, protons and neutrons are just the same particle. This symmetry is a sort of rotation, given by a unitary transformation U which transforms N_α as

$$N'_\alpha(x) = U^\beta{}_\alpha N_\beta(x) \qquad U^\beta{}_\alpha = \left(e^{i\frac{\omega\cdot\tau}{2}}\right)^\beta_\alpha \tag{7.3}$$

8 The Gell-Mann model of particles

What Gell-Mann proposed in 1964 was that any particle could be seen as made of 3 new light particles called u, d, s, the **quarks**. In this model the three quarks form three states

$$|u\rangle = |1\rangle \equiv (1,0,0) \qquad |d\rangle = |2\rangle \equiv (0,1,0) \qquad |s\rangle = |3\rangle \equiv (0,0,1) \tag{8.1}$$

Much like the proton and the neutron are states of a single entity which differ only by the isospin which, in turn, it's the result of an underlying symmetry of nature, the group symmetry of the quarks is $SU(3)_f$, where f standa for **flavour**. The more general physical state, such as a particle, is an arbitrary linear combination of these three quarks

$$q(x) = u(x)\,|1\rangle + d(x)\,|2\rangle + s(x)\,|3\rangle = \begin{pmatrix} u(x) \\ d(x) \\ s(x) \end{pmatrix} \tag{8.2}$$

where $u(x), d(x), s(x)$ are fermionic fields end are choose whether we want them to be eigenvectors or not of some physical quantum number.

In $SU(3)$ there're only two diagonal operator which form the Cartan algebra, namely the third isospin component τ_3 and the hypercharge Y, given by the Gell-Mann matrices

$$\tau_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad Y = \frac{1}{3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \tag{8.3}$$

Forming the Cartan algebra of $SU(3)$ means that $[\tau_3, Y] = 0$. Moreover, by hypotesis, $SU(3)$ is a symmetry of the theory, thus the hamiltonian H must commute with the generators, and in particular with the Cartan algebra. This implies that we can find final states which are simultaneously eigenvectors of energy-momentum, isospin and hypercharge, or by using the Gell-Mann Nishijima formula

$$Q = \tau_3 + \frac{Y}{2} \tag{8.4}$$

we can define the eigenstates based upon energy-momentum and charge. Thus for the three quarks the quantum numbers are

$$(\tau_u, Y_u, Q_u) = \left(\frac{1}{2}, \frac{1}{3}, \frac{2}{3}\right) \qquad (\tau_d, Y_d, Q_d) = \left(-\frac{1}{2}, \frac{1}{3}, -\frac{1}{3}\right) \qquad (\tau_s, Y_s, Q_s) = \left(0, -\frac{2}{3}, -\frac{1}{3}\right) \tag{8.5}$$

We can now classify the baryons, made by three quarks, and the mesons, made by two quarks, on the basis of the quantum numbers of their constituents. For example, a proton is made by two up and a down quark

$$P = uud \tag{8.6}$$

and it's electric charge is $Q_p = 1 = 2\ Q_u + Q_d$. The proton, as seen before, has isospin 1/2. In a. non relativistic picture we can imagine that the proton is a wave function, written in terms of the quark wave functions, all in a s wave configuration, being a spin half particle, and has the form

$$\underset{\substack{\uparrow \\ \text{Spin projection}}}{P^{\alpha}} \sim (u^{\uparrow}d^{\downarrow} - u^{\downarrow}d^{\uparrow})u^{\alpha=\uparrow,\downarrow} \tag{8.7}$$

This is a possible proton state: the wave function of the quarks is antisymmetric in both the spin and the isospin and thus this corresponds to zero angular momentum and isotopic spin. Thus the spin/isospin of the proton is fully given by the external quark.

Within $SU(3)$, mesons and baryons must be organised in irreducible representations of the group. Within a given multiplet, in the absence of symmetry breaking, the baryons are degenerate, with a mass of the order of the scale of strong interactions $\Lambda_{QCD} \sim M_P \sim 4\pi f_\pi \sim 1$ GeV, where M_p is the mass of the proton and f_π is the pion decay constant which will be described later on.

8.1 Baryons

Baryons are particles made up of three quarks, so they must come from the product of three fundamental representations of $SU(3)$ which can be decomposed in the following irreducible representations

$$\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} = \mathbf{1} \oplus \mathbf{8} \oplus \mathbf{8'} \oplus \mathbf{10} \tag{8.8}$$

The lightest baryons are classified within one of the octets (figure 5), namely they belong to the adjoint representation. In the quark model one can imagine that, in order to minimize the energy, they correspond to state where all the quarks have zero orbital angular momentum and the spin of the hadron is just given by the spin of the three quarks. Within the octet, it is possible to classify the states on the basis of the isospin which is a subgroup of $SU(3) \geqslant SU(2)$. According to the idea by Heisenberg, proton and neutrons form are in the same $\tau = 1/2$ doublet, as are Ξ^- and Ξ^0, which however are made up of two strange quarks. The $\Sigma^0, \Sigma^+, \Sigma^-$ form a triplet of isotopic spin $\tau = 1$. The Λ^0 is an isotopic spin singlet $\tau = 0$. The strangeness is given by

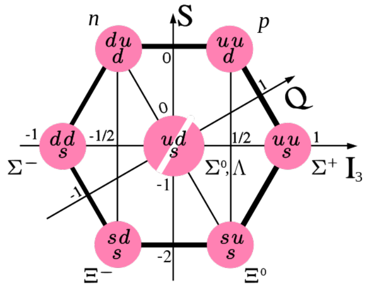


Figure 5. The light baryon spin-$\frac{1}{2}$ octet.

$$S = Y - B \tag{8.9}$$

where B is the baryon number and is given by the quadratic Casimir operators of $SU(3)$

$$B = \frac{1}{3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{8.10}$$

which is just a constant times the identity matrix, by virtue of Schur's lemma. So every quark has baryon number 1/3.

The octet is nearly degenerate in mass, as it should, with a mass

$$m_{\mathbf{8}} \approx 1100 \text{ MeV}. \tag{8.11}$$

If we take the quark masses to be zero, the masses of the hadrons should be around the rage of strong interactions, namely 900 MeV. If we switch on the masses of the quarks, the correction will be very small since

$$m_u \approx 2,3 \text{ MeV} \qquad m_d \approx 4,8 \text{ MeV} \qquad m_s \approx 95 \text{ MeV}. \tag{8.12}$$

Moreover, beside the mass corrections, there are the EM corrections which break isospin symmetry. But this correction is very very small since $\alpha = 1/137$. So the $SU(2)$ symmetry of the $SU(3)$ octet is a very good one, since the corrections are very small.

Remark. One would think that the **8** and the **8′** would be degenerate in mass since, for example, in the hydorgen atom the representations with $l = 0$ and $l = 1$ are degenerate. This result is just an accident since in the hydrogen atom there's another underlying symmetry: the Runge-Lenz symmetry. This symmetry is easily broken whenever we add relativistic corrections.

Beside the octet, the lightest baryons are classified within the decuplet (figure 6). The decuplet baryons have spin 3/2. The decuplet is divided as follows: there are four baryons with isotopic spin 3/2 with strangeness 0, the Δs. The average mass of the tetraplet is

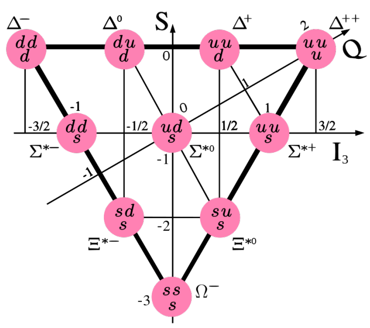


Figure 6. The light spin-$\frac{3}{2}$ decuplet.

$$m_\Delta = 1232 \text{ MeV}. \tag{8.13}$$

Remark. Let's consider the Δ^{++} particle. The wave function describing it will be

$$\psi(x) = \psi_{spatial}(x)\psi_{flavour}(x)\psi_{spin}(x) \tag{8.14}$$

and must be antisymmetric under particle exchange, according to Pauli's principle. In the ground state the spatial wave function is symmetric as the flavour one, since it's made up of 3 up quarks. This particle has +3/2 spin projection, so the spin wave function is symmetric. This wave function violates Pauli's principle for fermions, so the physicist Greenberg introduced a new quantum number, in order to solve this problem, the **colour**. This new quantum number must be antisymmetric under particle exchange

$$\Delta^{++} = u_a^{\uparrow} u_b^{\uparrow} u_c^{\uparrow} \varepsilon^{abc} \tag{8.15}$$

Then we have the isotopic triplet 1/2 with strangeness −1, the Σ^*s. The average mass of the triplet is

$$m_{\Sigma^*} = 1386 \text{ MeV}. \tag{8.16}$$

There's the Ξ^*s doublet with $I_3 = \pm 1$ and $S = -2$, with an average mass of

$$m_{\Xi^*} = 1533 \text{ MeV} \tag{8.17}$$

and, last but not least, the singlet state with $I_3 = -3$ and $S = -2$, the famous Ω^-, with a mass

$$m_{\Omega^-} = 1672 \text{ MeV}. \tag{8.18}$$

Remark. Remember that to every particle there's a corresponding antiparticle, so there exists a Ω^+ with chage $Q_{\Omega^+} = +1$ and $Y_{\Omega^+} = +2$.

Not all the members of the possible $SU(3)$ multiplets have been observed. In some cases these particles are unstable and decay via strong interaction. This means that their lifetime is very small and consequently their width is very large. In this cases it's difficult to speak of particles since they are very hard to identify. Particles which only decay weakly or electromagnetically, instead, have a much longer lifetimes and are much easier to identify as narrow resonances.

8.2 Mesons

Light mesons are made up of a quark-antiquark pair, therefore they correspond to the tensor product of two $SU(3)$, which is decomposible in the following way

$$\mathbf{3} \otimes \bar{\mathbf{3}} = \mathbf{1} \oplus \mathbf{8} \tag{8.19}$$

where the octet is the traceless tensor

$$M^i{}_j = q^i \bar{q}_j - \frac{1}{3} \delta^i{}_j q^k \bar{q}_k \tag{8.20}$$

and the singlet is the trace. Mesons come in a variety of types depending on their spin: pseudoscalar mesons, which have spin-parity $J^P = 0^-$, scalar mesons, with $J^P = 0^+$, vector mesons, with spin parity $J^P = 1^-$, pseudovector mesons, with spin-parity $J^P = 1^+$, and tensor mesons with $J^p = 2^+$. Let us first discuss the pseudoscalar mesons (figure 7).

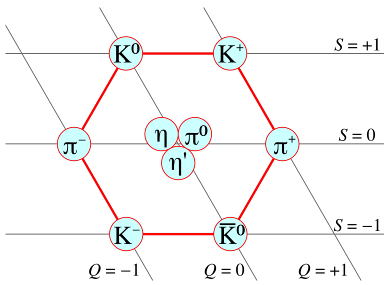


Figure 7. Pseudoscalar meson $J^P = 0^-$ multiplet.

Take as an example the positive charged pion, formed by a par up-antidown quark

$$\pi^+ = u\bar{d} \tag{8.21}$$

Its isotopic spin is given by

$$|u\rangle \oplus |\bar{d}\rangle = \left|\frac{1}{2}, \frac{1}{2}\right\rangle \oplus \left|\frac{1}{2}, \frac{1}{2}\right\rangle = |1, 1\rangle \tag{8.22}$$

and the electric charge is given by

$$Q_{\pi^+} = Q_u + Q_{\bar{d}} = \frac{2}{3} + \frac{1}{3} = 1. \tag{8.23}$$

Similarly, the isospin of the positive charged kaon $K^+ = u\bar{s}$ is easily found in much the same way

$$|u\rangle \oplus |\bar{s}\rangle = \left|\frac{1}{2}, \frac{1}{2}\right\rangle \oplus |0, 0\rangle = \left|\frac{1}{2}, \frac{1}{2}\right\rangle \tag{8.24}$$

and the electric charge

$$Q_{K^+} = Q_u + Q_{\bar{s}} = \frac{2}{3} + \frac{1}{3} = 1 \tag{8.25}$$

since the charge of the antistrange is the same as the charge of the antidown. The two zero-isospin mesons correspond to a member of the octet denoted as η, and the singlet η'. They are given by the following quark-antiquark combinations

$$\eta \propto \bar{q}\lambda_8 q = \frac{\bar{u}u + \bar{d}d - 2\bar{s}s}{\sqrt{6}} \qquad \eta' \propto \bar{q}\mathbb{1}q = \frac{\bar{u}u + \bar{d}d + \bar{s}s}{\sqrt{3}} \tag{8.26}$$

In figure 8 we give the octet of vector mesons.

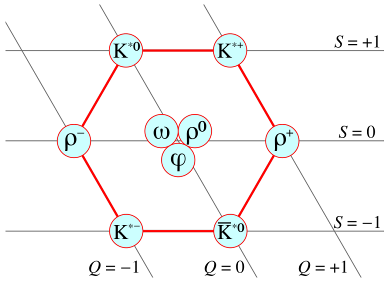


Figure 8. Vector meson $J^P = 1^-$ multiplet.

PART

IV

$\lambda\phi^4$ Theory

We will begin to study a simple toy model which will give us some basic formalism on the perturbative expansion for interacting field. This toy theory is the so called $\lambda\phi^4$ theory, and it's the theory of a real (or complex) scalar field with a self interaction term. Although it might seem a purely pedagogical example, this kind of theory enter in the physics of the real world. For example it can describe the self interaction of the Higgs field.

9 Quick digression: the Yukawa interaction

Before starting with the quartic theory, it may seems reasonable to mention the Yukawa interaction, named after Hideki Yukawa, that is an interaction between a scalar field (or pseudoscalar field) ϕ and a Dirac field ψ of the type

$$\mathcal{L}_{int} = Y\bar{\psi}\phi\psi \tag{9.1}$$

where Y corresponds to the dimensionless coupling constant of the interaction. A Yukawa interaction can be used to describe the nuclear force between nucleons, which are fermions, mediated by pions, which are pseudoscalar mesons. Moreover it's also used in the Standard Model to describe the coupling between the Higgs field and massless quark and lepton fields, i.e. the fundamental fermion particles. Only with the introduction spontaneous symmetry breaking we could see these fermions acquire a mass, but we will treat this discussion later on in the next chapters.

10 The real scalar field

10.1 Building up the perturbative expansion

We first consider a Klein-Gordon field, which lagrangian is

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{m^2}{2}\phi^2. \tag{10.1}$$

This lagrangian defines a free field theory. The only interesting quantity we can extract from this theory is the free propagator which is determined by the quadratic part in the lagrangian

$$D(p) = \frac{i}{p^2 - m^2 + i\epsilon} \qquad \Delta_F(x) = \int \frac{d^4p}{(2\pi)^4}\frac{e^{-ip_\mu x^\mu}}{p^2 - m^2 + i\epsilon} \tag{10.2}$$

From this theory, we cannot extract anything else. We therefore introduce a self-interaction term

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{m^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4. \tag{10.3}$$

This is the only possible term we can add without disrupting renormalizability, dimensionality and lower boundedness of the energy.
The equation of motion arising from this lagrangian is

$$\left(\partial_\mu\partial^\mu + m^2\right)\phi = -\frac{\lambda}{3!}\phi^3 \tag{10.4}$$

which is not easily solvable. We therefore turn to perturbation theory to find an approximate solution for the propagator of the theory.

There's now a very important remark: while, for a free theory, the propagator is given by the vacuum expectation value of time ordered product fields, like

$$\langle 0| T\{\phi_1\phi_2\} |0\rangle = \tag{10.5}$$

for an interacting theory, the ground state of the theory need not be the vacuum $|0\rangle$. Therefore, for whatever process we need to evaluate, we'll need to calculate something like

$$\langle \Omega| T\{\phi_1\phi_2\} |\Omega\rangle \tag{10.6}$$

where now $|\Omega\rangle$ is the real ground state of the theory. However, even in the case of interacting theory, we can restrain ourselves to the calculation of vacuum expectation values, with some caveats that we'll see later, since the expectation value of eq 10.6 can be written as[1]

$$\langle \Omega| T\{\phi(x)\phi(y)\} |\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} \frac{\langle 0| T\Big\{\phi_I(x)\phi_I(y) \exp\Big[-i\int_{-T}^{T} dt\, H_I(t)\Big]\Big\} |0\rangle}{\langle 0| T\Big\{\exp\Big[-i\int_{-T}^{T} dt\, H_I(t)\Big]\Big\} |0\rangle} \tag{10.7}$$

[1] This is a really deep formula. A proof for it can be found on Peskin-Schröder.

The vacuum expectation values are then calculated using Wick's theorem, which we recall here

Theorem 10.1 (Wick's theorem).

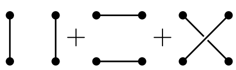
$$T\{\phi_1 \cdots \phi_n\} = :\phi_1 \cdots \phi_n: + \text{ all possible contractions of two fields} \tag{10.8}$$

where $:\phi_1 \cdot \phi_n:$ is the normal ordering. As an example, the T-product of three fields becomes

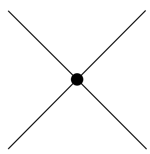
$$T\{\phi_1\phi_2\phi_3\} = :\phi_1\phi_2\phi_3 + \contraction{}{\phi_1}{}{\phi_2}\phi_1\phi_2\phi_3 + \contraction{}{\phi_1}{\phi_2}{\phi_3}\phi_1\phi_2\phi_3 + \phi_1\contraction{}{\phi_2}{}{\phi_3}\phi_2\phi_3 : \tag{10.9}$$

In the free theory, if we wanted to calculate the 4-point function, we would have constructed, using Wick's theorem, all the possible diagrams with two lines joining the four space-time points. Diagrammatically this boils down to

$$\langle 0| T\{\phi_1\phi_2\phi_3\phi_4\} |0\rangle = \tag{10.10}$$



Note that in the third diagram the lines do not intersect. In the self interacting theory we have another contribution from the interaction vertex as we can see on figure 9.



$= -i\lambda$

Figure 9. Self interaction vertex for the $\lambda\phi^4$-theory.

So let us return to the evaluation of the two point function $\langle \Omega| T\{\phi_1\phi_2\} |\Omega\rangle$ by using the formula 10.7. For now let us focus only on the numerator, we'll treat the denominator later on. The numerator can be expanded in power series

$$\langle 0| T\Big\{\phi_1\phi_2 + \phi_1\phi_2\Big[-i\int d^4z \frac{\lambda}{4!}\phi^4\Big] + \cdots\Big\} |0\rangle \tag{10.11}$$

The first term just gives the free field propagator. The second term, which for clarity we write down as

$$S^{(1)} = \langle 0| T\Big\{\phi_1\phi_2\Big[-i\int d^4z \frac{\lambda}{4!}\phi(z)\phi(z)\phi(z)\phi(z)\Big]\Big\} |0\rangle \tag{10.12}$$

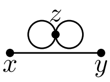
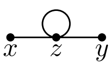
can be evaluated using Wick's theorem with a caveat: since we're calculating vacuum expectation values, all the terms, in which a pair of non contracted fields appear, cancel since, by definition, their normal ordering gives zero acting on the vacuum. By this mean, remain only terms in which all the fields are contracted. There are 15 ways of contracting 6 fields in pairs (the combination are easily found using the binomial coefficients). Fortunately only two of them are really different, all the others are topologically the same diagrams. For completeness sake I'll mention some of them

$$T\left\{\phi_1\phi_2\left[-i\int d^4z\frac{\lambda}{4!}\phi(z)\phi(z)\phi(z)\phi(z)\right]\right\} = \left(-\frac{i\lambda}{4!}\right)\int d^4z\,\phi_1\phi_2\left(\phi_3\phi_3\phi_3\phi_3 + \phi_3\phi_3\phi_3\phi_3 + \phi_3\phi_3\phi_3\phi_3\right) +$$
$$+\left(-\frac{i\lambda}{4!}\right)\int d^4z\,\left(\phi_1\phi_2\phi_3\phi_3\phi_3\phi_3 + \phi_1\phi_2\phi_3\phi_3\phi_3\phi_3 + \phi_1\phi_2\phi_3\phi_3\phi_3\phi_3 + \phi_1\phi_2\phi_3\phi_3\phi_3\phi_3 + \cdots\right) \tag{10.13}$$

From this, we can easily see that we have two diagrams that can be made in many different ways. For the calculation of the vacuum expectation value, diagrams that are topologically the same, give the exact same result. As a Feynman diagram, this two factors are

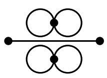
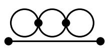
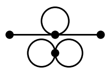
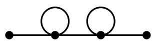
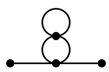
(10.14)

Note that the first one is the product of two diagrams, namely a vacuum-vacuum diagram with a free propagator. Diagrams such as this one are called **disconnected**. The second diagram is called **tadpole**.
The first order expansion of the two point function thus is given by the sum of all the diagrams times the interaction vertex, times a combinatorial factor which depends on how the ways we can construct the same diagrams. In formulas we have

$S^{(1)} = 3$  $+ 12$  $=$

$$= 3\left(\frac{i\lambda}{4!}\right)i\Delta_F(x-y)\int d^4z\,i\Delta_F(z-z)i\Delta_F(z-z) + 12\left(\frac{i\lambda}{4!}\right)\int d^4z\,i\Delta_F(x-z)i\Delta_F(z-z)i\Delta_F(y-z)$$
$$= \frac{i\lambda}{8}i\Delta_F\int d^4z\,i\Delta_F i\Delta_F + \frac{i\lambda}{2}\int d^4z\,i\Delta_F i\Delta_F i\Delta_F \tag{10.15}$$

We could easily go now to second order by making up all the possible diagrams with 2 external vertices and 2 internal vertices

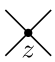
$S^{(2)} = c_1^{(2)}$  $+ c_2^{(2)}$  $+ c_3^{(2)}$  $+ c_4^{(2)}$  $+ c_5^{(2)}$ 

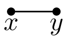
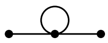
(10.16)

and so on to further orders.

We can begin now to write down the Feynman rules for the $\lambda\phi^4$-theory

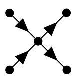
Remark. Position space Feynman rules:

- For every vertex  there is a factor $-i\lambda\int d^4z$
- For every n lines that can be exchanged without changing the diagram there is a factor $1/n!$. The factor $n!$ is called **symmetry factor**

- For every line  there is a propagator $i\Delta_F(x-y)$
- For every line there is a factor 1
- For every tadpole  there is a factor $\frac{1}{2}$

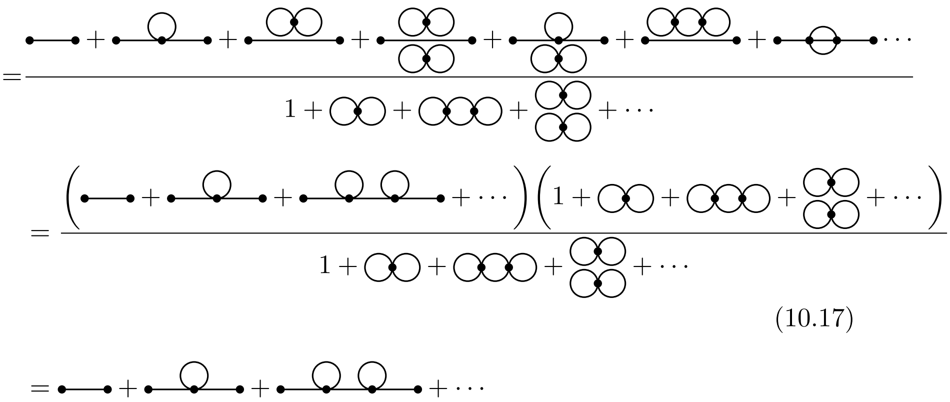
One way to interpret these rules is to think of the vertex factor $-i\lambda$ as the amplitude of the emission and/or absorption of particles at a vertex. The integral $\int d^4z$ instructs us to sum over all points in space-time where this process can occur. This is just the result of superposition.
It is convenient most of the times to evaluate Feynman diagrams in momentum space. The Feynman rules do not change much, we just have to take the Fourier transform of the various factors which will leave us with some delta functions which impose momentum conservation at vertices.

Remark. Feynman rules in momentum space

- For every vertex  there is a factor $-i\lambda$
- For every line  there is a propagator $D(p)$
- For every external line  there is a plane wave factor $e^{-ipx}D(p)$ if the momentum is entering the vertex, or a factor $e^{ipx}D(p)$ if the momentum is exiting the vertex
- At every vertex we must impose momentum conservation $(2\pi^4)\delta^{(4)}(\sum_{\text{initial}} p - \sum_{\text{final}} p)$
- On every loop we integrate on the momentum of the loop $\int \frac{d^4q}{(2pi)^4}$
- Divide by the symmetry factor

10.2 The grand finale

It's easy to see now that the denominator of the 10.7 contains only vacuum-vacuum diagrams. Moreover, we can see from equation 10.15, that there are some diagrams that are just free propagators times vacuum-vacuum diagram or tadpoles with vacuum-vacuum diagrams. This concept will help us understand the final form of the propagator in an interacting scalar theory. In fact, if we write down using Feynman diagrams the equation 10.7 we would get something like this

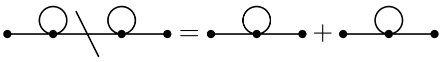
$\langle\Omega|\, T\{\phi_1\phi_2\}\, |\Omega\rangle =$ 

(10.17)

Incredibly enough, vacuum diagrams do not contribute[2] to the 2 point functions for a $\lambda\phi^4$-

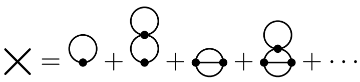
[2] Again, a more "formal" proof can be found on Peskin-Schroeder.

theory, since their contribution is eliminated by the denominator! You can even see how powerful the Feynman diagrams are since we got this result just by computing the essential parts and then adapting to all the others. This result works quite generally for all n-point functions.
Now that we know that the propagator of an interacting theory is just given by the infinite sum of its connected diagrams, we can do a further simplification.
Whenever a diagram can be cut into diagrams that still make sense, we call that diagram a **reducible** diagram. An example of a reducible diagram is the two tadpole diagram which can be cut into two one-tadpole diagrams



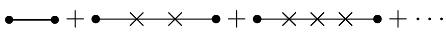
(10.18)

Diagrams that cannot be cut are called **irreducible diagrams**. If we cut off the external legs to the diagrams we get the so called **amputated diagrams**. Without giving the proof to this, which is quite lengthy and can be found on Peskin Schroeder, we conclude the following: for the evaluation of n-point functions only the sum of all **irreducible, amputated** diagrams has to be evaluated. If we define

$$D_0^{-1}(p^2)D(p^2)D_0^{-1}(p^2) =$$ 

(10.19)

then the propagator is given by the following sum

$$\langle\Omega|\, T\phi_1\phi_2\, |\Omega\rangle =$$ 

(10.20)

Therefore, if we define the amputated diagram series

$$-i\Delta(p) = \sum_{\substack{\text{All possible}\\ \text{irriducible diagrams}}} \binom{\text{Value of the irriducible}}{\text{amputated diagram}} =$$ 

(10.21)

which is called **self energy**, and let us evaluate the propagator for the interacting theory

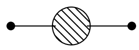
$$\begin{aligned}
\langle\Omega|\, T\phi_1\phi_2\, |\Omega\rangle =& \frac{i}{p^2 - m^2 + i\epsilon} \\
&+ \frac{i}{p^2 - m^2 + i\epsilon}(-i\Delta(p))\frac{i}{p^2 - m^2 + i\epsilon} \\
&+ \frac{i}{p^2 - m^2 + i\epsilon}(-i\Delta(p))\frac{i}{p^2 - m^2 + i\epsilon}(-i\Delta(p))\frac{i}{p^2 - m^2 + i\epsilon} + \cdots \\
&= \frac{i}{p^2 - m^2 + i\epsilon}\left(\frac{1}{1 - (-i\Delta(p))}\right)\frac{i}{p^2 - m^2 + i\epsilon} \\
&= \frac{i}{p^2 - m^2 - \Delta(p) + i\epsilon}
\end{aligned} \tag{10.22}$$

where we have used the formula for the geometric series. We can go back to the propagator in position space just by Fourier transform

$$\langle\Omega|\, T\phi_1\phi_2\, |\Omega\rangle = \int \frac{d^4p}{(2\pi)^4}\frac{i}{p^2 - m^2 - \Delta(p) + i\epsilon}e^{-ip(x-y)} \tag{10.23}$$

To comprehend the result we notice that the mass of a particle is the pole of the propagator. From the result 10.22 we conclude that the interaction of the field with the vacuum changes the mass that the particle would have if it didn't interact with it.

We come to the big conclusion that

Free theory	Interacting theory
$\langle 0\| T\phi_1\phi_2 \|0\rangle$	$\langle \Omega\| T\phi_1\phi_2 \|\Omega\rangle$
	
$\dfrac{i}{p^2 - m_0^2 + i\epsilon}$	$\dfrac{i}{p^2 - m^2 + i\epsilon}$ (10.24)

where the real mass is now $m^2 = m_0^2 + \Delta(p)$.

11 The complex scalar field

Just a few words for the case of a complex, self-interacting, scalar field. We know that the complex scalar lagrangian, that is

$$\mathcal{L} = (\partial_\mu \phi^\dagger)(\partial^\mu \phi) - m^2 \phi^\dagger \phi \tag{11.1}$$

can be constructed from two real scalar fields with their complex combination

$$\begin{aligned} \phi(x) &= \frac{1}{\sqrt{2}}(\sigma(x) + i\pi(x)) \qquad & \phi^\dagger(x) &= \frac{1}{\sqrt{2}}(\sigma(x) - i\pi(x)) \\ \sigma(x) &= \frac{1}{\sqrt{2}}(\phi(x) + \phi^\dagger(x)) \qquad & \pi(x) &= \frac{1}{i\sqrt{2}}(\phi(x) - \phi^\dagger(x)) \end{aligned} \tag{11.2}$$

so that the lagrangian is just the sum of two real lagrangians

$$\mathcal{L} = \frac{1}{2}(\partial^\mu \sigma)(\partial_\mu \sigma) + \frac{1}{2}(\partial^\mu \pi)(\partial_\mu \pi) - \frac{m^2}{2}\sigma^2 - \frac{m^2}{2}\pi^2. \tag{11.3}$$

In this theory, the free propagators of the two fields are identical since they have the same mass

$$\begin{aligned} \langle 0| T\{\sigma_1\sigma_2\} |0\rangle &= \frac{1}{2}\langle 0| T\left\{\phi_1\phi_2^\dagger\right\} |0\rangle + \frac{1}{2}\langle 0| T\left\{\phi_1^\dagger\phi_2\right\} |0\rangle = \int \frac{d^4p}{(2\pi)^4}\frac{i}{p^2 - m^2 + i\epsilon}e^{-ip(x-y)} \\ \langle 0| T\{\pi_1\pi_2\} |0\rangle &= \frac{1}{2}\langle 0| T\left\{\phi_1\phi_2^\dagger\right\} |0\rangle + \frac{1}{2}\langle 0| T\left\{\phi_1^\dagger\phi_2\right\} |0\rangle = \int \frac{d^4p}{(2\pi)^4}\frac{i}{p^2 - m^2 + i\epsilon}e^{-ip(x-y)} \end{aligned} \tag{11.4}$$

To introduce the interaction term we notice that there are two possible terms which preserve the hermiticity of the lagrangian

$$-\lambda(\phi^\dagger\phi)^2 \qquad Y\left(\left(\phi^\dagger\right)^2 + (\phi)^2\right)^2 \tag{11.5}$$

Between them only the first one is globally $U(1)$ invariant, so we choose that.

Consider then the complex scalar, self-interacting, lagrangian

$$\mathcal{L} = (\partial^\mu \phi^\dagger)(\partial_\mu \phi) - m^2\phi^\dagger\phi - \lambda(\phi^\dagger\phi)^2 \tag{11.6}$$

which is equivalent to the two real scalar field, self interacting lagrangian

$$\mathcal{L} = \frac{1}{2}\partial^\mu\sigma\partial_\mu\sigma + \frac{1}{2}\partial^\mu\pi\partial_\mu\pi - \frac{m^2}{2}\left(\sigma^2 + \pi^2\right) - \frac{\lambda}{4}\left(\sigma^2 + \pi^2\right)^2 \tag{11.7}$$

The global $U(1)$ transformation on the complex field ϕ acts on the real fields as follows

$$\phi(x) \to e^{i\theta}\phi(x) = (\cos\theta + i\sin\theta)\frac{\sigma + i\pi}{\sqrt{2}} = \frac{1}{\sqrt{2}}[(\sigma\cos\theta - \pi\sin\theta) + i(\sigma\sin\theta + \pi\cos\theta)]$$
$$\phi^\dagger(x) \to \phi^\dagger(x)e^{-i\theta} = \frac{1}{\sqrt{2}}[(\sigma\cos\theta - \pi\sin\theta) - i(\sigma\sin\theta + \pi\cos\theta)] \tag{11.8}$$

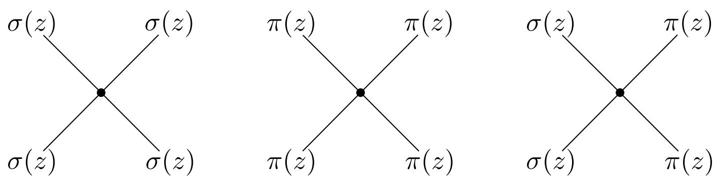
which can be equivalently written down as

$$\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}\begin{pmatrix} \phi(x) \\ \phi^\dagger(x) \end{pmatrix} \qquad \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \sigma(x) \\ \phi(x) \end{pmatrix} \tag{11.9}$$

Note that the potential term is invariant under rotations and that, since the two field, while rotated, are being mixed together, they have to be degenerate in mass.
The interaction lagrangian is therefore

$$\mathcal{L}_I = -\frac{\lambda}{4}(\sigma^2 + \pi^2)^2 = -\frac{\lambda}{4}(\sigma^4 + 2\sigma^2\pi^2 + \pi^4) \tag{11.10}$$

from which we get three possible interaction vertices



(11.11)

The Feynman rules for the first two diagrams are the same as for the real scalar field with a different symmetry factor- For the third vertex we can observe that contractions between π and σ fields are zero. As an example, the interaction vertex for the first two diagrams leads to a factor -6λ, which comes from the 4! possible combination of the lines times the constant $-\lambda/4$, whereas the last diagram leads to a factor $-\lambda$, which comes from the possible the 2 possible combination of the lines, both for π and σ, times the same factor.

12 Scalar electrodynamics

12.1 The theory

What if we impose the $U(1)$ symmetry to be local instead of global? We know from QED that imposing a gauge symmetry is equivalent to substituting to the normal derivative the following covariant derivative

$$\partial_\mu \mapsto D_\mu = \partial_\mu - ieA_\mu \tag{12.1}$$

where A_μ is the 4-potential, such that

$$D_\mu\phi \to e^{ie\theta(x)}D_\mu\phi \tag{12.2}$$

for which the gauge transformation becomes

$$\phi(x) \to e^{ie\theta(x)}\phi(x) \quad \phi^\dagger(x) \to \phi^\dagger(x)e^{-ie\theta(x)} \quad A_\mu(x) \to A_\mu(x) - \partial_\mu\theta(x) \tag{12.3}$$

and therefore the lagrangian is gauge invariant by construction

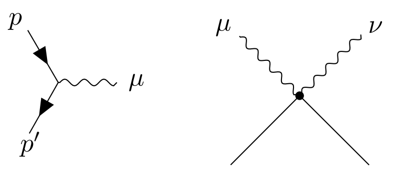
$$\begin{aligned}\mathcal{L} &= (D^\mu\phi^\dagger)(D_\mu\phi) - m^2\phi^\dagger\phi = (\partial^\mu + ieA^\mu)\phi^\dagger(\partial_\mu - ieA_\mu)\phi - m^2\phi^\dagger\phi \\ &= \partial^\mu\phi^\dagger\partial_\mu\phi + ieA^\mu(\phi^\dagger\partial_\mu\phi - \partial_\mu\phi^\dagger\phi) + e^2\phi^\dagger\phi A^\mu A_\mu - m^2\phi^\dagger\phi \\ &= \partial^\mu\phi^\dagger\partial_\mu\phi - m^2\phi^\dagger\phi + ej_\mu A^\mu + e^2\phi^\dagger\phi A^\mu A_\mu \end{aligned} \tag{12.4}$$

where $j_\mu = \phi^\dagger\partial_\mu\phi - \partial_\mu\phi^\dagger\phi = \phi^\dagger\overleftrightarrow{\partial}_\mu\phi$. In accordance to QED we have to add the kinetic term for the EM-field, completing the scalar electrodynamics lagrangian

$$\mathcal{L} = \partial^\mu\phi^\dagger\partial_\mu\phi - m^2\phi^\dagger\phi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} + ej_\mu A^\mu + e^2\phi^\dagger\phi A^\mu A_\mu \tag{12.5}$$

Note that in this considerations we explicitly put the charge e which identifies the different representations of the $U(1)$ symmetry group.

From the lagrangian 12.5 we see two interaction factors $ej_\mu A^\mu$ and $e^2\phi^\dagger\phi A^\mu A_\mu$. These two factors give rise to two different interaction vertices. The first factor is, in momentum space, $ie(p_\mu - p'_\mu)$ and the second one is $2ie^2\delta_{\mu\nu}$, where the factor 2 comes from the possibility of interchanging the two photon fields. These vertices are given by the following Feynman diagrams



(12.6)

There's a better way of defining the interactions terms through the use of a gauge covariant current

$$j_\mu = ie\phi^\dagger\overleftrightarrow{D}_\mu\phi \tag{12.7}$$

12.2 A taste of divergencies

We see that scalar electrodynamics is constructed in a analogous manner to quantum electrodynamics. Apart from the fact that in QED we have only one interaction term

$$-ie\bar{\psi}\not{A}_\mu\psi \tag{12.8}$$

while in scalar electrodynamics we have two interaction vertices, the perturbative expansion of the S-matrix in scalar electrodynamics is divergent, while in QED is convergent. What do we mean by this? Thanks to perturbation theory, the cross section of a certain process can be evaluated perturbatively in powers of a given scale α, which in QED is $1/137$

$$\sigma = \sigma_0 + \alpha\sigma_1 + \alpha^2\sigma_2 + \cdots \tag{12.9}$$

This series is convergent in QED while is not in scalar electrodynamics. The reason can be easily found as follows: in every theory, the number of Feynman diagrams grows up factorially with every perturbative order. However, in QED, the statistics of fermion fields helps us eliminate lots of diagrams since whenever we change two fermionic lines the diagram takes a negative sign. In scalar theory the statistics follows the Bose-Einstein distribution, so this cancellation does not happen.

However, the series expansion for scalar electrodynamics is Borel summable, that is

$$\sum_k c_k \alpha^k \to \infty \quad \text{but} \quad \sum_k \frac{c_k}{k!}\alpha^k < \infty \tag{12.10}$$

and this allows us to reconstruct the function for every value of *alpha*.
Contrary to all of this, QCD is not even Borel summable. Despite this, perturbation theory can give us some useful information even in this case. About that, consider the following integral

$$I(\lambda) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx \, \exp\left\{-\frac{x^2}{2} - \lambda x^4\right\} \quad \lambda > 0 \tag{12.11}$$

Perturbation theory helps us evaluate $I(\lambda)$ for small values of λ

$$I(\lambda) \sim \frac{1}{\sqrt{2\pi}}\sum_n \frac{(-\lambda)^n}{n!}\int_{-\infty}^{\infty} dx e^{-x^2/2} x^{4n} = \sum_n (-1)^n \frac{\lambda^n}{n!}\langle x^n \rangle \tag{12.12}$$

Therefore we have to evaluate the Gaussian moments

$$\langle x^n \rangle = \int \frac{dx}{\sqrt{2\pi}} e^{-x^2/2} x^{4n} = 2^{2n} 2 \int_0^{\infty} \frac{dx}{\sqrt{2\pi}} e^{x^2/2}\left(\frac{x^2}{2}\right)^{2n} = \frac{2^{2n}}{\sqrt{\pi}}\Gamma\left(\frac{4n+1}{2}\right) = (4n-1)!! \tag{12.13}$$

and so

$$I(\lambda) \sim \sum_n (-1)^n \frac{\lambda^n}{n!}(4n-1)!! \tag{12.14}$$

This sum diverges for every possible value of λ. Despite this, from the result found we can extract some useful information about the series.
Firstly we observe that the reason for which the series is divergent is that the integral converges for $\lambda > 0$ but diverges for $\lambda < 0$ and so the convergence radius is zero, as we obtained. If we plot our result as a function of the perturbative order n we get a graph which will explode for big enough n, but the plot will arrive to a minimum before exploding. Truncating the series to the term in which the plot has a minimum gives us the better estimate for the integral. This point where we truncate our expansion is called **cutoff**. Generally we need to know how the series behaves before doing perturbation theory.

Although divergencies seem unphysical, they are not. Consider as an example a system near phase transition at a certain critical temperature T_c. Imagine to expand the physical interesting quantity based on temperature. The impossibility of expanding the series after $T > T_c$ is a result of the phase transition, where we cannot define physical quantities. The convergence radius of the perturbative series will help us to estimate the value of T_c.

PART V

Noether Theorem For Internal Symmetries

13 Quantum Noether's theorem

In this short chapter we further develop Noether theorem in the case of internal symmetries for a continuous group of transformations.

We know that any transformation of a Lie group can be written in exponential form

$$D(\alpha) = \exp\left\{i\alpha_A t^A\right\} \tag{13.1}$$

where t^A are the generators of the group and satisfy the Lie algebra of the group

$$\left[t^A, t^B\right] = if^{ABC}t^C \tag{13.2}$$

Let's take a lagrangian $\mathcal{L}(\phi^a(x), \partial_\mu\phi^a(x))$, where a is an internal index. Since we're considering transformations that act on internal quantities, the coordinates remain invariant under these transformations, so the action variation is such that $\delta S = 0$ and it translates directly to the variation of the lagrangian $\delta\mathcal{L} = 0$.

Consider then the transformation induced on the fields[3]

$$\phi^a(x) \to \phi'^a(x) = \left(\exp\{i\alpha_A t^A\}\right)^a{}_b\phi^b(x) \approx \phi^a(x) + i\alpha_A\left(t^A\right)^a{}_b\phi^b(x) \tag{13.3}$$

then the variation of the lagrangian induced by this transformation is

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\phi^a}\delta\phi^a + \frac{\partial\mathcal{L}}{\partial{\phi^a}_{,\mu}}\delta{\phi^a}_{,\mu} = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial{\phi^a}_{,\mu}}\delta\phi^a\right) + \left(\frac{\partial\mathcal{L}}{\partial\phi^a} - \partial_\mu\frac{\partial\mathcal{L}}{\partial{\phi^a}_{,\mu}}\right)\delta\phi^a \tag{13.4}$$

considering the field ϕ^a as a solution to the Euler-Lagrange's equations, this leads to

$$\delta\mathcal{L} = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial{\phi^a}_{,\mu}}\delta\phi^a\right) = i\alpha_A\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial{\phi^a}_{,\mu}}(t^A)^a{}_b\phi^b\right) = -\alpha_A\partial_\mu j^{\mu,A} \tag{13.5}$$

and so

$$\frac{\delta\mathcal{L}}{\delta\alpha^A} = -\partial_\mu j^{\mu,A} \qquad \text{where: } j^{\mu,A} = -i\left(t^A\right)^a{}_b\frac{\partial\mathcal{L}}{\partial{\phi^a}_{,\mu}}\phi^b \tag{13.6}$$

If the transformation is a symmetry of the system, we get the conserved current

$$\frac{\delta\mathcal{L}}{\delta\alpha^A} = -\partial_\mu j^{\mu,A} = 0 \tag{13.7}$$

which tells us that we get a number of conserved currents which is equal to the number of the generators of the group.
The conserved charge is easily found

$$Q^A = \int d^3x j^{0,A} = -i\int d^3x\frac{\partial\mathcal{L}}{\partial\partial_0\phi^a}(t^A)^a{}_b\phi^b = -i\int d^3x\,\Pi_a(t^A)^a{}_b\phi^b \tag{13.8}$$

[3] It's very important to remember where the indices actually run

$$a, b = 1, \cdots, d$$

where d is the dimension of the representation and

$$A = 1, \cdots, D$$

where D is the number of generators.

and this, being conserved, implies

$$\frac{dQ^A}{dt} = 0 \implies \left[Q^A, H\right] = 0 \tag{13.9}$$

that is, the Hamiltonian is simultaneously diagonalizable using the conserved charges. We can therefore classify each state using the eigenvalues of the generators which belong to the Cartan subalgebra of the considered group. In fact

$$\begin{aligned}\left[H, Q^A\right] &= -i\int d^3x\left[H, \Pi_a (t^A)^a{}_b\phi^b\right] = -i(t^A)^a{}_b\int d^3x\left\{[H,\Pi_a]\phi^b + \Pi_a\left[H,\phi^b\right]\right\}\\ &= (t^A)^a{}_b\int d^3x\left\{\partial_0\Pi_a\phi^b + \Pi_a\partial_0\phi^b\right\} = i\partial_0(-i)\int d^3x\Pi_a(t^A)^a{}_b\phi^b\\ &= i\partial_0 Q^A = 0\end{aligned} \tag{13.10}$$

The conservation of the charge has several implications. First this means that, given a certain rank of the group, you may classify the physical states as eigenstates of the energy-momentum, of the invariant mass and of as many internal quantum numbers as the rank of the group is. For example, if the symmetry is flavour $SU(3)$, corresponding to the up, down and strange quarks, which has rank two, we can classify the physical states on the basis of their isotopic spin and hypercharge (or electric charge). This includes obviously the single particles states. Moreover, in any physical process, since Q^A is conserved the charge of the initial states must be the same of the final ones.

Since the charges are now part of the Cartan subalgebra, they can be taken as generators of the group, let's see why. We first compute the commutator of the charge operator with the field operator

$$\begin{aligned}\left[\hat{Q}^A, \hat{\phi}^a(x)\right] &= i\int d^3y\left[\hat{\Pi}_a(y)(t^A)^a{}_c\hat{\phi}^c(y), \hat{\phi}^b(x)\right]\\ &= i\int d^3y (t^A)^a{}_c\hat{\Pi}_a(y)\left[\hat{\phi}^c(y), \hat{\phi}^b(x)\right]\\ &+ \int d^3y (t^A)^a{}_c\left[\hat{\Pi}_a(y), \hat{\phi}^b(x)\right]\hat{\phi}^c(y)\\ &= i(t^A)^a{}_c\int d^3y\ \delta^b{}_a(-i)\delta^{(3)}(x-y)\phi^c(x)\\ &= (t^A)^b{}_c\phi^c(x)\end{aligned} \tag{13.11}$$

which means that we can write the transformation for the field in two equivalent ways

$$\phi'^a(x) = \left(e^{i\alpha^A t^A}\right)^a{}_b\phi^b = \left(e^{i\alpha^A Q^A}\phi\, e^{-i\alpha^A Q^A}\right)^a \tag{13.12}$$

where $\exp\left(i\alpha^A t^A\right)$ is a matrix and ϕ is a complex vector, whereas in the last term on the right hand side, ϕ can be seen as a quantum field. The proof of the statements is easily done by using infinitesimal transformations

$$\phi'^a = \left(1 + i\alpha^A Q^A\right)\phi^a\left(1 - i\alpha^A Q^A\right) = \phi^a + i\alpha^A[Q^a, \phi^a] = \phi^a + i\alpha^A(t^A)^a{}_b\phi^b \tag{13.13}$$

Lastly we note that

$$\left[Q^A, Q^B\right] = if^{AB}{}_C Q^C \tag{13.14}$$

which can be easily proven plugging in the definition of the charge in 13.8.

Fermi Theory of Weak interactions

PART

VI

14 Introduction

14.1 Building up the theory

Soon after the discovery of radiation by Henri Becquerel, there were many theories that tried to explain such a process. Firstly it was assumed that the fast electrons ejected in the radiation were part of the nucleus before the decay. This theory was rejected with the discovery of the neutron by Chadwick. It became evident that the electron is created at the instant that the neutron transformed into a proton.
Another difficulty came around when the spectrum of the emitted electrons was measured and resulted to be continuous. Since the initial and final nuclei have well defined energies, this would mean a violation of energy conservation. Pauli therefore, in 1930, proposed that another particle would have to be ejected in the decay. This particle nowadays is called **neutrino** and the full neutron decay is given by

$$n \to p + e^- + \bar{\nu} \tag{14.1}$$

This particle should carry no charge, because of charge conservation, and should have low mass. Since the neutron, proton and electron carry spin $\frac{1}{2}$, conservation of angular momentum requires that also the new particle should carry spin $\frac{1}{2}$.
The first one to conceive a quantitative theory of β decay (so it was called the decay of the neutron) was Fermi in 1934. He did this by postulating that the decay process can be described by adding to the Hamiltonian an interaction term containing the wave functions of the four free particles

$$H_F = H_n^0 + H_p^0 + H_e^0 + H_\nu^0 + \sum_i C_i \int d^3x \left(\bar{u}_p \hat{O}_i u_n\right)\left(\bar{u}_e \hat{O}_i u_\nu\right) \tag{14.2}$$

Here u_p, u_n, u_e, u_ν denote the wave functions of the four particles and the bar denotes the Dirac adjoint $\bar{u} = \gamma_0 u^\dagger$. The $\hat{O}_i$ are appropriate operators that characterize the interactions and are weighted by come constants C_i.
Since the electron in β decay is emitted with high energy with respect to its mass and the neutrino is practically massless, they are relativistic and therefore their wave functions are solution to the free Dirac equation

$$\left(i\partial\!\!\!/ - m\right)u = 0 \tag{14.3}$$

We now concentrate to solely the interaction term which is given by the Hamiltonian density

$$\mathcal{H}_F = \sum_i C_i \left(\bar{u}_p \hat{O}_i u_n\right)\left(\bar{u}_e \hat{O}_i u_\nu\right) \tag{14.4}$$

A question arises: what are the operators $\hat{O}_i$? The answer was found in the deep experimental evidences in the years following the proposed theory.

Firstly, the Hamiltonian density has to be Lorentz invariant, this means that the quantity

$$\left(\bar{u}_p \hat{O}_i u_n\right)\left(\bar{u}_e \hat{O}_i u_\nu\right) \tag{14.5}$$

has to be a Lorentz scalar. The only possible quantities that make it a Lorentz scalar are the following bilinear covariants

$$1 \qquad \gamma^\mu \qquad \sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu] \qquad \gamma^\mu\gamma_5 \qquad \gamma_5 \tag{14.6}$$

or, in practice, a scalar (S), a vector (V), a tensor (T), an axial vector (A) and a pseudoscalar (P).
Since in nuclear β decay protons and neutrons move non-relativistically, the matrix elements can be simplified in the nucleonic part of the Hamiltonian. As it's well known, the Dirac 4-spinor can be decomposed into two components ϕ and ξ where the latter is a 2-spinor and, in the non-relativistic limit, is much smaller than the other 2-spinor and so can be neglected. In this limit, from the product $\bar{u}_p \Gamma u_n$, where $\Gamma = \{1, \gamma^\mu, \sigma^{\mu\nu}, \gamma^\mu\gamma_5, \gamma_5\}$, what remains is

$$\begin{aligned} &\bar{u}_p u_n,\ \bar{u}_p \gamma^\mu u_n \to \phi_p^\dagger \phi_n \\ &\bar{u}_p \sigma^{\mu\nu} u_n,\ \bar{u}_p \gamma^\mu \gamma_5 u_n \to \phi_p^\dagger \boldsymbol{\sigma} \phi_n \\ &\bar{u}_p \gamma_5 u_n \to 0 \end{aligned} \tag{14.7}$$

The relevant non-vanishing cases ar called **Fermi** transitions S, V, and the **Gamow-Teller** transitions T, A. In the latter obviously the spin of the nucleus may change, while it does not for Fermi transitions. Both cases are actually observed in nature, that is, the Fermi Hamiltonian 14.4 must contain some combination of $S-V$ and $T-A$ couplings. It can be shown that oscillation would occur in the electron spectrum if S and V couplings where to be simultaneously present, and the same would be true for T and A. Since this oscillations are not observed in nature, the only possible couplings can be between $S-T$, $S-A$, $V-T$, $V-A$. Measurement of several lifetimes of several nuclei lead to the conclusion that the coupling constants of both Fermi and Gamow-Teller transitions are about equal in magnitude

$$G_F \approx 10^{-4} \text{ MeV fm}^{-1} \implies G_F \approx 10^{-11} \text{ MeV}^{-2} \tag{14.8}$$

In the following years many other particles that decay weakly were discovered and in all cases almost the same constant G_F appears. One can talk about the universal Fermi constant.

14.2 Parity is not conserved?!

The K^+ meson discovery displayed a serious problem. It was found that this particle, beside the decay in $\pi^0\mu^+\nu$, decays in the following ways

$$K^+ \to \pi^0 + \pi^+ \qquad K^+ \to \pi^+ + \pi^+ + \pi^0 \tag{14.9}$$

It was well known that the pion had negative parity. Since all pions are emitted in s-wave the total parity of the two decays is not the same.
This violation of parity was soon proved in the beautiful experiment by Wu at al. using the decay of ^{60}Co into ^{60}Ni. They observed that the direction of the emitted electron in the β decay of ^{60}Co was predominantly opposite to the nuclear spin[4]. This fact can be rephrased as follows: electrons are predominantly polarized opposite to the direction of their motion, that is, they have **negative helicity**. Following this experiment, the same

[4] The spin of the nucleus was aligned using a strong magnetic field at a very low temperature. The experimental apparatus that made possible to arrive at 0.01K in the 1960 was marvelous. I urge you to gave it a look.

measurement were done for positrons in β^+ decays. What it was found is that positrons have **positive elicity**.
There only remains to find the elicity of the neutrino. Careful measurement led by Goldhaber et al., using the electron capture of ^{152}Eu [5] proved that neutrino is always emitted with negative helicity and anti-neutrino with positive helicity. From experiments we have concluded that only a single helicity appears: electrons and neutrinos are always left-handed while positrons and anti-neutrino are always right-handed.
Now we return to the question of the exact form of the Fermi Hamiltonian 14.4. Experiments showed that the part containing electrons and neutrino spinors, should only contain the part of the wave function with negative helicity. In the relativistic limit we can construct the projection operators

[5] Since it would be impossible to measure directly the helicity of the neutrino emitted, they instead measured the photon emitted by the subsequent de-excitation of the ^{152}Sm atom.

$$\hat{P}_{\pm} = \frac{1}{2}(1 \pm \gamma_5) \tag{14.10}$$

which are called projection operators on states of **positive and negative chirality**. For neutrinos this form of the operators is exact. For electron, being massive, they are good if the electron momentum is high enough[6]. According to this considerations we must replace the spinors by their components with negative chirality

[6] This implies that the statement that electrons have only positive helicity is only approximately correct.

$$\bar{u}_e \hat{O}_i u_\nu \to \overline{(\hat{P}_- u)} \hat{O}_i (\hat{P}_- u_\nu) \tag{14.11}$$

From the barred component we find

$$\begin{aligned} \overline{P_- u} &= (P_- u)^\dagger \gamma^0 = u^\dagger P_-^\dagger \gamma^0 = u^\dagger \left(\frac{1-\gamma_5}{2}\right)^\dagger \gamma^0 \\ &= u^\dagger \frac{1-\gamma_5}{2}\gamma^0 = u^\dagger \gamma^0 \frac{1+\gamma_5}{2} = \bar{u} P_+ \end{aligned} \tag{14.12}$$

where in the last line we have used the property $\{\gamma^0, \gamma_5\} = 0$. Therefore we have to see how the operator $\hat{O}_i$ transforms under $\hat{P}_+ \hat{O}_i \hat{P}_-$, for the cases found before. It can be easily evaluated that

$$\begin{aligned} &\hat{P}_+(1)P_- = 0 & &\hat{P}_+ \hat{\gamma}^\mu \hat{P}_- = \frac{1}{2}\gamma^\mu(1-\gamma_5) \\ &\hat{P}_+ \sigma^{\mu\nu} O_i \hat{P}_- = 0 & &\hat{P}_+ \gamma^\mu \gamma_5 \hat{P}_- = -\frac{1}{2}\gamma^\mu(1-\gamma_5) \\ &\hat{P}_+ \gamma_5 \hat{P}_- = 0 \end{aligned} \tag{14.13}$$

from which it's obvious that the only relevant contributions are the $V-A$ type. Lorentz invariance requires that even the nucleonic part of the Fermi Hamiltonian has to be $V-A$ type. Extensive experimental analysis has led to the conclusion that the correct form for the nucleonic part is given by

$$\bar{u}_p \gamma^\mu (g_V + g_A \gamma_5) u_n = g_V \bar{u}_p \left(1 - \frac{g_A}{g_V}\gamma_5\right) u_n \tag{14.14}$$

with

$$g_A / g_V = -1.255 \pm 0.006 \tag{14.15}$$

This takes into a fact that protons and neutrons are composite particles.
The complete expression for the Hamiltonian interaction term is therefore given by

$$\mathcal{H}_F = -\frac{G_F}{\sqrt{2}} g_V \left[\bar{p}\,\gamma^\mu \left(1 - \frac{g_A}{g_V}\gamma_5\right) n\right] [\bar{e}\,\gamma_\mu (1-\gamma_5)\nu_e] = -\frac{G_F}{\sqrt{2}} j^\mu_{(H)} j^{(L)}_\mu \tag{14.16}$$

where, in analogy with the electromagnetic current $j^\mu = e\bar{\psi}\gamma^\mu\psi$, we defined the hadronic current $j^\mu_{(H)}$ and the leptonic current $j^\mu_{(L)}$.

Through measurement of the decay of the neutron was found that

$$\Gamma \sim G_F^2 \times \text{"Phase space volume"} \sim G_F^2 \Delta m^5 \tag{14.17}$$

Using the Fermi theory one could construct in the same way the interaction Hamiltonian for other processes like

Muon decay

$$\mu^- \to e^- + \bar{\nu}_e + \nu_\mu \qquad H_F = -\frac{G_F}{\sqrt{2}}\bar{\nu}_\mu\gamma^\mu(1-\gamma_5)\mu\bar{e}^-\gamma_\mu(1-\gamma_5)\nu_e$$

Inverse β decay

$$\mu^+ + n \to p + \bar{\nu}_\mu \qquad H_F = -\frac{G_F}{\sqrt{2}} g_V \bar{p}\gamma^\mu(1 - \frac{g_A}{g_V}\gamma_5)n\bar{\mu}\gamma_\mu(1-\gamma_5)\nu_\mu \tag{14.18}$$

where for the first process

$$\Gamma \sim G_F^2 m_\mu^5 \tag{14.19}$$

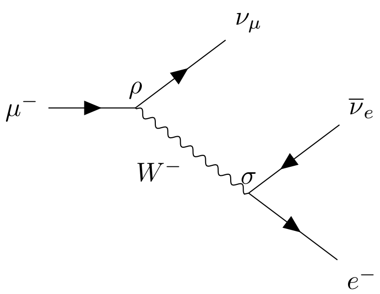
Despite being an effective theory, the Fermi model works not only in finding with a good approximation the rates but also the distribution of energy of the particles in the interaction.
Unfortunately, Fermi theory is not renormalizable since the operator that make up the Hamiltonian have an overall dimension of 6. This required a better theory in which, by some limits, one would get the Fermi effective theory since it gives sensitive results and so it must be right in some way.

15 The Muon decay

15.1 From the SM to the Fermi effective theory

Now we know how to evaluate the decay rate of the muon in the framework of the Standard Model. The theory of the Standard Model is quite extensive and is proven itself to be a quite precise theory. What we want to see is that, given the Standard Model theory of the muon decay, we can get, by some limit, the Fermi theory.

As described in the Standard Model, the muon decay is given at tree level by the following diagram



$$= \left(-\frac{ig_W}{\sqrt{2}}\right)^2 \bar{u}(p_{\nu_\mu})\gamma^\rho\frac{1-\gamma_5}{2}u(p_{\mu^-})W^-_{\rho\sigma}(q)\bar{u}(p_{e^-})\gamma^\sigma\frac{1-\gamma_5}{2}v(p_{\bar{\nu}_e}) \tag{15.1}$$

where the propagator of the W^- boson with momentum q is given by

$$W^-_{\rho\sigma}(q) = \frac{i\left(-g_{\rho\sigma} + \frac{q_\rho q_\sigma}{M_W^2}\right)}{q^2 - M_W^2 + i\epsilon} \tag{15.2}$$

With this we have now two parts to the matrix element which are given by the propagator

$$\begin{aligned} A =& \left(-\frac{ig_W}{\sqrt{2}}\right)^2 \bar{u}(p_{\nu_\mu})\gamma^\rho \frac{1-\gamma_5}{2} u(p_{\mu^-}) \frac{-ig_{\rho\sigma}}{q^2 - M_W^2 + i\epsilon} \bar{u}(p_{e^-})\gamma^\sigma \frac{1-\gamma_5}{2} v(p_{\bar{\nu}_e}) \\ &+ \left(-\frac{ig_W}{\sqrt{2}}\right)^2 \bar{u}(p_{\nu_\mu})\gamma^\rho \frac{1-\gamma_5}{2} u(p_{\mu^-}) \frac{iq_\sigma q_\rho/M_W^2}{q^2 - M_W^2 + i\epsilon} \bar{u}(p_{e^-})\gamma^\sigma \frac{1-\gamma_5}{2} v(p_{\bar{\nu}_e}) \end{aligned} \tag{15.3}$$

It's easy to prove that the first factor can be neglected in the limit in which $q = m_\mu \approx 100$ MeV and using the fact that $M_W \approx 100$ GeV. Taking only the second factor then[7]

[7] We'd make use of the relations

$$\not{p}u(p) = mu(p)$$
$$\bar{u}(p)\not{p} = -m\bar{u}(p)$$

and the same goes for $v(p)$ and $\bar{v}(p)$. Moreover we'll use

$$\{\gamma^\mu, \gamma_5\} = 0$$

$$\begin{aligned} &\left(-\frac{ig_W}{\sqrt{2}}\right)^2 \bar{u}(p_{\nu_\mu})\gamma^\rho \frac{1-\gamma_5}{2} u(p_{\mu^-}) \frac{iq_\sigma q_\rho/M_W^2}{q^2 - M_W^2 + i\epsilon} \bar{u}(p_{e^-})\gamma^\sigma \frac{1-\gamma_5}{2} v(p_{\bar{\nu}_e}) \\ =& \left(-\frac{ig_W}{\sqrt{2}}\right)^2 \bar{u}(p_{\nu_\mu})\not{q} \frac{1-\gamma_5}{2} u(p_{\mu^-}) \frac{i/M_W^2}{q^2 - M_W^2 + i\epsilon} \bar{u}(p_{e^-})\not{q} \frac{1-\gamma_5}{2} v(p_{\bar{\nu}_e}) \\ =& \left(-\frac{ig_W}{\sqrt{2}}\right)^2 \bar{u}(p_{\nu_\mu})\left(\not{p}_\mu - \not{p}_\nu\right) \frac{1-\gamma_5}{2} u(p_{\mu^-}) \frac{i/M_W^2}{q^2 - M_W^2 + i\epsilon} \bar{u}(p_{e^-})\left(\not{p}_e - \not{p}_\nu\right) \frac{1-\gamma_5}{2} v(p_{\bar{\nu}_e}) \\ =& \left(-\frac{ig_W}{\sqrt{2}}\right)^2 \bar{u}(p_{\nu_\mu})\not{p}_\mu \frac{1-\gamma_5}{2} u(p_{\mu^-}) \frac{i/M_W^2}{q^2 - M_W^2 + i\epsilon} \bar{u}(p_{e^-})\not{p}_e \frac{1-\gamma_5}{2} v(p_{\bar{\nu}_e}) \\ -& \left(-\frac{ig_W}{\sqrt{2}}\right)^2 \bar{u}(p_{\nu_\mu})\not{p}_\mu \frac{1-\gamma_5}{2} u(p_{\mu^-}) \frac{i/M_W^2}{q^2 - M_W^2 + i\epsilon} \bar{u}(p_{e^-})\not{p}_\nu \frac{1-\gamma_5}{2} v(p_{\bar{\nu}_e}) \\ -& \left(-\frac{ig_W}{\sqrt{2}}\right)^2 \bar{u}(p_{\nu_\mu})\not{p}_\nu \frac{1-\gamma_5}{2} u(p_{\mu^-}) \frac{i/M_W^2}{q^2 - M_W^2 + i\epsilon} \bar{u}(p_{e^-})\not{p}_e \frac{1-\gamma_5}{2} v(p_{\bar{\nu}_e}) \\ +& \left(-\frac{ig_W}{\sqrt{2}}\right)^2 \bar{u}(p_{\nu_\mu})\not{p}_\nu \frac{1-\gamma_5}{2} u(p_{\mu^-}) \frac{i/M_W^2}{q^2 - M_W^2 + i\epsilon} \bar{u}(p_{e^-})\not{p}_\nu \frac{1-\gamma_5}{2} v(p_{\bar{\nu}_e}) \end{aligned} \tag{15.4}$$

using the properties of anticommutation and the operation of the momentum on the wave functions, the final result is

$$\left(\frac{-ig_W}{\sqrt{2}}\right)^2 \left\{\frac{m_\mu - m_\nu}{M_W^2}\bar{u}(p_{\nu_\mu})u(p_\mu) + \frac{m_\mu + m_\nu}{M_W^2}\bar{u}(p_{\nu_\mu})\gamma_5 u(p_\mu)\right\} \frac{i}{q^2 - M_W^2 + i\epsilon} \left\{\frac{m_e + m_\nu}{M_W^2}\bar{u}(p_e)v(p_{\nu_e}) + \frac{m_e - m_\nu}{M_W^2}\bar{u}(p_e)\gamma_5 v(p_{\nu_e})\right\} \tag{15.5}$$

Since $m_e, m_\nu, m_\mu \ll M_W$, we can neglect all the terms which is what we were searching.

Now only one term remains

$$A = \left(-\frac{ig_W}{\sqrt{2}}\right)^2 \bar{u}(p_{\nu_\mu})\gamma^\rho \frac{1-\gamma_5}{2} u(p_{\mu^-}) \frac{ig_{\rho\sigma}}{M_W^2} \bar{u}(p_{e^-})\gamma^\sigma \frac{1-\gamma_5}{2} v(p_{\bar{\nu}_e}) \tag{15.6}$$

from which we find that, in the limit in which $q \ll M_W$, the propagator is given by

$$W_{\rho\sigma} = \frac{ig_{\rho\sigma}}{M_W^2} \tag{15.7}$$

In real space the propagator 15.7 is

$$W(x-y) = i\int \frac{d^4q}{(2\pi)^4}\frac{g_{\rho\sigma}}{M_W^2}e^{-iq(x-y)} = \frac{g_{\rho\sigma}}{M_W^2}\delta^{(4)}(x-y) \tag{15.8}$$

which means that the interaction is practically localized in one point. In this limit

$$A = \frac{g_W^2}{8M_W^2}\left[\bar{u}(p_{\nu_\mu})\gamma^\rho(1-\gamma_5)u(p_{\mu^-})\right]\left[\bar{u}(p_{e^-})\gamma_\rho(1-\gamma_5)v(p_{\bar{\nu}_e})\right] \tag{15.9}$$

which is exactly the Fermi theory, where now $G_F = \sqrt{2}g_W^2/8M_W^2$. This consideration proves that even non renormalizable theories are useful as an effective theory and they constitute a certain limit of a more general theory. In the limit of small energies $q/M_W \ll 1$, we get back from the SM theory to the Fermi theory.

15.2 Decay width in the Fermi theory

We would like to evaluate the decay rate of the muon using Fermi theory. Starting from

$$d\Gamma = \frac{1}{N\sigma_i}\sum_{\sigma_i}\sum_{\sigma_f}\frac{(2\pi)^4\delta^{(4)}\left(p-\sum_f p_f\right)}{2m}|M_{fi}|^2\prod_f\frac{d^3p_f}{(2\pi)^3 2E_f} \tag{15.10}$$

where we sum over all polarizations of incoming and outgoing particles. Obviously we take the transition amplitude from the Fermi theory

$$\begin{aligned} M_{fi} &= i\frac{G_F}{\sqrt{2}}\bar{u}(p_e)\gamma^\mu(1-\gamma_5)v(p_{\nu_e})\bar{u}(p_{\nu_\mu})\gamma_\mu(1-\gamma_5)u(p_\mu) \\ M_{fi}^* &= -i\frac{G_F}{\sqrt{2}}\left[\bar{u}(p_e)\gamma^\mu(1-\gamma_5)v(p_{\nu_e})\bar{u}(p_{\nu_\mu})\gamma_\mu(1-\gamma_5)u(p_\mu)\right]^* \end{aligned} \tag{15.11}$$

where the complex conjugate of the matrix element can be easily evaluated using the properties of the gamma matrices[8]. Using a general bilinear covariant Γ we have the following

[8] Namely

$$\gamma^0\gamma^{\mu\dagger}\gamma^0 = \gamma^\mu$$

and

$$\gamma^0\gamma^0 = 1$$

$$[\bar{u}_1\Gamma u_2]^* = \left[u_1^\dagger\gamma^0\Gamma u_2\right]^\dagger = u_2^\dagger\Gamma^\dagger\gamma^{0\dagger}u_1^\dagger = u_2^\dagger\gamma^0\gamma^0\Gamma^\dagger\gamma^0 u_1^\dagger = \bar{u}_2\bar{\Gamma}u_1 \tag{15.12}$$

where

$$\bar{\Gamma} = \gamma^0\Gamma^\dagger\gamma^0 \tag{15.13}$$

which in our case $\Gamma = \gamma^\mu(1-\gamma_5)$ yields

$$\gamma^0(1-\gamma_5)^\dagger\gamma^{\mu\dagger}\gamma^0 = \gamma^0\gamma^{\mu\dagger}\gamma^0(1-\gamma^5) = \gamma^\mu(1-\gamma_5) \tag{15.14}$$

The modulus square of the transition matrix becomes, rearranging the terms in the product

$$\frac{G_F^2}{2}\left[\bar{u}(p_{\nu_\mu})\gamma^\rho(1-\gamma_5)u(p_\mu)\bar{u}(p_\mu)\gamma^\sigma(1-\gamma_5)u(p_{\nu_\mu})\right]\left[\bar{u}(p_e)\gamma_\rho(1-\gamma_5)v(p_{\nu_e})\bar{v}(p_{\nu_e})\gamma_\sigma(1-\gamma_5)u(p_e)\right] \tag{15.15}$$

Since we're summing over all polarizations, we can use Casimir's trick with the completeness relations and get

$$\begin{aligned} |A|^2 = \frac{G_F^2}{2}\,\mathrm{Tr}&\left[\gamma^\rho(1-\gamma_5)(\not{p}_\mu + m_\mu)\gamma^\sigma(1-\gamma_5)(\not{p}_{\nu_\nu} + m_{\nu_\mu})\right] \\ &\mathrm{Tr}\left[\gamma_\rho(1-\gamma_5)(\not{p}_{\nu_e} - m_{\nu_e})\gamma^\sigma(1-\gamma_5)(\not{p}_e + m_e)\right] \end{aligned} \tag{15.16}$$

We could have got the same result by directly manipulating the Feynman diagram, closing all fermion lines and joining up the two loops with a propagator to ensure that the particles be on-shell. The resulting diagram is given in the sidenote[9] where the ρ and σ are two weak vertices. The reason to this is a consequence of the **Cutkosky cutting rules**.

Let's now calculate the two traces[10] neglecting the mass of the neutrino

$$
\begin{aligned}
&Tr\Big[\gamma^\rho(1-\gamma_5)(\not p^{(1)}+m^{(1)})\gamma^\sigma(1-\gamma_5)\not p^{(2)}\Big] \\
&= \mathrm{Tr}\Big[\gamma^\rho(1-\gamma_5)\not p^{(1)}\gamma^\sigma(1-\gamma_5)\not p^{(2)}\Big] + \mathrm{Tr}\Big[\gamma^\rho(1-\gamma_5)m^{(1)}\gamma^\sigma(1-\gamma_5)\not p^{(2)}\Big] \\
&= \mathrm{Tr}\Big[\gamma^\rho(1-\gamma_5)\not p^{(1)}\gamma^\sigma(1-\gamma_5)\not p^{(2)}\Big] + \cancel{\mathrm{Tr}\Big[\gamma^\rho(1-\gamma_5)(1+\gamma_5)m^{(1)}\gamma^\sigma\not p^{(2)}\Big]} \\
&= 2\,\mathrm{Tr}\Big[\gamma^\rho(1-\gamma_5)\not p^{(1)}\gamma^\sigma\not p^{(2)}\Big] \\
&= 2p^{(1)}_\alpha p^{(2)}_\beta\left\{\mathrm{Tr}\left[\gamma^\rho\gamma^\alpha\gamma^\sigma\gamma^\beta\right] - \mathrm{Tr}\left[\gamma^\rho\gamma_5\gamma^\alpha\gamma^\sigma\gamma^\beta\right]\right\} \\
&= 2p^{(1)}_\alpha p^{(2)}_\beta\left\{4(g^{\rho\alpha}g^{\sigma\beta}-g^{\rho\sigma}g^{\alpha\beta}+g^{\rho\beta}g^{\alpha\sigma})+4i\epsilon^{\rho\alpha\sigma\beta}\right\} \\
&= 8\Big(p_1^\rho p_2^\sigma + p_1^\sigma p_2^\rho - \mathbf{p}_1\cdot\mathbf{p}_2\, g^{\rho\sigma} + ip^{(1)}_\alpha p^{(2)}_\beta\epsilon^{\rho\alpha\sigma\beta}\Big)
\end{aligned}
\tag{15.17}
$$

The same goes for the other trace, we just have to plug in $(1,2)=(\mu,\nu_\mu)$ or $(1,2)=(e,\nu_e)$. Taking the product of the two traces cancels out many terms since there'll be a lot of contractions between symmetric tensors and the completely antisymmetric Levi-Civita tensor. Taking this into account the result we get is

$$
\begin{aligned}
\frac{1}{2}\sum_\sigma\sum_{\sigma'}|M_{fi}|^2 &= \frac{G_F^2}{4}64\Big(p^\rho_\mu p^\sigma_{\nu_\mu}+p^\sigma_\mu p^\rho_{\nu_\mu}-\mathbf{p}_\mu\cdot\mathbf{p}_{\nu_\mu}g^{\rho\sigma}+ip^\mu_\alpha p^{\nu_\mu}_\beta\epsilon^{\rho\alpha\sigma\beta}\Big) \\
&\quad\times\left(p^e_\rho p^{\nu_e}_\sigma + p^e_\sigma p^{\nu_e}_\rho - \mathbf{p}_e\cdot\mathbf{p}_{\nu_e}g_{\rho\sigma}+ip^\gamma_e p^\delta_{\nu_e}\epsilon_{\rho\gamma\sigma\delta}\right) \\
&= \frac{G_F^2}{4}64\Big(2(\mathbf{p}_e\cdot\mathbf{p}_{\nu_\mu})(\mathbf{p}_{\nu_e}\cdot\mathbf{p}_\mu)+2(\mathbf{p}_e\cdot\mathbf{p}_\mu)(\mathbf{p}_{\nu_e}\cdot\mathbf{p}_{\nu_\mu})-p^\mu_\alpha p^{\nu_\mu}_\beta p^\gamma_e p^\delta_{\nu_e}\epsilon^{\rho\alpha\sigma\beta}\epsilon_{\rho\gamma\sigma\delta}\Big)
\end{aligned}
\tag{15.18}
$$

Using the property

$$
\epsilon^{\rho\alpha\sigma\beta}\epsilon_{\rho\gamma\sigma\delta}=\epsilon^{\rho\sigma\alpha\beta}\epsilon_{\rho\sigma\gamma\delta}=-2\Big(\delta^\alpha{}_\gamma\delta^\beta{}_\delta-\delta^\alpha{}_\delta\delta^\beta{}_\gamma\Big)
\tag{15.19}
$$

we get

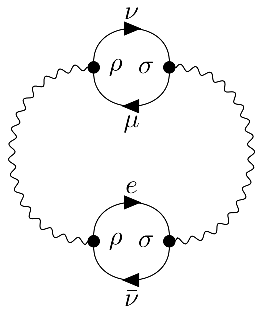
$$
\begin{aligned}
&= \frac{G_F^2}{4}64\Big(2(\mathbf{p}_e\cdot\mathbf{p}_{\nu_\mu})(\mathbf{p}_{\nu_e}\cdot\mathbf{p}_\mu)+2(\mathbf{p}_e\cdot\mathbf{p}_\mu)(\mathbf{p}_{\nu_e}\cdot\mathbf{p}_{\nu_\mu})+2p^\mu_\alpha p^{\nu_\mu}_\beta p^\gamma_e p^\delta_{\nu_e}\Big(\delta^\alpha{}_\gamma\delta^\beta{}_\delta-\delta^\alpha{}_\delta\delta^\beta{}_\gamma\Big)\Big) \\
&= \frac{G_F^2}{4}64\big((\mathbf{p}_e\cdot\mathbf{p}_{\nu_\mu})(\mathbf{p}_{\nu_e}\cdot\mathbf{p}_\mu)+2(\mathbf{p}_e\cdot\mathbf{p}_\mu)(\mathbf{p}_{\nu_e}\cdot\mathbf{p}_{\nu_\mu})+2(\mathbf{p}_e\cdot\mathbf{p}_{\nu_\mu})(\mathbf{p}_{\nu_e}\cdot\mathbf{p}_\mu)-2(\mathbf{p}_e\cdot\mathbf{p}_\mu)(\mathbf{p}_{\nu_e}\cdot\mathbf{p}_{\nu_\mu})\big) \\
&= \frac{G_F^2}{4}256(\mathbf{p}_e\cdot\mathbf{p}_{\nu_\mu})(\mathbf{p}_{\bar\nu_e}\cdot\mathbf{p}_\mu)
\end{aligned}
\tag{15.20}
$$

From the full differential decay rate

$$
\begin{aligned}
d\Gamma = \frac{1}{2}\frac{1}{2m_\mu}\frac{G_F^2}{2}(256)(\mathbf{p}_e\cdot\mathbf{p}_{\nu_\mu})(\mathbf{p}_{\bar\nu_e}\cdot\mathbf{p}_\mu)(2\pi)^4\,\cdot \\
\cdot\,\delta^{(4)}(q-p_{\bar\nu_e}-p_{\nu_\mu})\frac{d^3p_e}{(2\pi)^3 2E_e}\frac{d^3p_{\bar\nu_e}}{(2\pi)^3 2E_{\bar\nu_e}}\frac{d^3p_{\nu_\mu}}{(2\pi)^3 2E_{\nu_\mu}}
\end{aligned}
\tag{15.21}
$$

Since, experimentally, it would be really hard to measure neutrinos, to get a sensitive result

[9]



[10] Remember that the trace of the product of any odd number of γ matrices vanishes. Moreover, the trace of the product of two or any odd number of γ times a γ_5 is zero.

we integrate over their momenta. What we're trying to solve is an integral of the following form

$$I_{\alpha\beta} = \int \frac{d^3p}{2E} \int \frac{d^3p'}{2E'} p_\alpha p'_\beta \, \delta^{(4)}(q - p - p') \tag{15.22}$$

This integral is manifestly Lorentz covariant. This is obvious because the delta function is Lorentz covariant as the integration factor, which is due to the equality

$$\frac{d^3p}{2p^0} = \int d^4p \, \delta(p^2 - m^2)\Theta(p^0) \tag{15.23}$$

which is indeed Lorentz covariant. Performing the integration, the variable q is just the only variable we're left with. The only two second rank, Lorentz invariant, tensors that we can construct are therefore $g_{\alpha\beta}$ and $q_\alpha q_\beta$. So

$$I_{\alpha\beta} = A(q^2) g_{\alpha\beta} + B(q^2) \frac{q_\alpha q_\beta}{q^2} \tag{15.24}$$

From this we can construct the following invariant forms

$$\begin{aligned} g^{\alpha\beta} I_{\alpha\beta} &= 4A(q^2) + B(q^2) \\ q^\alpha q^\beta I_{\alpha\beta} &= \left[A(q^2) + B(q^2)\right] q^2 \end{aligned} \tag{15.25}$$

We distinguish now two cases: the first one in which q is time-like, that is $q^2 > 0$. With this condition we can always perform a Lorentz transformation in which

$$\tilde{q}^\mu = \Lambda^\mu{}_\nu q^\nu \equiv (\tilde{q}^0, 0) \tag{15.26}$$

defines the reference frame. So we start in a general reference frame in which

$$\begin{aligned} g^{\alpha\beta} I_{\alpha\beta} &= \int \frac{d^3p}{2E} \int \frac{d^3p'}{2E'} p'_\alpha p^\alpha \delta^{(4)}(\tilde{q}^0 - p^0 - p'^0) \\ &= \int \frac{d^3p}{2E} \int \frac{d^3p'}{2E'} \delta^{(4)}(q - p - p') \frac{1}{2}\left[(p + p')^2 - p'^2 - p^2\right] \\ &= \frac{1}{2}(q^2 - m^2 - m'^2) \int \frac{d^3p}{2E} \int \frac{d^3p'}{2E'} \delta^{(4)}(q - p - p') \end{aligned} \tag{15.27}$$

By using the definition of the delta function

$$\begin{aligned} I^\alpha{}_\alpha &= \frac{1}{2} q^2 \int \frac{d^3p}{2E} \int \frac{d^3p'}{2E'} \delta^{(4)}(q - p - p') \\ &= \frac{1}{2} q^2 \int \frac{d^3p}{2E} \int \frac{d^3p'}{2E'} \delta^{(3)}(q - p - p') \delta(q^0 - E - E') \\ &= \int \frac{d^3p}{2E} \frac{1}{2E'} \delta(q^0 - E - E') \end{aligned} \tag{15.28}$$

Now we switch to the center of mass reference frame. In this case, as we said, we have

$$q = (q^0, 0) \qquad p = (\sqrt{p^2 + m^2}, \mathbf{p}) \qquad p' = (\sqrt{p^2 + m'^2}, -\mathbf{p}) \tag{15.29}$$

and so

$$\int \frac{d^3p}{2E} \frac{1}{2E'} \delta(q^0 - E - E') = \int \frac{dp d\Omega p^2}{4EE'} \delta(q^0 - \sqrt{m^2 + p^2} - \sqrt{m'^2 + p^2}) \tag{15.30}$$

The delta function can be rewritten finding its zeros

$$m'^2 + p^2 = (q^0 - \sqrt{m^2+p^2})^2 = {q^0}^2 + m^2 + p^2 - 2q^0\sqrt{m^2+p^2}$$

$$\sqrt{m^2+p^2} = \frac{{q^0}^2+m^2+p^2}{2q^0} \implies p^2 = \left(\frac{{q^0}^2+m^2+p^2}{2q^0}\right)^2 - m^2 \tag{15.31}$$

Its derivative gives

$$\frac{d}{dp}\Big(q^0 - \sqrt{m^2+p^2} - \sqrt{m'^2+p^2}\Big) = \frac{p}{E} + \frac{p}{E'} = \frac{p(E+E')}{EE'} = \frac{pq^0}{EE'} \tag{15.32}$$

so that we get in the integral

$$\begin{aligned}
&\int \frac{dpp^2 d\Omega}{4EE'}\delta(q^0 - \sqrt{m^2+p^2} - \sqrt{m'^2+p^2}) \\
&= 4\pi \int \frac{dpp^2}{4EE'}\frac{EE'}{q^0 p}\delta\left(p - \sqrt{\left(\frac{{q^0}^2+m^2-m'^2}{2q^0}\right)^2 - m^2}\right)\Theta(q^0) \\
&= \frac{\pi}{q^0}\left(\sqrt{\left(\frac{{q^0}^2+m^2-m'^2}{2q^0}\right)^2 - m^2}\right)\Theta(q^0) \\
&= \pi\sqrt{\frac{{q^0}^4 + m^4 + m'^4 - 2{q^0}^2m^2 - 2{q^0}^2m'^2 - 2m^2m'^2}{4{q^0}^4}}\Theta(q^0)
\end{aligned} \tag{15.33}$$

In the limit case in which the neutrino are massless, the result becomes

$$I^\alpha{}_\alpha = \frac{\pi}{4}q^2\Theta(q^0) \qquad q^2 > 0 \tag{15.34}$$

Now, for the second integral we had, again in the center of mass frame, $q = (q, 0)$

$$\begin{aligned}
q^\alpha q^\beta I_{\alpha\beta} &= \int \frac{d^3p}{2E}\int\frac{d^3p'}{2E'}\delta^{(4)}(q-p-p')(q^\alpha p_\alpha)(q^\beta p'_\beta) \\
&= \int \frac{d^3p}{2E}\int\frac{d^3p'}{2E'}\delta^{(4)}(q-p-p')(q^0 E)(q^0 E') \\
&= \frac{q^2}{4}\int d^3p\, d^3p' \delta^{(4)}(q-p-p') \\
&= \frac{q^2}{4}\int d^3p\, d^3p' \delta(q^0 - E - E')\delta^{(3)}(q-p-p') \\
&= \frac{q^2}{4}4\pi\int dp\, p^2\delta(q^0 - E - E')
\end{aligned} \tag{15.35}$$

Again, in the limit of massless neutrinos, $p = E$ and $p' = E'$, so

$$q^\alpha q^\beta I_{\alpha\beta} = \pi q^2 \int dE E^2 \frac{1}{2}\delta\Big(E - \frac{q}{2}\Big) = \pi\frac{q^4}{8} \tag{15.36}$$

Then, by using the relations 15.25 we get

$$I^\alpha{}_\alpha(q^2) = 4A(q^2) + B(q^2) = \frac{\pi}{4}q^2 \qquad q^\alpha q^\beta I_{\alpha\beta}(q^2) = A(q^2) + B(q^2) = \frac{\pi}{8}q^2$$

$$\begin{cases} A(q^2) = \frac{\pi}{24}q^2 \\ B(q^2) = \frac{\pi}{12}q^2 \end{cases} \tag{15.37}$$

Then

$$I_{\alpha\beta}(q^2) = \frac{\pi}{24}\left(q^2 g_{\alpha\beta} + 2q_\alpha q_\beta\right) \tag{15.38}$$

We didn't analyze the case in which the vector q is space-like. In this case the integral will be always zero.

We can now reinsert that in the formula for the decay rate 15.21

$$d\Gamma = \frac{32G_F^2}{(2\pi)^5 m_\mu} p^e_\alpha p^\mu_\beta I^{\alpha\beta}\left((p^\mu - p^e)^2\right)\frac{d^3p_e}{2E_e} \tag{15.39}$$

Plugging in the value for the integral

$$\begin{aligned} d\Gamma &= \frac{32G_F^2}{(2\pi)^5 m_\mu}\frac{\pi}{24}\left[(p^\mu - p^e)^2(\mathbf{p}_e\cdot\mathbf{p}_\mu) + 2\,(p^\mu - p^e)\cdot\mathbf{p}_e(p^\mu - p^e)\cdot\mathbf{p}_\mu\right]\frac{d^3p_e}{2E_e} \\ &= \frac{G_F^2}{3\pi m_\mu}\left[3(m_\mu^2 + m_e^2)(\mathbf{p}_\mu\cdot\mathbf{p}_e) - 4(\mathbf{p}_\mu\cdot\mathbf{p}_e)^2 - 2m_e^2 m_\mu^2\right]\frac{d^3p_e}{(2\pi)^3 2E_e} \end{aligned} \tag{15.40}$$

In the reference frame of the muon

$$d\Gamma = \frac{G_F^2}{3\pi m_\mu}\left[3(m_\mu^2 + m_e^2)m_\mu E_e - 4m_\mu^2 E_e^2 - 2m_\mu^2 m_e^2\right]\frac{d^3p_e}{(2\pi)^3 2E_e} \tag{15.41}$$

For simplicity we can study the limit in which the electron mass can be neglected $m_e \ll m_\mu$. After we have passed from the factor d^3p_e to the factor dE_e, we get

$$d\Gamma = \frac{G_F^2}{24\pi^4 m_\mu}(3m_\mu^3 E_e - 4m_\mu^2 E_e^2)4\pi\frac{E_e^2 dE_e}{2E_e} = \frac{G_F^2}{12\pi^3 m_\mu}(3m_\mu^3 - 4m_\mu^2 E_e)E_e^2 dE_e \tag{15.42}$$

In the end, what we get is

$$\frac{d\Gamma}{dE} = \frac{G_F^2}{12\pi^3 m_\mu}E^2(3m_\mu^3 - 4m_\mu^2 E) \tag{15.43}$$

Since we know the kinematic limits for the energy of the electron to be $E_0 = m_e \approx 0$ and $E_1 = m_\mu/2$, we can easily integrate and get the final result

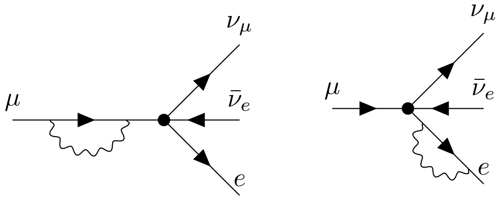
$$\Gamma \approx \frac{G_F^2}{12\pi^3 m_\mu}\left(E^3 m_\mu^3 - E^4 m_\mu^2\right)\Big|_{E_1} = \frac{G_F^2 m_\mu^5}{192\pi^3} \tag{15.44}$$

If we considered the mass of the electron we would get a correction of the order $(m_e/m_\mu)^2$

$$\Gamma = \frac{G_F^2 m_\mu^5}{192\pi^3}\left(1 + O\left(\frac{m_e^2}{m_\mu^2}\right)\right) \tag{15.45}$$

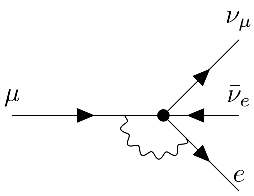
This result is in really good match with the experiments if we put back the masses of the neutrinos.

The result 15.44 is used to define the value of the Fermi constant G_F. In order to do that we have to take into account even radiative corrections, which come from the electromagnetic interactions of the electron and the muon. The relevant processes are the following



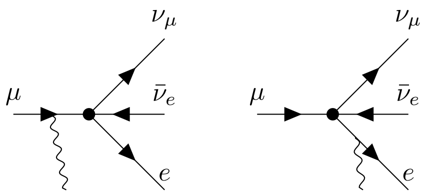
(15.46)

which are the self energy corrections, and



(15.47)

which is a vertex correction, and finally



(15.48)

which are the bremsstrahlung corrections. The bremsstrahlung diagram has to be included since, owing to the vanishing photon mass, photons with arbitrary small energies may be emitted. On the other hand, because of the limited experimental resolution, it is impossible to distinguish the muon decay accompanied by an extremely soft photon from the decay without radiation. This contribution exactly cancels the divergent terms in the self-energy diagrams for very soft photons, the so called infrared divergences. The calculation of the corrections leads to a modification of the decay rate

$$\Gamma = \frac{G_F^2 m_\mu^5}{192\pi^3}\left(1 + \frac{\alpha}{4\pi}\left(\pi^2 - \frac{25}{4}\right) - 8\frac{m_e^2}{m_\mu^2} + \cdots\right) \tag{15.49}$$

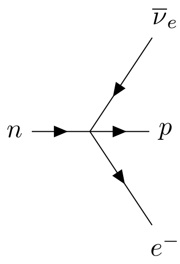
in which we see that radiative corrections are of greater importance than the correction due to the non-zero mass of the electron. With this formula, it has been found experimentally that

$$G_F = (1.166\,37 \pm 0.000\,02) \times 10^{-11}\ \mathrm{MeV}^{-2} \tag{15.50}$$

16 The Neutron decay

From Fermi Hamiltonian for the neutron decay given in equation 14.16 we see that the interaction is of the following form[11]

[11] We'll see later that even this is the result of the low energy limit of the SM lagrangian. In the Standard Model the decay is based on the interactions between the quarks that make up the proton and the neutron, mediated by the vector boson W^- which in turn decays into an electron-antineutrino pair.



$$= -\frac{G_F}{\sqrt{2}} g_V \left[\bar{p}\,\gamma^\mu \left(1 - \frac{g_A}{g_V}\gamma_5\right) n\right] [\bar{e}\,\gamma_\mu (1-\gamma_5)\nu_e]$$

This semileptonic decay is considered one of the most important decays, especially in order to study and comprehend all the ideas concerning the electroweak theory and to better understand the quark model proposed by Gell-Mann.
In this section we will see only the accounts concerning the cross section because the study of this decay is the same made for the muon decay. We have only to take into account of some differences:

- There is the factor $\dfrac{g_A}{g_V} \neq 1$
- In the kinematics you have proton and neutron almost at rest, so when you write the energy of the proton in a non relativistic formula $E_p \simeq m_p + \frac{p_p^2}{2m_p}$, you can neglect the momentum
- There is only one term depending on the angle between the electron and the neutrino, $\mathbf{p}_e \cdot \mathbf{p}_{\bar{\nu}_e} = E_e E_{\bar{\nu}_e} - p_e p_{\bar{\nu}_e} \cos\theta$. Since we know $p_p^2 = (p_n - p_e - p_{\bar{\nu}_e})^2$, then we can write the scalar product independent of the angle, $\mathbf{p}_e \cdot \mathbf{p}_{\bar{\nu}_e} = E_e E_{\bar{\nu}_e}$

16.1 Cross section in the Fermi theory

The formula of the cross section is

$$d\sigma = \frac{1}{\phi}\frac{1}{N\sigma_i}\sum_{\sigma_i}\sum_{\sigma_f}\frac{(2\pi)^4\delta^{(4)}\left(p - \sum_f p_f\right)}{2m}|M_{fi}|^2 \prod_f \frac{d^3 p_f}{(2\pi)^3 2E_f} \tag{16.1}$$

that is the same formula of the decay width except for the factor ϕ that represents the flux.
Now let's evaluate ϕ. We know

$$\phi = 4\sqrt{(\mathbf{p}_1 \cdot \mathbf{p}_2)^2 - p_1^2 p_2^2} \tag{16.2}$$

and represents the collision of a particle with energy E_1 with another particle at rest with mass m_2. Knowing that

$$(\mathbf{p}_i) = (E_i, E_i \overrightarrow{v_i}) \tag{16.3}$$

we can write

$$\begin{aligned}\phi^2 &= 16\left[E_1^2 E_2^2 (1 - \overrightarrow{v_1}\overrightarrow{v_2})^2 - m_1^2 m_2^2\right] \\ &= 16\left[E_1^2 E_2^2 (1 - 2\overrightarrow{v_1}\overrightarrow{v_2} + v_1^2 v_2^2) - m_1^2 m_2^2\right]\end{aligned} \tag{16.4}$$

Exploiting the formula $(\overrightarrow{v_1} - \overrightarrow{v_2})^2 = v_1^2 + v_2^2 - 2\overrightarrow{v_1}\overrightarrow{v_2}$, we can invert it and get to

$$\phi^2 = 16\big[E_1^2E_2^2(\overrightarrow{v_1} - \overrightarrow{v_2})^2 + E_1^2E_2^2(1 - v_1^2 - v_2^2 + v_1^2v_2^2) - m_1^2m_2^2\big] \tag{16.5}$$

Now let's focus on the second term

$$\begin{aligned}
&E_1^2E_2^2(1 - v_1^2 - v_2^2 + v_1^2v_2^2)\\
&= E_2^2(E_1^2 - p_1^2 - E_1^2v_2^2 - p_1^2v_2^2)\\
&= E_2^2\big(m_1^2 - (E_1^2 - p_1^2)v_2^2\big)\\
&= E_2^2\big(m_1^2 - m_1^2v_2^2\big)\\
&= m_1^2(E_2^2 - v_2^2)\\
&= m_1^2m_2^2
\end{aligned} \tag{16.6}$$

This term cancels out with the third term, so in the end we have

$$\phi = 4\sqrt{E_1^2E_2^2(\overrightarrow{v_1} - \overrightarrow{v_2})^2} = 4E_1E_2\,|\overrightarrow{v_1} - \overrightarrow{v_2}| \tag{16.7}$$

PART

VII

Gell-Mann Levy Model

In the previous chapter we have analyzed the muon decay, a particular case of the Fermi theory in which only lepton interactions were present, that is in the Fermi Hamiltonian only lepton currents were present. This approach completely neglected the case of the β decay from which we started. A full theory of β decays requires also lepton currents to appear in the Hamiltonian, but it's not enough. This requires also the study of strong interaction.

The sigma model is an effective theory to describe the interactions between nucleons. The theory encodes the vector and axial vector isospin symmetries of the strong forces and the implication of the breaking of these symmetries for the weak axial current. It's build up by three pseudoscalar fields $\pi_i(x)$ for $i = 1, 2, 3$ and a scalar field $\sigma(x)$, plus the fields of the proton $p(x)$ and the neutron $n(x)$. From the three pseudoscalar mesons it's possible to construct the three pions

$$\pi^{\pm} = \frac{\pi_1 \pm i\pi_2}{\sqrt{2}} \qquad \pi^0 = \pi_3 \tag{16.8}$$

17 The bosonic sector

17.1 Vector and Axial symmetries

The bosonic sector of the theory is given by the lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \sigma \partial^\mu \sigma + \partial_\mu \vec{\pi} \partial^\mu \vec{\pi}) - \frac{m^2}{2}(\sigma^2 + \vec{\pi} \cdot \vec{\pi}) - \frac{\lambda}{4}(\sigma^2 + \vec{\pi} \cdot \vec{\pi})^2 \tag{17.1}$$

If we introduce the following notation in which all the fields are in a vector

$$\Phi(x) = \begin{pmatrix} \sigma(x) \\ \pi_1(x) \\ \pi_2(x) \\ \pi_3(x) \end{pmatrix} \qquad \Phi(x) = \begin{pmatrix} \pi_1(x) \\ \pi_2(x) \\ \pi_3(x) \\ \sigma(x) \end{pmatrix} \tag{17.2}$$

we can rewrite the lagrangian in a compact way as

$$\mathcal{L} = \frac{1}{2}\partial_\mu \Phi^T \partial^\mu \Phi - V(\Phi^T \Phi) \tag{17.3}$$

In this form it's evident that the theory is $O(4)$ invariant, that is

$$O \in O(4) \implies \Phi'_i = O_{ij}\Phi^j \implies \mathcal{L}'(\Phi') = \mathcal{L}(\Phi) \tag{17.4}$$

Consider now the following infinitesimal transformation that we'll call A and V

$$\begin{aligned} &\text{V} \qquad \delta\sigma = 0 \quad \delta\pi_i = -\epsilon_{ijk}\alpha_j \pi_k = -\vec{\alpha} \times \vec{\pi} \\ &\text{A} \qquad \delta\sigma = \vec{\beta} \cdot \vec{\pi} \quad \delta\vec{\pi} = -\sigma\vec{\beta} \end{aligned} \tag{17.5}$$

with $\vec{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$ and $\vec{\beta} = (\beta_1, \beta_2, \beta_3)$ the parameters of the transformation.

Let's now check that these are in fact symmetries of the lagrangian 17.1, starting from the axial symmetry

$$\begin{aligned}\delta\mathcal{L} &= -(\vec{\alpha}\times\partial_\mu\vec{\pi})\partial^\mu\vec{\pi} + \delta V = \frac{\delta V}{\delta(\Phi^T\Phi)}\left(\frac{\delta(\Phi^T\Phi)}{\delta\sigma}\delta\sigma + \frac{\delta V}{\delta(\Phi^T\Phi)}\frac{\delta(\Phi^T\Phi)}{\delta\pi_i}\delta\pi_i\right) \\ &= \frac{\delta V}{\delta(\Phi^T\Phi)}(-2\vec{\pi}\cdot(\vec{\alpha}\times\vec{\pi})) = 0 \end{aligned} \tag{17.6}$$

since the cross product with π creates a vector which is orthogonal to π itself. For the axial transformation we have

$$\delta\mathcal{L} = -\frac{1}{2}\vec{\beta}\cdot\partial_\mu\vec{\pi}\partial^\mu\sigma - \frac{1}{2}\partial_\mu\sigma\vec{\beta}\cdot\partial^\mu\vec{\pi} + \delta V = \frac{\delta V}{\delta(\Phi^T\Phi)}\left(2\sigma\vec{\beta}\cdot\vec{\pi} - 2\sigma\vec{\beta}\cdot\vec{\pi}\right) = 0 \tag{17.7}$$

Using Nöether's theorem we can therefore find two conserved currents

$$\begin{aligned} j^\mu_{a,V} &= -\frac{\partial\mathcal{L}}{\partial\pi_{k,\mu}}\frac{\delta\pi_k}{\delta\alpha^a} = \partial^\mu\pi_k(-\epsilon_{aik}\pi_i) = (\partial^\mu\vec{\pi}\times\vec{\pi})_a \\ j^\mu_{b,A} &= \frac{\partial\mathcal{L}}{\partial\Phi_{i,\mu}}\frac{\delta\Phi_i}{\delta\beta^b} = (\partial^\mu\pi_b)\sigma - (\partial^\mu\sigma)\pi_b \end{aligned} \tag{17.8}$$

from which we can find six conserved charges

$$\begin{aligned} Q^V_a &= \int d^3x(\partial_0\vec{\pi}\times\vec{\pi})_a \\ Q^A_b &= \int d^3x[(\partial_0\pi_b)\sigma - (\partial_0\sigma)\pi_b] \end{aligned} \tag{17.9}$$

17.2 The Lie algebra of the conserved charges

As we have seen in the chapter on the generalized Nöether theorem, this two charges are the generators of the rotations on the quantized fields. If we compute the commutator of the charges we'll find the algebra. Using the quantization rules for the fields

$$\begin{aligned} [\pi_i(\mathbf{x},t),\dot{\pi}_i(\mathbf{y},t)] &= i\delta_{ij}\delta^3(\mathbf{x}-\mathbf{y}) \\ [\sigma(\mathbf{x},t),\dot{\sigma}(\mathbf{y},t)] &= i\delta^3(\mathbf{x}-\mathbf{y}) \end{aligned} \tag{17.10}$$

we'll have

$$\begin{aligned} \left[Q^V_i,Q^V_j\right] &= i\epsilon_{ijk}Q^k_V \\ \left[Q^A_i,Q^A_j\right] &= i\epsilon_{ijk}Q^k_V \\ \left[Q^V_i,Q^A_j\right] &= i\epsilon_{ijk}Q^k_A \end{aligned} \tag{17.11}$$

which is a complicated mixed algebra (like the Lorentz algebra), that can be decomposed in the same manner as the Lorentz algebra by introducing new operators

$$Q^a_{R,L} = \frac{Q^a_V \pm Q^a_A}{2} \tag{17.12}$$

Using this operators in the algebra 17.11 we find

$$\begin{aligned} \left[Q^a_{R,L},Q^b_{R,L}\right] &= \frac{1}{4}\left[Q^a_V \pm Q^a_A, Q^b_V \pm Q^b_A\right] = \frac{1}{4}\left\{\left[Q^a_V,Q^b_V\right] \pm \left[Q^a_V,Q^b_A\right] \mp \left[Q^a_A,Q^b_V\right] + \left[Q^a_A,Q^b_A\right]\right\} \\ &= \frac{i\epsilon_{abc}}{4}\{Q^c_V \pm Q^c_A \pm Q^c_A + Q^c_V\} = i\epsilon_{abc}Q^c_{R,L} \end{aligned} \tag{17.13}$$

and, easily enough

$$\left[Q_R^a, Q_L^b\right] = 0 \tag{17.14}$$

This way we separated the $SO(4)$ algebra in two parts

$$SO(4) \sim SU(2)_A \times SU(2)_V \sim SU(2)_{V+A} \times SU(2)_{V-A} \sim SU(2)_R \times SU(2)_L \tag{17.15}$$

In both bases every state carries two quantum numbers, each of which is a member of the multiplet for the two symmetries. We'll see why the notation R, L is convenient. We can therefore write a generic transformation as

$$\begin{aligned} U(\alpha,\beta) &= \exp\left(i\alpha_a Q_a^V + i\beta_b Q_b^A\right) = \exp\left(i\alpha_a(Q_a^R + Q_a^L) + i\beta_b(Q_b^R + Q_b^L)\right) \\ &= \exp\left(i(\alpha_a+\beta_a)Q_a^R + i(\alpha_b+\beta_b)Q_b^L\right) \\ &= \exp\left(i\theta_a^R Q_a^R + i\theta_b^L Q_b^L\right) = \exp\left(i\theta_a^R Q_a^R\right)\exp\left(i\theta_b^L Q_b^L\right) \\ &= U_R(\theta_R)U_L(\theta_L) \end{aligned} \tag{17.16}$$

where in the last line we used the fact that the two generators commute so there're no commutator terms in the Backer-Campbell-Haussdorff formula.

17.3 The spinorial representation of the sigma model

In order to understand how to transform a meson state, let us first introduce the spinorial representation of the field Φ. We introduce the field Σ defined as

$$\Sigma = \sigma\mathbb{1} + 2i\vec{\tau}\cdot\vec{\pi} \qquad \Sigma^\dagger = \sigma\mathbb{1} - 2i\vec{\tau}\cdot\vec{\pi} \tag{17.17}$$

where $\mathbb{1}$ is the identity matrix and $\vec{\tau}$ is the isotopic spin acting on the internal indices with

$$\tau_x = \frac{1}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \qquad \tau_y = \frac{1}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} \qquad \tau_z = \frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \tag{17.18}$$

which are the Pauli matrices satisfying the relations

$$[\tau_i, \tau_j] = i\epsilon_{ijk}\tau^k \qquad \{\tau_i, \tau_j\} = \frac{1}{2}\delta_{ij} \tag{17.19}$$

In this representation the field is given by a 2×2 matrix

$$\Sigma = \begin{pmatrix}\sigma + i\pi_3 & i(\pi_1 - i\pi_2)\\ i(\pi_1 + i\pi_2) & \sigma - i\pi_3\end{pmatrix} \tag{17.20}$$

with the following properties

$$\sigma = \frac{1}{2}\operatorname{Tr}(\Sigma\mathbb{1}) \qquad \pi_i = (-i)\operatorname{Tr}(\Sigma\tau_i) \qquad \det\Sigma = \sigma^2 + \vec{\pi}\cdot\vec{\pi} \tag{17.21}$$

Let's consider this transformation of the field Σ

$$\Sigma' = e^{i\alpha_i\tau_i}\Sigma e^{-i\alpha_i\tau_i} \approx \Sigma + i\alpha_i[\tau_i, \Sigma] \tag{17.22}$$

which component-wise gives

$$\begin{aligned} \sigma' &= \frac{1}{2}\operatorname{Tr}(\Sigma\mathbb{1}) = \frac{1}{2}\operatorname{Tr}(\Sigma\mathbb{1}) + \frac{i\alpha_i}{2}\operatorname{Tr}([\tau_i,\Sigma]) \\ &= \frac{1}{2}\operatorname{Tr}(\Sigma\mathbb{1}) + \frac{i\alpha_i\sigma}{2}\operatorname{Tr}([\tau_i,\mathbb{1}]) - \alpha_i\pi_k\operatorname{Tr}([\tau_i,\tau_k]) = \frac{1}{2}\operatorname{Tr}(\Sigma\mathbb{1}) = \sigma \end{aligned} \tag{17.23}$$

and

$$\begin{aligned}\pi'_j &= (-i)\,\mathrm{Tr}(\Sigma'\tau_j) = (-i)\,\mathrm{Tr}(\Sigma\tau_j) + \alpha_i\,\mathrm{Tr}([\tau_i,\Sigma]\tau_k)\\ &= \pi_j + \alpha_i\,\mathrm{Tr}([\tau_j,\tau_i]\Sigma) = \pi_j + i\alpha_i\epsilon_{jik}\,\mathrm{Tr}(\tau_k\Sigma)\\ &= \pi_j - \alpha_i\epsilon_{jik}\pi_k = \pi_j - (\vec{\alpha}\times\vec{\pi})_j \end{aligned} \tag{17.24}$$

which means that the transformation 17.22 is just the V transformation 17.5. In an analogous way, the following transformation

$$\Sigma' = e^{-i\beta_i\tau_i}\Sigma e^{-i\beta_i\tau_i} \approx \Sigma - i\beta_i\{\tau_i,\Sigma\} \tag{17.25}$$

which again, component-wise realizes the before defined transformations

$$\begin{aligned}\sigma' &= \frac{1}{2}\,\mathrm{Tr}(\Sigma\mathbb{1}) = \frac{1}{2}\,\mathrm{Tr}(\Sigma\mathbb{1}) - \frac{i\beta_i}{2}\,\mathrm{Tr}(\{\tau_i,\Sigma\})\\ &= \sigma - i\beta_i\,\mathrm{Tr}(\{\tau_i,\Sigma\}) = \sigma + \vec{\beta}\cdot\vec{\pi}\end{aligned} \tag{17.26}$$

and

$$\begin{aligned}\pi'_j &= (-i)\,\mathrm{Tr}(\Sigma'\tau_j) = (-i)\,\mathrm{Tr}(\Sigma\tau_j) + (-i)^2\beta_i\,\mathrm{Tr}(\{\tau_i,\Sigma\}\tau_j)\\ &= \pi_j - \beta_i\,\mathrm{Tr}(\{\tau_j,\tau_i\}\Sigma) = \pi_j - \frac{1}{2}\beta_j\,\mathrm{Tr}(\Sigma\mathbb{1})\\ &= \pi_j - \beta_j\sigma\end{aligned} \tag{17.27}$$

which is the A transformation defined in 17.5. But how do we relate this to the R, L transformations? Let's take an A transformation after a V one

$$\Sigma' = U_A^\dagger(\beta)U_V(\alpha)\Sigma U_V^\dagger(\alpha)U_A^\dagger(\beta) = e^{-i\beta_i\tau_i}e^{i\alpha_i\tau_i}\Sigma e^{-i\alpha_i\tau_i}e^{-i\beta_i\tau_i} \tag{17.28}$$

In the case of $\alpha_i = \beta_i$ (R) we find

$$\Sigma' = \Sigma U_R^\dagger \tag{17.29}$$

and in the case of $\alpha_i = -\beta_i$ (L) we find

$$\Sigma' = U_L\Sigma \tag{17.30}$$

so that a general transformation is given by

$$\Sigma' = U_L\Sigma U_R^\dagger \qquad \Sigma'^\dagger = U_R\Sigma^\dagger U_L^\dagger \tag{17.31}$$

We can now write the most general $SU(2)_R \times SU(2)_L$ renormalizable lagrangian with the Σ field noting that

$$\begin{aligned}\Sigma\Sigma^\dagger &\implies \text{Invariant under } U_R\\ \Sigma^\dagger\Sigma &\implies \text{Invariant under } U_L\\ \mathrm{Tr}\big(\Sigma\Sigma^\dagger\big) &\implies \text{Invatiante under both}\end{aligned} \tag{17.32}$$

which means that the lagrangian will be

$$\mathcal{L} = \frac{1}{4}\,\mathrm{Tr}\big(\partial_\mu\Sigma^\dagger\partial^\mu\Sigma\big) - \frac{m^2}{4}\,\mathrm{Tr}\big(\Sigma^\dagger\Sigma\big) - \frac{\lambda}{16}\big(\mathrm{Tr}\big(\Sigma^\dagger\Sigma\big)\big)^2 \tag{17.33}$$

We could have even added a term like

$$-\frac{g}{16}\,\mathrm{Tr}\big(\Sigma^\dagger\Sigma\Sigma^\dagger\Sigma\big) \overset{SU(2)}{=} -K\big(\mathrm{Tr}\big(\Sigma^\dagger\Sigma\big)\big)^2 \tag{17.34}$$

but since $SU(2)$ is pseudoreal[12], the last equality holds and so we don't add any new term to the lagrangian.

[12] The fact that the group is pseudoreal implies the following equality

$$(\sigma^a)^* = -\sigma^2\sigma^a\sigma^2$$

and together with the ciclicity of the trace gives the last equality.

18 The fermionic sector

18.1 Weyl spinors and nucleonic field

We have since now treated only the bosonic part of the lagrangia accounting for the three pseudoscalar masons and a scalar meson. The ultimate goal is to build up an effective theory of nucleon interactions, so we have now to study the nucleonic part of the lagrangian.
We know that a nucleon is given in term of an isospin dublet

$$N_\alpha(x) = \begin{pmatrix} p_\alpha(x) \\ n_\alpha(x) \end{pmatrix} \tag{18.1}$$

where p_α and n_α are the fermion fields of the proton and the neutron respectively. At this point we want to find again the vector and axial transformations for the spinor fields which are given in general by

$$\begin{aligned} &V \qquad \psi' = e^{i\alpha_i\tau_i}\psi \qquad \bar\psi' = \bar\psi e^{-i\alpha_i\tau_i} \\ &A \qquad \psi' = e^{i\alpha_i\tau_i\gamma_5}\psi \qquad \bar\psi' = \bar\psi e^{-i\alpha_i\tau_i\gamma_5} \end{aligned} \tag{18.2}$$

It's clear that there's no change of sign in the axial transformation of ψ and $\bar\psi$ do to the fact that $\{\psi_5, \gamma^\mu\} = 0$.
Note that to be a symmetry of the theory, it's generators should commute with the Lorentz generators which is always a symmetry. The only symmetries acting on Dirac indices that commute with $S^{\mu\nu}$ are, in fact, $\mathbb{1}$ and γ_5 which give in fact the two transformations in 18.2.

We knot that a dirac spinor transforms with the representation $\left(\frac{1}{2},0\right)\otimes\left(0,\frac{1}{2}\right)$ of the Lorentz group so that it can be decomposed into two Weyl spinors: a right handed spinor ψ_R which transforms as a doublet in $\left(\frac{1}{2},0\right)$ and as a singlet in $\left(0,\frac{1}{2}\right)$ and a left handed spinor ψ_L which transforms as a doublet in $\left(0,\frac{1}{2}\right)$ and as a singlet in $\left(\frac{1}{2},0\right)$. This is what us called as the **chiral representation**

$$\psi_L = \frac{(1-\gamma_5)}{2}\psi = P_-\psi \qquad \frac{(1+\gamma_5)}{2}\psi = P_+\psi \tag{18.3}$$

where $P_\pm$ are projection operators onto the right handed chirality or left handed chirality. Let's see how the two Weyl spinors transform under the V, A transformations using infinitesimal transformation

$$\psi' \approx \left(1 + i\frac{i\alpha_i\tau_i}{2} + i\frac{\beta_i\tau_i\gamma_5}{2}\right)\psi \tag{18.4}$$

which can be projected onto the Weyl spinor using the relation $P_- + P_+ = \mathbb{1}$

$$\begin{aligned} P_+\psi' + P_-\psi' &= \left(1 + i\frac{\alpha_i\tau_i}{2} + i\frac{\beta_i\tau_i\gamma_5}{2}\right)(P_+\psi + P_-\psi) \\ \psi'_L &= \left(1 + i\frac{\alpha_i\tau_i}{2} - i\frac{\beta_i\tau_i}{2}\right)\psi_L \qquad \psi'_R = \left(1 + i\frac{\alpha_i\tau_i}{2} + i\frac{\beta_i\tau_i}{2}\right)\psi_R \end{aligned} \tag{18.5}$$

where in the last line we used the property of the γ_5 matrix

$$P_\pm \gamma_5 = \pm P_\pm \tag{18.6}$$

We therefore conclude, using the operators defined in 17.29 and 17.30, that

$$\begin{array}{lll} \psi'_R = U_R \psi_R & \psi'_L = \psi_L & \text{Under } U_R \\ \psi'_L = U_L \psi_L & \psi'_R = \psi_R & \text{Under } U_L \\ \text{Doublet} & \text{Singlet} & \end{array} \tag{18.7}$$

In the notation of $SU(2)_L \times SU(2)_R \equiv (n_L, m_R)$ the two Weyl spinors and the bosonic Σ field are given by the representations

$$\psi_L \in (2,1) \qquad \psi_R \in (1,2) \qquad \Sigma \in (2,\bar{2}) \tag{18.8}$$

So there is no γ_5 associated with the V, A transformations, but only two different fields which rotates differently under R, L.

18.2 The fermion lagrangian

We have now to build up the fermionic $SU(2)_R \times SU(2)_L$ invariant lagrangian with operators with dimensions ≤ 4. The only factor that respects this requirement is the kinetic factor

$$\mathcal{L} = i\bar{\psi}_R \partial\!\!\!/ \psi_R + i\bar{\psi}_L \partial\!\!\!/ \psi_L \tag{18.9}$$

Not even a mass term is possible since

$$m(\bar{\psi}_R \psi_L + \bar{\psi}_L \psi_R) \tag{18.10}$$

since is not invariant under R, L transformations. In fact

$$m(\bar{\psi}_R U_R^\dagger U_L \psi_L + \bar{\psi}_L U_L^\dagger U_R \psi_R) \tag{18.11}$$

is clearly different from the term 18.10. The only possible interaction terms between fermions are

$$g(\bar{\psi}\gamma^\mu \psi)(\bar{\psi}\gamma_\mu \psi) \qquad g(\bar{\psi}\gamma^\mu \gamma_5 \psi)(\bar{\psi}\gamma_\mu \gamma_5 \psi) \tag{18.12}$$

but this have dimension 6 so we do not consider them.

The conserved vector and axial currents for the spinor fields are

$$\begin{aligned} j^\mu_{V,a} &= -\frac{\delta\mathcal{L}}{\delta(\partial_\mu \psi)} \frac{\delta\psi}{\delta\alpha_a} = \bar{\psi}\gamma^\mu \tau_a \psi \\ j^\mu_{A,a} &= -\frac{\delta\mathcal{L}}{\delta(\partial_\mu \psi)} \frac{\delta\psi}{\delta\beta_a} = \bar{\psi}\gamma^\mu \gamma_5 \tau_a \psi \end{aligned} \tag{18.13}$$

19 The full theory

19.1 The interaction term

A general lagrangian has to take into consideration the meson part, the fermion part and the interaction

$$\mathcal{L} = \mathcal{L}_{Bose} + \mathcal{L}_{Fermi} + \mathcal{L}_{Int} \tag{19.1}$$

where we have discussed the first two parts before and are given by 17.33 and 18.9. What remains to find is the interaction lagrangian.

The only possible interaction term which, again is renormalizable and has the full $SU(2)_R \times SU(2)_L$ symmetry, is given by

$$\mathcal{L}_{Int} = Y\left[\bar{\psi}_L \Sigma \psi_R + \bar{\psi}_R \Sigma^\dagger \psi_L\right] \tag{19.2}$$

where if we want to explicitly have it in terms of protons and neutron fields

$$\begin{aligned} &= Y\left[\bar{\psi}\frac{(1+\gamma_5)}{2}(\sigma\mathbb{1} + 2i\vec{\tau}\cdot\vec{\pi})\frac{(1+\gamma_5)}{2}\psi + \bar{\psi}\frac{(1-\gamma_5)}{2}(\sigma\mathbb{1} - 2i\vec{\tau}\cdot\vec{\pi})\frac{(1-\gamma_5)}{2}\psi\right] \\ &= Y\left[\sigma\psi\bar{\psi} + 2i\vec{\pi}\cdot\left(\bar{\psi}\gamma_5\vec{\tau}\psi\right)\right] \end{aligned} \tag{19.3}$$

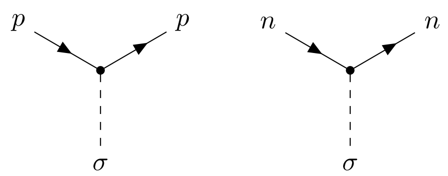
We see that there are are in fact two terms in the interaction, let's study them separately.

19.2 The nucleon scalar meson interaction vertex

The first term consist in the interaction of the proton and neutron field and the scalar field σ

$$Y\sigma\psi\bar{\psi} = Y(p\bar{p} + n\bar{n})\sigma \tag{19.4}$$

This vertex looks like the electromagnetic one, since there are an incoming proton (neutron), an outgoing proton (neutron) and an emitted meson σ. The main difference between this vertex and the electromagnetic one is that σ is a scalar, so there is no need of introducing γ_μ and the vertex is simply $-iY$. The diagrams associated to this term are the following



(19.5)

19.3 The nucleon pion interaction vertex

The second term is more interesting since it represent the nucleon-pion interactions. First let us write $\vec{\tau}\cdot\vec{\pi}$ as

$$\vec{\tau}\cdot\vec{\pi} = \tau_1\pi_1 + \tau_2\pi_2 + \tau_3\pi^0 = \frac{1}{\sqrt{2}}(\pi^+\tau_- + \pi^-\tau_+) + \pi_3\tau_3 \tag{19.6}$$

where

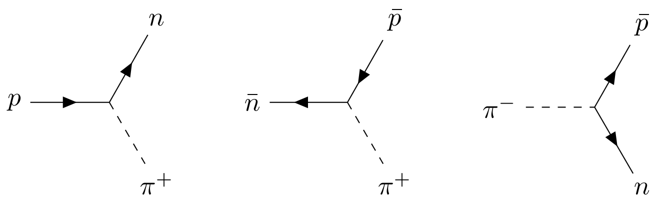
$$\pi^\pm = \frac{\pi_1 \pm i\pi_2}{\sqrt{2}} \qquad \pi_3 = \pi^0 \qquad \tau_\pm = \tau_1 \pm i\tau_2 \tag{19.7}$$

Using this notation, the second term in the interaction lagrangian 19.3 can be rewritten as

$$\begin{aligned} Y2i\vec{\pi}\cdot(\bar{\psi}\gamma_5\vec{\tau}\psi) &= 2iY\begin{pmatrix}\bar{p} & \bar{n}\end{pmatrix}\left[\frac{1}{\sqrt{2}}(\pi^+\tau_- + \pi^-\tau_+) + \pi^0\tau_3\right]\gamma_5\begin{pmatrix}p\\ n\end{pmatrix} \\ &= iY\left[\sqrt{2}(\bar{n}\gamma_5 p\pi^+ + \bar{p}\gamma_5 n\pi^-) + (\bar{p}\gamma_5 p - \bar{n}\gamma_5 n)\pi^0\right] \end{aligned} \tag{19.8}$$

The factor $\sqrt{2}$ that we find in the term that coulpes charged pions with the nucleons derives from the isospin combination while the i is necessary for the hermiticity of the lagrangian. In fact $(\pi^+\bar{\pi}\gamma_5)^\dagger = -\pi^-\bar{n}\gamma_5 p$.

The first term corresponds to the vertex where the incoming proton p is annihilated with the creation of a neutron n and a pion π^+ in the final state. The total electric charge is conserved at the vertex. This term also represents the annihilation of an antineutron with the creation of an antiproton $\bar{p}$ and a π^+, or the annihilation of a virtual π^- into a neutron n and a antiproton $\bar{p}$. All this diagram are found by the first one just by taking the particles from the initial state to the final state and changing it into it's antiparticle. The Feynman rule corrisponding to this vertex is $\sqrt{2}Y\gamma_5$. Some of the diagrams associated with it are the following



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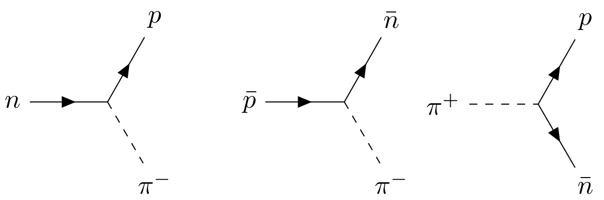
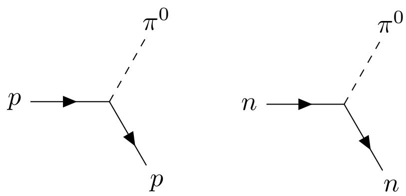
The second term corresponds to a vertex with the annihilation of an incoming neutron and the creation of a proton p and a π^- in the final state. The same term also represents the annihilation of an antiproton $\bar{p}$ to give an antrineutron $\bar{n}$ and a π^-, or the annihilation of a π^+ into a proton-antineutron pair. The Feynman rules for these vertices is the same as before. Some of the Feynman diagrams are shown



(19.10)

The third term corresponds to two possible vertices, one with the emission of a neutral pion π^0 by a proton, the other the emission of a π^0 by a neutron. All the other crossed channels are also present as in the previous cases. The Feynman rule for this vertex in these last cases is $\pm Y\gamma_5$. Note that between the coupling of the charged pions and the neutral there is a factor $\sqrt{2}$, which is simply a Clebsch-Gordan coefficient for the isospin, as said before. The Feynman diagrams corresponding to the third vertex are shown



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PART
VIII

Spontaneous Symmetry Breaking

Spontaneous symmetry breaking is a very general phenomena characterized by the fact that the action has a symmetry, local or global, but the quantum theory, instead of having a unique vacuum state which respects this symmetry, has several degenerate vacua that transform into each other under the action of the symmetry group.
When a global symmetry is spontaneously broken, in the spectrum of the theory, there is a massless particle for each broken symmetry generator. We'll see that in particular the pions are the Goldstone bosons of the spontaneously broken $SU(2)$ axial symmetry of QCD. They would be exactly massless if the symmetry were exact. Since it's only approximate, they are just lighter than the other hadrons.
When a local symmetry is spontaneously broken, the gauge field becomes massive and the would-be Goldstone boson is turns into a third physical degree of freedom of the massive spin-1 gauge field. This mechanism gives an effective mass to the gauge bosons $W^{\pm}$ and Z^0 of the electroweak theory.

20 SSB in statistical mechanics

20.1 The Landau-Ginzburg theory of phase transitions

Spontaneous symmetry breaking is a phenomena that can be found all over in physics. To introduce the concept of SSB we first take a look at a very simple model of ferromagnetism.

Let us consider a solid with some magnetization density $\vec{\mathbf{M}}(\vec{\mathbf{x}})$[13] The energy given by the interaction of the magnetic dipoles

$$H = -\sum_{x,v} J(x,v) M(x) \cdot M(x+v) \approx -\sum_{x,v} J(|v|) M(x) \cdot M(x+v) \tag{20.1}$$

where for simplicity we considered that the magnetization is homogeneous and isotropic. If we further simplify by considering only nearest neighbours interaction with equal coupling, the energy becomes

$$H = -J \sum_{x,i} M(x) \cdot M(x+ai) \tag{20.2}$$

where a is the lattice spacing.
A potential term can be added to this energy since we can consider the energy depending also to the total magnetization of the solid. This interaction term has to be rotationally invariant to reflect the fact that the dipole moments can align in any possible direction. The only rotationally invariant term of $M(x)$is $M(x) \cdot M(x)$ and so

$$H = -J \sum_{x,i} M(x) \cdot M(x+ai) + V(M(x) \cdot M(x)) \tag{20.3}$$

What we are now interested in is the region of **phase transition**, particularly second order phase transition. The relevant parameter for phase transition in a ferromagnet is the **Curie temperature**. It's well known that above Curie temperature, magnetism ceases to exist in a ferromagnet. A diagram of the second order phase transition is given in figure 10.
So around T_C, magnetization is small and we can therefore expand 20.3 is Taylor series

[13] To simplify my work, I'll write $M(x)$ instead of $\vec{\mathbf{M}}(\vec{\mathbf{x}})$. This won't change in any way the theory, since we could have considered a uniaxial ferromagnet instead of a 3-dimensional one.

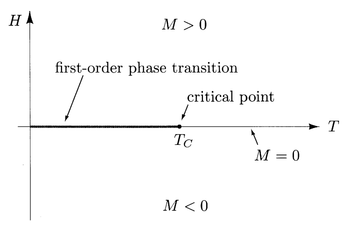


Figure 10. Phase diagram in the $H - T$ plane for a uniaxial ferromagnet

$$H \approx -\frac{Ja^2}{2}\left(\frac{M(x)-M(x+ai)}{a}\right)^2 + B(T)(M\cdot M) + C(T)(M\cdot M)^2 \tag{20.4}$$

In the limit of $a \to 0$, so if we look at the macroscopic level, and by redefining the magnetization to incorporate the constant J, the energy becomes

$$H = \frac{1}{2}(\nabla M)^2 + B(T)(M\cdot M) + C(T)(M\cdot M)^2 \tag{20.5}$$

Since the system wants to minimize it's energy, it'll go in the configuration of M in which H is minimized. Clearly, M has to be a minima of the potential part of 20.5. In this case

$$0 = \frac{\partial V}{\partial M} = 2B(T)M + 4C(T)M^3 \tag{20.6}$$

and here comes the catch: this minima clearly depends on the specific sign of the constants B and C. This constant, in field theory, are the mass squared and the possible quartic coupling, so it's important to understand this fact.
If B and C are both positive, the only solution of 20.6 is $M = 0$. However, if $C > 0$ but B is negative below some temperature T_C, we have a non-trivial solution for $T < T_C$, as shown in figure 11. More concretely, approximate for $T \approx T_C$

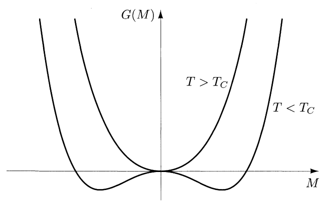


Figure 11. Behaviour of the potential energy at temperatures above and belowthe critical temperature.

$$B(T) = b(T - T_C) \qquad C(T) = c > 0 \tag{20.7}$$

Then the solution to 20.6 is

$$M = \begin{cases} 0 & \text{for } T > T_C \\ \pm\left[\frac{b}{2c}(T-T_C)\right]^{1/2} & \text{for } T > T_C \end{cases} \tag{20.8}$$

This is the qualitative behaviour that we expect at a critical point: a sharp, non continuous, transition between two states.

20.2 The choice of the vacuum

What we have found is that below critical temperature, there are two possible minima if the system is one dimensional, and an infinite quantity of minima lying on a circle if the system is three dimensional.
This same result could have been found starting from 20.5 by a suitable name of the constants

$$H = \frac{1}{2}(\nabla M)^2 + \frac{m^2(T)}{2}M^2 + \frac{\lambda}{4}M^4 \tag{20.9}$$

together with the following step when the temperature is below T_C

$$m^2(T) \to -\mu^2 \qquad \text{When } T < T_C \tag{20.10}$$

so that the new energy acquires more minima

$$H = \frac{1}{2}(\nabla M)^2 - \frac{\mu^2}{2}M^2 + \frac{\lambda}{4}M^4 \tag{20.11}$$

The spontaneous breaking of symmetry comes whenever we apply an external magnetic field so that we put the system in **one** of all the possible minima. A ferromagnet below critical temperature immersed in an external magnetic field has energy

$$H = \frac{1}{2}(\nabla M)^2 - \frac{\mu^2}{2}M^2 + \frac{\lambda}{4}M^4 - B\cdot M \tag{20.12}$$

This additional term lowers one of the possible minima making it the only absolute minima. The system will now be permanently magnetized sine the transition probability between the absolute minima and the relative ones is very small.

> **Remark.** Below critical temperature it becomes thermodynamically favorable to develop a non-zero magnetization, and in this new vacuum $M \neq 0$ and the full $SO(3)$ symmetry is **broken** to the subgroup $SO(2)$ of rotations around the magnetization axis.

The original invariance of the system is now reflected in the fact that, instead of a single vacuum state, there is a whole family of vacua related to each other by rotations, since the magnetization can in principle be developed in any direction. However, the system will choose one of this states as it's vacuum state, which is exactly what we have done by putting in an external magnetic field. The symmetry is then said to be spontaneously broken by the choice of the vacuum.

21 The mass term

Suppose for now the have a theory described by the following lagrangian

N.B. This part wasn't in Martinelli's lectures but I think that it's instructive to take a look at it nonetheless.

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + e^{-(\alpha\phi)^2} \tag{21.1}$$

where α is some real constant. Where is the mass term in this lagrangian? At first glance it is not clear if there is one. Of course the field ϕ could be massless without any problem, but this is not the case. In fact, if we expand in series the exponential

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + 1 - \alpha^2\phi^2 + \frac{1}{2}\alpha^4\phi^4 - \frac{1}{6}\alpha^6\phi^6 + \cdots \tag{21.2}$$

the quadratic term looks just like a mass term. Evidently the lagrangian 21.1 describes a partile of mass

$$m = \sqrt{2}\alpha \tag{21.3}$$

and the higher order terms represent various couplings. This is only one example in which the mass term in a lagrangian can be disguised.
One other very subtle example which we've used before is given by the following lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{\mu^2}{2}\phi^2 - \frac{\lambda}{4}\phi^4 \tag{21.4}$$

Here μ and λ are real constants. The second term looks like a mass term but with the wrong sign! This lagrangian is exactly the same as the one we used before in 20.11. If that's a mass term, then m should be imaginary, which is nonsense. But let us stick with this for a moment and try to reason on how we could interpret this lagrangian. To answer that question we must understand that Feynman calculus is really a perturbation procedure, in which we start from the ground state, the vacuum, and treat fields as flactuations about that state. For most of the lagrangian seen up until now, the ground state was always the trivial one $\phi = 0$. For this lagrangian we find it by minimizing the potential term

$$U(\phi) = -\frac{\mu^2}{2}\phi^2 + \frac{\lambda}{4}\phi^4 \tag{21.5}$$

which has two minima

$$\phi = \pm\frac{\mu}{\sqrt{\lambda}} \tag{21.6}$$

The Feynman calculus must be formulated in terms of deviation from one or the other ground state. We therefore have to choose one and for that we introduce a new field variable

$$\eta = \phi - \frac{\mu}{\sqrt{\lambda}} \tag{21.7}$$

In terms of the new field η the lagrangian 21.4 becomes

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}\partial_\mu\eta\partial^\mu\eta + \frac{\mu^2}{2}\left(\eta + \frac{\mu}{\sqrt{\lambda}}\right)^2 - \frac{\lambda}{4}\left(\eta + \frac{\mu}{\sqrt{\lambda}}\right)^4 \\ &= \frac{1}{2}\partial_\mu\eta\partial^\mu\eta - \mu^2\eta^2 - \mu\sqrt{\lambda}\eta^3 - \frac{\lambda}{4}\eta^4 \end{aligned} \tag{21.8}$$

leaving out the constant term which does not change anything. We see now a remarkable thing: the second term now is a mass term with the right sign. So this lagrangian describes a particle of mass

$$m = \sqrt{2}\mu \tag{21.9}$$

and the other terms describe various interactions. I emphasize that these lagrangians describe exactly the **same physica system**, all we have done is to change the notation for the field.

Remark. The example we have just considered illustrates the phenomena of spontaneous symmetry breaking. The original lagrangian was even in ϕ: it was invariant under the transformation $\phi \to -\phi$. But the reformulated lagrangian is not even in η; the symmetry has been broken. It happened because the vacuum, whichever of the two one, does not share the symmetry of the lagrangian.

The symmetry broken here is just a discrete symmetry. More interesting things happen when a continuous symmetry is broken, which we'll see later on.

22 SSB In the linear sigma model

22.1 Symmetry breaking in the bosonic sector

As discussed before we can apply the same ideas to the sigma model where the three pseudoscalar mesons and the sigma meson are described by the lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma + \frac{1}{2}\partial_\mu\vec{\pi}\cdot\partial^\mu\vec{\pi} - \frac{m^2}{2}(\sigma^2 + \vec{\pi}\cdot\vec{\pi}) - \frac{\lambda}{4}(\sigma^2 + \vec{\pi}\cdot\vec{\pi})^2 \tag{22.1}$$

This lagrangian is invariant under the continuous symmetry group $SU(2)_V \times SU(2)_A$. In this theory, all the bosons have the exact same mass m. This theory has only one vacuum which is in $\sigma = 0$ and $\pi = 0$. If now the mass term goes negative $m^2 \to -\mu^2$ we're dealing with a set of vacua, connected to one another by some symmetry. To break this symmetry we put an additional term $-h\sigma$ and study what happens.[1] The lagrangian is now

$$\mathcal{L} = \frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma + \frac{1}{2}\partial_\mu\vec{\pi}\cdot\partial^\mu\vec{\pi} + \frac{\mu^2}{2}(\sigma^2 + \pi^2) - \frac{\lambda}{4}(\sigma^2 + \pi^2)^2 + h\sigma \tag{22.2}$$

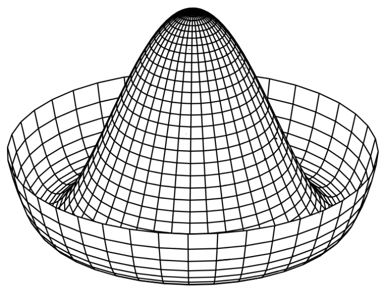


Figure 12. Symmetric potential $V(\sigma^2 + \pi^2)$.

[1]This is like adding the external magnetic field term in the ferromagnet to put it in one specific magnetization state, i.e. by fixing a real vacuum.

We need to find the new vacuum of the theory which is easily done by minimizing the potential

$$\begin{aligned}\frac{\partial U}{\partial \sigma} &= -\mu^2\sigma + \lambda\sigma(\sigma^2+\pi^2) - h = 0 \\ \frac{\partial U}{\partial \pi_i} &= -\mu^2\pi_i + \lambda\pi_i(\sigma^2+\pi^2) = 0 \implies \vec{\pi} = 0\end{aligned} \tag{22.3}$$

call the solution to the first equation v

$$\vec{\pi} = 0 \implies -\mu^2 v + \lambda v^3 - h = 0 \tag{22.4}$$

To find the condition of minimization we study the second derivatives

$$\begin{aligned}\left.\frac{\partial^2 U}{\partial \sigma^2}\right|_{\pi=0,\sigma=v} &= -\mu^2 + 3\lambda v^2 = m_\sigma^2 \\ \left.\frac{\partial^2 U}{\partial \pi_i \partial \pi_j}\right|_{\pi=0,\sigma=v} &= -\mu^2\delta_{ij} + 2\lambda v^2\delta_{ij} = m_\pi^2 \\ \left.\frac{\partial^2 U}{\partial \sigma \partial \pi_i}\right|_{\pi=0,\sigma=v} &= 0\end{aligned} \tag{22.5}$$

From the equation defining v we find the following relation

$$-\mu^2 v + \lambda v^3 - h = 0 \implies \lambda v^2 = \frac{h}{v} + \mu^2 \tag{22.6}$$

that, together with the equations 22.5, defines the mass matrix

$$\frac{\partial^2 U}{\partial \Phi_i \partial \Phi_j} = \begin{pmatrix} 2\lambda v + \frac{h}{v} & & & \\ & \frac{h}{v} & & \\ & & \frac{h}{v} & \\ & & & \frac{h}{v} \end{pmatrix} \tag{22.7}$$

The minima is where all the eigenvalues of this matrix are positive, which will give us four positive mass terms. The only way to be positive is when h and v have the same sign. Let's now redefine the fields in the stable vacuum $\sigma \to \sigma + v$ and $\pi \to \pi + 0$. Putting it all back in the lagrangian, the kinetic term remains the same, while the potential part will be given by

$$\begin{aligned}&\frac{\mu^2}{2}\left[(\sigma+v)^2+\pi^2\right] - \frac{\lambda}{4}\left[(\sigma+v)^2+\pi^2\right]^2 + h(\sigma+v) \\ &= \frac{\mu^2}{2}\sigma^2 + \frac{\mu^2}{2}v^2 + \frac{\mu^2}{2}\pi^2 + \mu^2 v\sigma - \frac{\lambda}{4}\sigma^4 - \frac{\lambda}{4}v^4 - \frac{\lambda}{4}\pi^4 \\ &- \lambda v^2\sigma^2 - \frac{\lambda}{2}\sigma^2\pi^2 - \frac{\lambda}{2}\sigma^2 v^2 - \lambda v\sigma^3 - \lambda v^3\sigma - \frac{\lambda}{2}v^2\pi^2 - \lambda\sigma v\pi^2 + h\sigma + hv\end{aligned} \tag{22.8}$$

Let's check term by term for the σ field

$$
\begin{aligned}
\sigma^4 \text{ term} \quad & -\frac{\lambda}{4}\sigma^4 \\
\sigma^3 \text{ term} \quad & -\lambda v \sigma^3 \\
\sigma^2 \text{ term} \quad & \frac{\mu^2}{2}\sigma^2 - \lambda v^2\sigma^2 - \frac{\lambda}{2}v^2\sigma^2 \\
& = -\frac{\mu^2}{2}\sigma^2 + \frac{h}{v}\sigma^2 - \mu^2\sigma^2 - \frac{h}{2v}\sigma^2 \\
& = -\frac{1}{2}\left(2\mu^2 + 2\frac{h}{v} + \frac{h}{v}\right)\sigma^2 \\
& = -\frac{1}{2}\left(2\lambda v + \frac{h}{v}\right)\sigma^2 = -\frac{m_\sigma^2}{2}\sigma^2 \\
\sigma \text{ term} \quad & \mu^2 v\sigma + h\sigma - \lambda v^3\sigma \\
& = (\mu^2 v + h - \lambda v^3)\sigma \\
& = (\lambda v^3 - h + h - \lambda v^3)\sigma = 0
\end{aligned}
\tag{22.9}
$$

and same for the π field

$$
\begin{aligned}
\pi^4 \text{ term} \quad & -\frac{\lambda}{4}\pi^4 \\
\pi^2 \text{ term} \quad & \frac{\mu^2}{2}\pi^2 - \frac{\lambda}{2}v^2\pi^2 \\
& = \frac{1}{2}(\mu^2 - \lambda v^2)\pi^2 \\
& = -\frac{h}{2v}\pi^2 = -\frac{m_\pi^2}{2}\pi^2
\end{aligned}
\tag{22.10}
$$

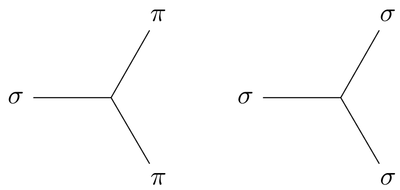
there only remains mixed interaction terms and the vacuum terms. All the calculations where based on the equivalence

$$
\lambda v^2 = \frac{h}{v} + \mu^2 \tag{22.11}
$$

It's clear now that we got two mass terms with the right sign and the two fields have different masses.
The full lagrangian is now given by the this factors

$$
\begin{aligned}
\mathcal{L}_{\text{free}} &= \frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma - \frac{m_\sigma^2}{2}\sigma^2 + \frac{1}{2}\partial_\mu\pi\partial^\mu\pi - \frac{m_\pi^2}{2}\pi^2 \\
\mathcal{L}_{\text{int}} &= -\frac{\lambda}{4}\sigma^4 - \frac{\lambda}{4}\pi^4 - \frac{\lambda}{2}\sigma^2\pi^2 - \lambda v(\sigma\pi^2 + \sigma^3) \\
\mathcal{L}_{vac} &= \frac{\mu^2}{2}v^2 - \frac{\lambda}{4}v^4 + hv
\end{aligned}
\tag{22.12}
$$

By comparing this new lagrangian 22.12 with the initial one 22.1 we see some striking differences: firstly the two fields have different real masses with $\Delta m = m_\sigma^2 - m_\pi^2 = 2\lambda v$. Secondly we have new interaction terms given by $\lambda v(\sigma\pi^2 + \sigma^3)$ which are found by the initial interaction terms by exchanging one σ term with v, so to give rise to the following vertices



(22.13)

Now we have to take the symmetry breaking parameter h to zero. In this limit we see that

$$\lim_{h\to 0} m_\pi^2 = 0 \tag{22.14}$$

Remark. The pion becomes massless! This is not a coincidence, it's a result of a general theorem known as Goldstone theorem, which we'll prove afterwards. The pion is now a Nambu-Goldstone (or Goldstone) boson.

In the same manner we can find the mass of the σ meson from the first equation of 22.3

$$-\mu^2 v + \lambda v^3 = 0 \implies v = +\sqrt{\frac{\mu}{\lambda}} \tag{22.15}$$

so that, from the first of 22.5 we find

$$m_\sigma^2 = 2\lambda\frac{\mu}{\lambda} \implies m_\sigma = \sqrt{2}\mu \tag{22.16}$$

What we got at the end is a theory in which there is a massive boson and three massless Goldstone bosons going from an $SU(2)_A \times SU(2)_V$ invariant theory to a $SU(2)_V$ theory. The first had 6 generators while the latter has only 3. It's not a case that there are exactly as much Goldstone bosons as the number of broken symmetry generators.

What is the physical meaning of the field $\vec{\mathbf{h}}$ which breaks explicitly the symmetry of our system? in the magnetic case was just the real external magnetic field, that we send to zero in order to study a spontaneously magnetized system in the absence of any external force. We have however many other physical examples where the symmetry is almost realised, although it is explicitly broken by a soft term. This is the case of hadronic physics. If all the light quarks, namely u, d and s, where to be degenerate in mass, the full $SU(3)$ symmetry would be exact and the lagrangian would be invariant under this symmetry. Under this hypothesis the masses of the full baryon octet would be degenerate with the proton mass. This situation corresponds to the cas in which $\vec{\mathbf{h}} = 0$. Since the mass of the quarks are note equal, however, the mass differences manifests as *magnetic fields* pointing in a given direction fixed by the strange to up-down mass difference

$$\mathcal{L}_{h=\Delta m_{\text{sud}}} = \Delta m_{\text{sud}}\bar{q}\lambda_8 q = \frac{2m_s - m_u - m_d}{3}(2\bar{s}s - \bar{u}u - \bar{d}d) \tag{22.17}$$

and by the mass isospin explicit symmetry breaking

$$\mathcal{L}_{h=\Delta m_{ud}} = \Delta m_{ud}\bar{q}\tau_3 q = \frac{m_d - m_u}{2}(\bar{d}d - \bar{u}u) \tag{22.18}$$

We note that the masses of the quarks cannot be derived within the theory of strong nor electroweak interactions. The origin of the masses of the quarks comes from very high scales, so that the. physical origin is beyond the Standard Model.

22.2 Yukawa interactions in the broken phase

Now we study the consequences of symmetry breaking in the fermion-boson interaction part of the sigma model

$$\mathcal{L} = -Y\left[\sigma\bar{\psi}\psi + 2i\pi_a\bar{\psi}\gamma_5\tau_a\psi\right] \tag{22.19}$$

What we found in the section on the interaction of the sigma model was that the fermions had to be massles to have a lagrangian with the same symmetry of the boson one, which clearly is not physically acceptable. Let's see now if symmetry breaking solves the problem. Using the shift $\sigma \to \sigma + v$ we get a mass term for the fermions

$$\mathcal{L} = -Y\left[\sigma\bar{\psi}\psi + v\bar{\psi}\psi + 2i\pi_a\bar{\psi}\gamma_5\tau_a\psi\right] \tag{22.20}$$

with $m_f = Yv$. Under symmetry breaking, fermions become massive with a mass m_f. This means that the field σ couples more to massive fermions. We'll see later that the σ particle is actually the Higgs boson.
Since the decay width is proportional to the square of the mass, we expect the Higgs, if $m_H > 2m_f$, to decay most of the times in heavy fermions. In fact this is exactily what we see experimentally. The most probable fermionic decay of the Higgs is into a bottom-antibottom pair. The decay into top fermions is not possible since $m_t > m_H$. Furthermore, the decays into e^+, e^- or μ^+, μ^- are practically negligible.

The symmetry breaking $SU(2)_A \times SU(2)_V \to SU(2)_V$ clearly influences also the conserved currents we found before. Since the only surviving symmetry is $SU(2)_V$ we would expect that current to remain unchanged, and in fact

$$j_V^{\mu,a} = \bar{\psi}\gamma^\mu\tau^a\psi + (\pi \times \partial^\mu\pi)^a \tag{22.21}$$

remains unchanged. This is also clear by looking at the symmetry broken lagrangian. Meanwhile, the axial part of the current won't remain conserved since under symmetry breaking $\sigma \to \sigma + v$

$$j_A^{\mu,a} = \bar{\psi}\gamma^\mu\gamma_5\tau^a\psi + (\partial^\mu\pi^a)(\sigma + v) - \pi^a(\partial^\mu\sigma) \tag{22.22}$$

we see that a new term appears in the current found before. By direct computation of the divergence of the current, in fact, one founds that

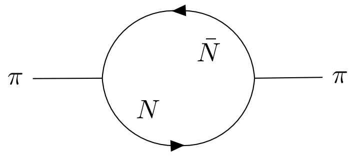
$$\begin{aligned}
\partial_\mu j_A^{\mu,a} &= -\frac{\delta\mathcal{L}}{\delta\alpha^a} = -\frac{\delta}{\delta\alpha^a}\left(-\frac{1}{2}m_\sigma^2\sigma^2 - \lambda v\sigma(\pi^2+\sigma^2) - m_f\bar{\psi}\psi\right)\\
&= m_\sigma^2\sigma\frac{\delta\sigma}{\delta\alpha^a} + \lambda v\frac{\delta\sigma}{\delta\alpha^a} + \lambda v\sigma\frac{\delta}{\delta\alpha^a} + m_f\frac{\delta}{\delta\alpha^a}\bar{\psi}\psi\\
&= m_\sigma^2\sigma\pi^a + \lambda v\pi^a(\sigma^2+\pi^2) + 2\lambda\sigma v(\pi^a\sigma - \pi^a\sigma) + 2im_f\bar{\psi}\tau^a\gamma_5\psi\\
&= 2im_f\bar{\psi}\tau^a\gamma_5\psi + m_\sigma^2\sigma\pi^a + \lambda v\pi^a(\sigma^2+\pi^2) \neq 0
\end{aligned} \tag{22.23}$$

clearly, this is not a conserved quantity.

Let us now make a calculation that might seem nonsensical but it will prove useful in the calculation of the pion decay width. We calculate the matrix element of the axial current between a pion and the vacuum in first order in perturbation theory

$$\langle 0|\, j_A^{\mu,a}\, |\pi^b\rangle \tag{22.24}$$

Firstly we note that the only term in the current 22.22 that gives a non-zero value between the vacuum and a pion is the term $v\partial^\mu\pi^a$ since the term with the fermion fields contains only creation and annihilation operators for fermions and therefore cannot annihilate the pion. in higher order this term can contribute to loop corrections like



(22.25)

The terms with pions and the σ boson contain some combination of creation and annihilation operators for the two fields and, even if the pion is annihilated, a *sigma* is created and viceversa. So the only possible useful term is the one which depends only on the pion field.
With this in mind let us evaluate the matrix element

$$\begin{aligned}\langle 0| J_A^{\mu,a} \left|\pi^b\right\rangle &= v \langle a| \partial^\mu \pi^a(x) \left|\pi^b\right\rangle = v\partial^\mu \langle 0| \pi^a(x) \left|\pi^b\right\rangle \\ &= v\partial^\mu \langle 0| e^{ipx}\pi^a(0)e^{-ipx} \left|\pi^b\right\rangle = v\partial^\mu e^{-ip_\pi x} \langle 0| \pi^a(0) \left|\pi^b\right\rangle \\ &= -ivp_\pi^\mu e^{-ip_\pi x} \langle 0| \pi^a(0) \left|\pi^b\right\rangle = -ivp_\pi^\mu \langle 0| \pi^a(x) \left|\pi^b\right\rangle \\ &= -ivp_\pi^\mu \delta^{ab} \equiv -ip_\pi^\mu f_\pi \delta^{ab}\end{aligned} \tag{22.26}$$

With this we find that the matrix element is non zero and is given by

$$\langle 0| j_A^{\mu,a} \left|\pi^b\right\rangle = -ip_\pi^\mu f_\pi \delta^{ab} \tag{22.27}$$

where f_π is the pion decay constant and is experimentally found to be around 132 MeV.
With the normalization of the states *a la Feynman* we have

$$\begin{aligned}&\langle p|p'\rangle = (2\pi)^3 2E_p \delta(p-p') \\ &[j] = 3 \qquad [|p\rangle] = -1 \\ &[\langle 0| j \left|\pi\right\rangle] = 3 - 1 = 2 \propto [m]\end{aligned} \tag{22.28}$$

The result we found 22.27 is generally valid outside perturbation theory. This is so since the result is a direct consequence of the Wigner-Eckart theorem. Since $\left|\pi^b\right\rangle$ is a pseudoscalar and $\partial^\mu \pi^a$ is a pseudovector, their product has to transform like a vector and since the only vector at our disposal is p^μ that matrix element has to depend on p^μ, which in fact does. Moreover the proportionality constant has to be a Lorentz invariant quantity and the only possible one here is $p^2 = m_\pi^2$.

23 Back to the Fermi theory

We're now able to justify the presence of the two different factors g_A and g_V in the Fermi Hamiltonian for the decay of the neutron

$$H = -\frac{G_F}{\sqrt{2}} g_V \bar{p}\gamma^\mu \left(1 - \frac{g_A}{g_V}\gamma_5\right) n\bar{e}\gamma_\mu(1-\gamma_5)\nu_e \tag{23.1}$$

We see that the Fermi Hamiltonian we find again the axial and vector currents, like in the Gell-Mann Levy model. In the lepton current we simply find $j_V - j_A$, but in the hadronic part there are two more factors so that we have $g_V j_V - g_A/g_V j_A$.
We said in the chapter on Fermi interaction that the factor $g_V \approx 1$ while the factor g_A was big. Let's justify this statement using spontaneous symmetry breaking.

In general we know that in the presence of massive fermions

$$\partial_\mu j^\mu_{a,V} = 0 \qquad \partial_\mu j^\mu_{a,A} = 2im_f\bar{\psi}\gamma_5\tau_a\psi \tag{23.2}$$

Let's calculate the following matrix element

$$\langle\pi^+(p')|\, J^\mu_{em}\, |\pi^+(p)\rangle \tag{23.3}$$

Thanks to Wigner-Eckart theorem we know that this quantity, beign a product of two pseudoscalars and a vector, has to be a vector quantity. The only vector quantities at our disposal are p^μ and p'^μ, the two four-momenta of the particles. Therefore the only form that the matrix element can have is, given $q = p - p'$

$$\langle\pi^+(p')|\, J^\mu_{em}\, |\pi^+(p)\rangle = A(q^2)p^\mu + B(q^2)p'^\mu = f^+(q^2)(p+p')^\mu + f^-(q^2)(p-p')^\mu \tag{23.4}$$

where we used a more convenient form using two functions f^+ and f^- called **form factors**. If we suppose that J^μ_{em} is a conserved current

$$\begin{aligned}
\langle\pi^+(p')|\, \partial_\mu J^\mu_{em}(x)\, |\pi^+(p)\rangle &= \partial_\mu \langle\pi^+(p')|\, J^\mu_{em}(x)\, |\pi^+(p)\rangle \\
&= \partial_\mu \langle\pi^+(p')|\, e^{ipx} J^\mu_{em}(0) e^{-ipx}\, |\pi^+(p)\rangle \\
&= \partial_\mu e^{-i(p-p')x} \langle\pi^+(p')|\, J^\mu_{em}(0)\, |\pi^+(p)\rangle \\
&= -iq_\mu e^{-iqx} \langle\pi^+(p')|\, J^\mu_{em}(0)\, |\pi^+(p)\rangle \\
&= -iq_\mu \langle\pi^+(p')|\, J^\mu_{em}(x)\, |\pi^+(p)\rangle \\
&= -iq_\mu \left[f^+(q^2)(p+p')^\mu + f^-(q^2)(p-p')^\mu\right] \\
&= -if^+(q^2)q_\mu(p+p')^\mu + f^-(q^2)q^2 = 0
\end{aligned} \tag{23.5}$$

Since $q^2 \neq 0$ the only possible solution is that $f^-(q^2) = 0$ if the current is conserved. Moreover, a conserved current gives us a conserved charge

$$Q_{em} = \int \mathrm{d}^3x\, J^0_{em} \tag{23.6}$$

Being conserved implies $[Q_{em}, H] = 0$ and so that we con construct a simultaneous eigenbasis of both operators. What we get is

$$Q_{em}\,|\pi^+(p)\rangle = q_{em}\,|\pi^+(p)\rangle \tag{23.7}$$

Using this conditions we can proceed to calculate the following

$$\langle\pi^+(p')|\, Q_{em}\, |\pi^+(p)\rangle = q_{em}\langle\pi^+(p')|\pi^+(p)\rangle = q_{em}2E_p(2\pi)^3\delta^3(p-p') \tag{23.8}$$

Furthermore, by using the definition of the charge

$$\begin{aligned}
\langle\pi^+(p')| \int \mathrm{d}^3x\, J^0_{em}(x)\, |\pi^+(p)\rangle &= \int \mathrm{d}^3x \langle\pi^+(p')|\, J^0_{em}(x)\, |\pi^+(p)\rangle \\
&= \int \mathrm{d}^3x\, e^{-i(E-E')t} e^{i(p-p')\cdot x} \langle\pi^+(p')|\, J^0_{em}(0)\, |\pi^+(p)\rangle \\
&= (2\pi)^3\delta^3(p-p') \langle\pi^+(p')|\, J^0_{em}(x)\, |\pi^+(p)\rangle
\end{aligned} \tag{23.9}$$

By comparing results 23.8 and 23.9 what we get is that

$$f^+(q^2 = 0) = q_{em} \tag{23.10}$$

This still holds in general for the $SU(3)$ currents. We conclude what follows

> **Remark.** If the $SU(3)$ vector symmetry were to be exact, the factor g_V should be equal to 1. But symmetry is slightly broken. A theorem by Gatto gives us the second order expansion to g_V
>
> $$g_V = 1 + \left(\frac{m_d - m_u}{\Lambda_{QCD}}\right)^2 \tag{23.11}$$
>
> The other factor g_A clearly cannot be equal to 1 since the symmetry is explicitly broken by the mass of the fermions.

More complex form factors arises whenever we search for complex matrix elements. For example, if we want to evaluate the matrix element of the EM-current between two protons, from the Wigner-Eckart theorem one finds that

$$\langle p(p')| J^\mu_{em}(x) |p(p)\rangle = \bar{u}(p')\left[F_1(q^2)\gamma^\mu + F_2(q^2)\frac{i\sigma^{\mu\nu}}{2m}q_\nu + F_3(q^2)q^\mu\right]u(p) \tag{23.12}$$

Whenever the current is conserved one easily finds that

$$\begin{aligned} iq_\mu \langle p(p')| J^\mu_{em}(x) |p(p)\rangle &= \bar{u}(p')\left[F_1(q^2)\not{q} + F_2(q^2)\frac{i\sigma^{\mu\nu}}{2m}q_\mu q_\nu + F_3(q^2)q^2\right]u(p) \\ &= \bar{u}(p')\left[F_1(q^2)\not{q} + F_3(q^2)q^2\right]u(p) = 0 \end{aligned} \tag{23.13}$$

where the $\sigma^{\mu\nu}$ factor disappeared since it's antisymmetric and is contracted with a symmetric tensor. Again from this we find $F_3(q^2) = 0$ if the current is conserved. Moreover, the last term is identically zero since the two protons have the same mass

$$\bar{u}(p')(\not{p} - \not{p}')u(p) = (-m_p + m_p)\bar{u}(p')u(p) = 0 \tag{23.14}$$

If the current is conserved we find again the same result as before, albeit in a more convoluted way

$$\begin{aligned} \langle p(p')| Q_{em} |p(p)\rangle &= q_{em} 2E_p (2\pi)^3 \delta^3(p - p') \\ &= \langle p(p')| J^0_{em}(0) |p(p)\rangle (2\pi)^3 \delta^3(p - p') \\ &= \bar{u}(p')\left[\gamma^0 F_1(q^2 = 0) + i\frac{\sigma^{0\nu}}{2m}q_0 F_2(q^2 = 0)\right]u(p)(2\pi)^3\delta^3(p - p') \\ &= u^\dagger(p)u(p)F_1(q^2 = 0) + \frac{\bar{u}(p)}{2m}\left(\gamma^0\gamma^\nu q_\nu - \gamma^\nu\gamma^0 q_\nu\right)u(p)F_2(q^2 = 0) \end{aligned} \tag{23.15}$$

Using the completeness relation

$$u^{r\dagger}(p)u^s(p) = 2E_p\delta^{rs} \tag{23.16}$$

and the algebra of the gamma matrices, we find

$$q_{em} = F_1(q^2 = 0) \tag{23.17}$$

which is exactly what we expect. The two form factors which remains are called **electric form factor** F_1 and **magnetic form factor** F_2.

24 SSB in a complex scalar theory

Just as a last example of spontaneous symmetry breaking, we consider the charged Klein-Gordon field

$$\mathcal{L} = (\partial_\mu \phi^\dagger)(\partial^\mu \phi) - V(\phi^\dagger \phi) \tag{24.1}$$

This theory has a global $U(1)$ symmetry.
In the broken phase we take the mass term to be imaginary

$$\mathcal{L} = (\partial_\mu \phi^\dagger)(\partial^\mu \phi) + \mu^2 \phi^\dagger \phi - \lambda(\phi^\dagger \phi)^2 \tag{24.2}$$

Notice that we can always write a complex scalar field as a complex combination of two real fields

$$\phi = \frac{\sigma + i\pi}{\sqrt{2}} \qquad \phi^\dagger = \frac{\sigma - i\pi}{\sqrt{2}} \tag{24.3}$$

In this representation, the complex lagarangian becomes exactily the Gell-Mann Levy lagrangian but with only two scalar fields instead of four. The real lagrangian is $O(2)$ invariant.

To spontaneously broke the symmetry we again add a term which chooses a vacuum

$$\mathcal{L} = (\partial_\mu \phi^\dagger)(\partial^\mu \phi) + \mu^2 \phi^\dagger \phi - \lambda(\phi^\dagger \phi)^2 - h\,\mathrm{Re}\{\phi\} \tag{24.4}$$

As for the Gell-Mann Levy model, we shift the field $\sigma \to \sigma + v$ so that the complex field shifts as

$$\phi \to \frac{v + \sigma + i\pi}{\sqrt{2}} \tag{24.5}$$

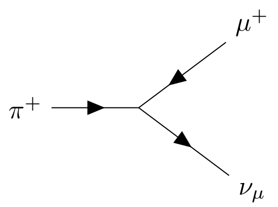
and by exactily the same computations as before, in the limit of no external field $h \to 0$ one finds the lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu \sigma \partial^\mu \sigma - \frac{1}{2} m_\sigma^2 \sigma^2 + \frac{1}{2}\partial_\mu \pi \partial^\mu \pi - \lambda v(\pi^2 \sigma + \sigma^3) - \frac{\lambda}{4}(\sigma^2 + \pi^2) \tag{24.6}$$

This models describes the dynamics of a massive scalar field σ and a single massless scalar field π. This is to be compared with the Gell-Mann Levy model, where the massless modes where three.
Note that we could discuss the spontaneous symmetry breaking by only writing down the potential, since it is enough that the symmetry is broken. Indeed the Goldstone theory is so general that we can derive it even without writing down the Lagrangian. The present discussion, based on the potential that we write in the Lagrangian, is valid within the framework of lowest order perturbation theory and can be easily extended to higher orders.

25 Digression: the Pion decay

Let's come back to the Fermi theory and evaluate the decay of a charged pion in a lepton couple. This decay[14] is represented by the following Feynman diagram



[14] This is true for the Fermi theory which, as we'll see, it's just a low energy limit of the SM. Actually the diagram is more complicated and contains the interaction between the quarks that make up the pion, mediated by the vector boson W$^+$.

while the Fermi Hamiltonian is

$$\begin{aligned}\mathcal{H}_F &= -\frac{G_F}{\sqrt{2}}\left\langle \nu_\mu\, \mu^+ \right| \bar{u}_\nu \gamma^\mu (1-\gamma_5) v_\mu \left| i f_\pi \partial_\mu \pi^\mu \right\rangle \\ &= -\frac{G_F}{\sqrt{2}} (i f_\pi \partial_\mu \pi^\mu)(\bar{u}_\nu \gamma^\mu (1-\gamma_5) v_\mu) \\ &= \frac{G_F}{\sqrt{2}} f_\pi p_\pi^\mu e^{-ip^\mu x} (\bar{u}_\nu \gamma^\mu (1-\gamma_5) v_\mu) \end{aligned} \tag{25.1}$$

25.1 Decay width

As we have done for the muon, let's evaluate the decay width for the charged pion exploiting the equation 15.10.
The modulus square of the transition matrix will be

$$\frac{G_F^2}{2} f_\pi^2\, p_\pi^\rho\, p_\pi^\sigma \left[\bar{u}(p_{\nu_\mu})\gamma^\rho (1-\gamma_5) v(p_\mu) \bar{v}(p_\mu) \gamma^\sigma (1-\gamma_5) u(p_{\nu_\mu})\right] \tag{25.2}$$

Summing over all polarizations, we use once again Casimir's trick with the completeness relations and get

$$|A|^2 = \frac{G_F^2}{2} f_\pi^2\, p_\pi^\rho\, p_\pi^\sigma\, \mathrm{Tr}\left[\gamma^\rho (1-\gamma_5)(\not{p}_\mu - m_\mu)\gamma^\sigma (1-\gamma_5)(\not{p}_{\nu_\nu} + m_{\nu_\mu})\right] \tag{25.3}$$

Now let's calculate the trace, neglecting the mass of the neutrino

$$\begin{aligned} &\mathrm{Tr}\left[\gamma^\rho (1-\gamma_5)(\not{p}_\mu - m_\mu)\gamma^\sigma (1-\gamma_5)\not{p}_{\nu_\mu}\right] \\ &= \mathrm{Tr}\left[\gamma^\rho (1-\gamma_5)\not{p}_\mu \gamma^\sigma (1-\gamma_5)\not{p}_{\nu_\mu}\right] + \mathrm{Tr}\left[\gamma^\rho (1-\gamma_5) m_\mu \gamma^\sigma (1-\gamma_5)\not{p}_{\nu_\mu}\right] \\ &= \mathrm{Tr}\left[\gamma^\rho (1-\gamma_5)\not{p}_\mu \gamma^\sigma (1-\gamma_5)\not{p}_{\nu_\mu}\right] + \cancel{\mathrm{Tr}\left[\gamma^\rho (1-\gamma_5)(1+\gamma_5) m_\mu \gamma^\sigma \not{p}_{\nu_\mu}\right]} \\ &= 2\,\mathrm{Tr}\left[\gamma^\rho (1-\gamma_5)\not{p}_\mu \gamma^\sigma \not{p}_{\nu_\mu}\right] \\ &= 2\, p_\alpha^\mu p_\beta^{\nu_\mu} \left\{\mathrm{Tr}\left[\gamma^\rho\gamma^\alpha\gamma^\sigma\gamma^\beta\right] - \mathrm{Tr}\left[\gamma^\rho\gamma_5\gamma^\alpha\gamma^\sigma\gamma^\beta\right]\right\} \\ &= 2\, p_\alpha^\mu p_\beta^{\nu_\mu} \left\{4(g^{\rho\alpha}g^{\sigma\beta} - g^{\rho\sigma}g^{\alpha\beta} + g^{\rho\beta}g^{\alpha\sigma}) + 4i\epsilon^{\rho\alpha\sigma\beta}\right\} \\ &= 8\Big(p_\mu^\rho p_{\nu_\mu}^\sigma + p_\mu^\sigma p_{\nu_\mu}^\rho - \mathbf{p}_\mu \cdot \mathbf{p}_{\nu_\mu}\, g^{\rho\sigma} + i p_\alpha^\mu p_\beta^{\nu_\mu} \epsilon^{\rho\alpha\sigma\beta}\Big) \end{aligned} \tag{25.4}$$

At the end what we get is

$$\begin{aligned} |A|^2 &= \frac{G_F^2}{2} f_\pi^2\, p_\pi^\rho\, p_\pi^\sigma\, 8\Big(p_\mu^\rho p_{\nu_\mu}^\sigma + p_\mu^\sigma p_{\nu_\mu}^\rho - \mathbf{p}_\mu \cdot \mathbf{p}_{\nu_\mu}\, g^{\rho\sigma} + i p_\alpha^\mu p_\beta^{\nu_\mu} \epsilon^{\rho\alpha\sigma\beta}\Big) \\ &= 4\, G_F^2 f_\pi^2 \big(2(\mathbf{p}_\pi \cdot \mathbf{p}_{\nu_\mu})(\mathbf{p}_\pi \cdot \mathbf{p}_\mu) - m_\pi^2 (\mathbf{p}_{\nu_\mu} \cdot \mathbf{p}_\mu)\big) \end{aligned} \tag{25.5}$$

where the term containing the antisymmetric Levi-Civita tensor disappears because it contracts with the symmetric tensor.
Let's see how we can write all the momenta, exploiting the conservation $p_\pi = p_\mu + p_{\nu_\mu}$ and neglecting the mass of the neutrino

$$\begin{aligned} \mathbf{p}_\pi \cdot \mathbf{p}_{\nu_\mu} &= \mathbf{p}_\mu \mathbf{p}_{\nu_\mu} \\ \mathbf{p}_\pi \cdot \mathbf{p}_\mu &= m_\mu^2 + \mathbf{p}_\mu \mathbf{p}_{\nu_\mu} \\ p_\pi^2 = m_\pi^2 &= m_\mu^2 + 2\, \mathbf{p}_\mu \mathbf{p}_{\nu_\mu} \end{aligned} \tag{25.6}$$

If we substitute into the formula of the transition amplitude, we get

$$\begin{aligned}|A|^2 &= 4\,G_F^2 f_\pi^2\left(2\left(\frac{m_\pi^2-m_\mu^2}{2}\right)\left(\frac{m_\pi^2+m_\mu^2}{2}\right)-m_\pi^2\left(\frac{m_\pi^2-m_\mu^2}{2}\right)\right)\\ &= 4\,G_F^2 f_\pi^2\left(\frac{m_\pi^2-m_\mu^2}{2}\right)(m_\pi^2+m_\mu^2-m_\pi^2)\\ &= 2\,G_F^2 f_\pi^2 m_\pi^2 m_\mu^2\left(1-\frac{m_\mu^2}{m_\pi^2}\right) \qquad (25.7)\end{aligned}$$

From the full differential decay rate

$$d\Gamma = \frac{1}{2}\frac{1}{2m_\pi}2\,G_F^2 f_\pi^2 m_\pi^2 m_\mu^2\left(1-\frac{m_\mu^2}{m_\pi^2}\right)(2\pi)^4\delta^{(4)}(p_\pi-p_\mu-p_{\nu_\mu})\frac{d^3p_\mu}{(2\pi)^3 2E_\mu}\frac{d^3p_{\nu_\mu}}{(2\pi)^3 2E_{\nu_\mu}} \qquad (25.8)$$

In this case it's easier to find the solution for the decay width. Let's solve the integrals of the momenta

$$I = \int\frac{d^3p_{\nu_\mu}}{2E_{\nu_\mu}}\int\frac{d^3p_\mu}{2E_\mu}\;\delta^{(4)}(p_\pi-p_\mu-p_{\nu_\mu}) \qquad (25.9)$$

Similarly to the muon decay, this integral is Lorentz covariant because the delta function is Lorentz covariant as the integration factor, which is due to the equality specified in equation 15.23. Now if we use directly that equation for the muon momentum, we get

$$\begin{aligned}I &= \int\frac{d^3p_{\nu_\mu}}{2E_{\nu_\mu}}\int d^4p_\mu\;\delta^{(4)}(p_\pi-p_\mu-p_{\nu_\mu})\;\delta(p_\mu^2-m_\mu^2)\Theta(p_\mu^0)\\ &= \int\frac{d^3p_{\nu_\mu}}{2E_{\nu_\mu}}\;\delta((p_\pi-p_{\nu_\mu})^2-m_\mu^2)\Theta(p_\mu^0) \qquad (25.10)\end{aligned}$$

In the reference frame of the pion, neglecting the mass of the neutrino

$$= \int\frac{d^3p_{\nu_\mu}}{2E_{\nu_\mu}}\;\delta(m_\pi^2-2E_{\nu_\mu}m_\pi-m_\mu^2)\Theta(p_\mu^0) \qquad (25.11)$$

Passing from the neutrino momentum to its energy[15], as we have done for the muon decay, we get

[15] From this moment, we assume energy is always positive

$$= 4\pi\int\frac{dE_{\nu_\mu}}{2E_{\nu_\mu}}\frac{E_{\nu_\mu}}{p_{\nu_\mu}}p_{\nu_\mu}^2\,\delta(m_\pi^2-2E_{\nu_\mu}m_\pi-m_\mu^2) \qquad (25.12)$$

Knowing the property for the delta composition with a function, at the end we get

$$\begin{aligned}I &= 2\pi\int dE_{\nu_\mu}E_{\nu_\mu}\frac{1}{2m_\pi}\delta\left(E_{\nu_\mu}-\frac{m_\pi^2-m_\mu^2}{2m_\pi}\right)\\ &= \pi\left(\frac{m_\pi^2-m_\mu^2}{2m_\pi^2}\right) \qquad (25.13)\end{aligned}$$

The final formula of the decay width will be

$$\Gamma = \frac{1}{2}\frac{1}{2m_\pi}\frac{2\,G_F^2}{(2\pi)^2} f_\pi^2 m_\pi^2 m_\mu^2 \left(1-\frac{m_\mu^2}{m_\pi^2}\right)\pi\left(\frac{m_\pi^2-m_\mu^2}{2m_\pi^2}\right)$$

$$= \frac{G_F^2}{16\pi} f_\pi^2\, m_\pi m_\mu^2 \left(1-\frac{m_\mu^2}{m_\pi^2}\right)^2 \tag{25.14}$$

If now we analyze the decay, we can notice that the theory allows us to consider the charged pion decaying into a positron-neutrino pair. If we compare the two decay widths in a ratio, we get a known result

$$\frac{\Gamma(\pi^+\to e^+\nu_e)}{\Gamma(\pi^+\to \mu^+\nu_\mu)} = \frac{m_e^2}{m_\mu^2}\frac{\left(1-\dfrac{m_e^2}{m_\pi^2}\right)^2}{\left(1-\dfrac{m_\mu^2}{m_\pi^2}\right)^2} \simeq 1,275\cdot 10^{-4} \tag{25.15}$$

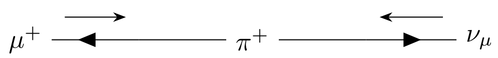
The probability that the charged pion decays into an antimuon-neutrino pair rather than a positron-neutrino pair is very high and, for this reason, the decay into an electronic pair is strongly suppressed.

Actually we could analyze the same problem starting from considerations on helicity. If we take the matrix element for this decay, after some semplification, we would get[16]

[16] θ_C is the Cabibbo angle and turns out form the mixing of quarks. We'll talk about this in the chapter on the Standard Model and the CKM matrix.

$$A_{fi} = -\frac{G_F f_\pi \cos\theta_C}{\sqrt{2}} m_l \bar{u}(\nu_l)(1+\gamma_5)v(l) \tag{25.16}$$

where the subscript l stands for lepton, which in general can be one from e, μ. Observe the strong dependence on the mass of the lepton and that it vanishes whenever $m_l = 0$. This can be understood by observing that charged weak currents depend only on left-handed quantities. In the massless limit, a left-handed operator describes a particle with helicity $-1/2$ and its antiparticle with helicity $+1/2$. So, if the neutrino is massless or its mass is so low that we can consider it massless, it's going to be left-handed. Since the pion has spin zero, conservation of angular momenta requires that also the antilepton has negative helicity. Remember that helicity is the component of the spin projected on the direction of the momentum of the particle and that in a decay, in the rest frame of the decaying particle, the two decay products are back-to-back. However, the antilepton created by a massless left-handed Weyl field has elicity $+1/2$. Therefore in the massless limit the process is forbidden because it would violate the conservation of angular momentum.



This result extends to the branching ratio found before: the decay into more massive particles is favorable[17] since it's further from conservation angular momentum violation.

[17] We cannot consider τ lepton because it has a rest mass much higher than the pion.

It's important to note that whenever we find such results as the one for the branching ratio, we should always ask ourselves if that specific result comes from symmetry considerations, since most of the times it does.

PART

IX

Goldstone Theorem

We have seen that there are three massless bosons in the broken Gell-Mann Levy model, where the initial symmetry is broken down as $SU(2)_A \times SU(2)_V \to SU(2)_V$. However there is only one massless boson in the broken down global $U(1)$ complex lagrangian, in which the real field symmetry breaks down as $SO(2) \to SO(1)$. Thus we may ask **how many** Goldstone bosons are generated whenever a continuous symmetry group is broken into a subgroup.
The result is what is known as Goldstone theorem. We'll state the formal theorem in the section of the non perturbative proof.

26 Perturbative analysis

In this first section we discuss this point in the framework of perturbation theory.
Consider a potential $V(\phi^i)$ depending on several fields with an internal index i which rotates as

$$\phi i\prime = \phi^i + i\alpha^A (t^A)^i{}_j \phi^j \qquad \delta\phi^i = i\alpha^A (t^A)^i{}_j \phi^j \tag{26.1}$$

If the theory is invariant under this transformation

$$V(\phi^i + i\alpha^A (t^A)^i{}_j \phi^j) - V(\phi^i) = 0 \tag{26.2}$$

By expanding up to first order one finds

$$\frac{\delta V}{\delta \phi^i}\delta\phi^i = i\alpha^A \frac{\delta V}{\delta\phi^i}(t^A)^i{}_j\phi^j = 0 \tag{26.3}$$

Since this equality has to be true for any α we have

$$\frac{\delta V}{\delta\phi^i}(t^A)^i{}_j\phi^j = 0 \qquad \forall A = 1, \cdots, \dim G \tag{26.4}$$

where $\dim G$ is the dimension of the group of the underlying symmetry. If we take another derivative in ϕ^k

$$\frac{\delta}{\delta\phi^k}\left[\frac{\delta V}{\delta\phi^i}(t^A)^i{}_j\phi^j\right] = \frac{\delta^2 V}{\delta\phi^k\delta\phi^i}(t^A)^i{}_j\phi^j + \frac{\delta V}{\delta\phi^i}(t^A)^i{}_k = 0 \tag{26.5}$$

Suppose that now we fix the vacuum expectation value on the vector $v \equiv (v_1, \cdots, v_D)$ where D is the dimension of the representation in which the field ϕ^i belongs. Since in the vacuum we have

$$\left.\frac{\delta V}{\delta\phi^i}\right|_{\phi^i = v^i} = 0 \tag{26.6}$$

the equation 26.5 evaluated on the vacuum becomes

$$\left.\frac{\delta^2 V}{\delta\phi^k\delta\phi^i}\right|_{\phi^i=v^i}(t^A)^i{}_j v^j = M^2_{ik}(t^A)^i{}_j v^j = 0 \tag{26.7}$$

since the second derivative evaluated on the vacuum is exactly the mass matrix. We now introduce the vector $w_A \equiv (w^1_A, \cdots, w^D_A)$, one for each generator with components

$w_A^i = (t^A)^i{}_j v^j$. With this the final equation becomes

$$M_{ik}^2 w_A^i = 0 \tag{26.8}$$

At this point there remains only two possibilities

$$\vec{\mathbf{w}}_A = 0 = (t^A)^i{}_j v^j \implies \text{the generator does not transform the vacuum}$$
$$\vec{\mathbf{w}}_A \neq 0 \implies \text{the generator does transform the vacuum}$$

in the latter $(t^A)^i{}_j v^j$ is an eigenstate of the mass operator with eigenvalue zero. Therefore we conclude that we have as many massless spinless[18] particles, which we'll call from now on NG-bosons, as the number of generators which do not leave the vacuum invariant.

[18] The spinless condition will be clarified in the non perturbative proof.

27 Non perturbative analysis

The proof that we're going to give now, does not rely on perturbation theory. This proof was given by Weinberg, Salam and Goldstone in their famous paper. This proof relies heavily on what is known as Källen-Lehmann spectral representation which we'll see in detail in the following section. From that, it will be easy to construct a proof which does not rely on perturbation theory since the Källen-Lehmann representation is a non-perturbative formula.

27.1 Asymptotic Theory

Although this is beyond the scope of the electroweak course, we'll highlight some concept which will be useful later on, and that will be covered in much more details in following courses.
Broadly speaking, quantum field theory can be thought of as based on the following elements: the possible states of a theory are generated from a unique vacuum state $|0\rangle$ by the action of **free fields** $\phi_{in}(x)$, which generates the Fock space of states; physical observables, such as the interacting field $\phi(x)$ can be expressed in terms of the free fields $\phi_{in}(x)$. The basic idea behind this setting is that the interacting fields, as well as other observables, can be found from the free field by switching adiabatically on and off the interaction as $x_0 \to \pm\infty$. This construction is clearly relevant to scattering processes where the incoming particles are to be thought as free particle, well separated in space, before interacting. Starting from the interaction field $\phi(x_0, \vec{\mathbf{x}})$ one should recover the free one $\phi_{in}(x_0, \vec{\mathbf{x}})$ when $x_0 \to -\infty$, but this generally occurs up to a wave-function renormalization constant $Z^{1/2}$

$$\lim_{x_0 \to -\infty} \phi(x_0, \vec{\mathbf{x}}) = Z^{1/2} \phi_{in}(x_0, \vec{\mathbf{x}}) \tag{27.1}$$

Analogously

$$\lim_{x_0 \to \infty} \phi(x_0, \vec{\mathbf{x}}) = Z^{1/2} \phi_{out}(x_0, \vec{\mathbf{x}}) \tag{27.2}$$

where $\phi_{out}(x)$ is a free field. The free fields satisfy the free KG-equation, in the case of scalar fields, while the interacting one **does not**. The free fields can be therefore written down as the usual expansion with creation and annihilation operators

$$\phi_{in/out}(x) = \int \frac{\mathrm{d}^3 p}{\sqrt{2E_p (2\pi)^3}} \Big(a_{in/out}(p) e^{-ipx} + a^\dagger_{in/out}(p) e^{ipx} \Big) \tag{27.3}$$

satisfying the usual commutation relations. The main important difference between free fields and interacting fields is that $\phi_{in/out}(x)\,|0\rangle$ creates one particle states whereas $\phi(x)\,|0\rangle$

creates multuparticle states due to the interaction.

27.2 Källen-Lehmann spectral representation

When we studied the $\lambda\phi^4$ theory, we have seen how the propagator of an interacting theory can be found through series expansion of the matrix elements of the interaction lagrangian between non-interacting vacuum. The formula we found was based, as said before, on a perturbative analysis method.
Now consider, for simplicity, the scalar field propagator $\langle\Omega| T\phi(x)\phi(0) |\Omega\rangle$ and insert in it a complete set of multiparticle states $|n\rangle$

$$\begin{aligned}\langle\Omega| T\phi(x)\phi(0) |\Omega\rangle &= \langle\Omega| \Theta(t)\phi(x)\phi(0) |\Omega\rangle + \langle\Omega| \Theta(-t)\phi(0)\phi(x) |\Omega\rangle \\ &= \sum_n \Theta(t) \langle\Omega| \phi(x) |n\rangle\langle n| \phi(0) |\Omega\rangle + \Theta(-t) \langle\Omega| \phi(0) |n\rangle\langle n| \phi(x) |\Omega\rangle \\ &= \sum_n e^{-ip_n x}\Theta(t) \langle\Omega| \phi(0) |n\rangle\langle n| \phi(0) |\Omega\rangle + e^{ip_n x}\Theta(-t) \langle\Omega| \phi(0) |n\rangle\langle n| \phi(0) |\Omega\rangle \\ &= \sum_n |\langle\Omega| \phi(0) |n\rangle|^2 \big(\Theta(t)e^{-ip_n x} + \Theta(-t)e^{ip_n x}\big) \end{aligned} \tag{27.4}$$

by inserting another identity in the mix, one gets

$$= \int \frac{\mathrm{d}^4 q}{(2\pi)^2} \big(\Theta(t)e^{-ip_n x} + \Theta(-t)e^{ip_n x}\big)(2\pi)^3 \sum_n |\langle\Omega| \phi(0) |n\rangle|^2 \delta^4(q - p_n) \tag{27.5}$$

The quantity

$$\rho(q) \equiv (2\pi)^2 \sum_n |\langle\Omega| \phi(0) |n\rangle|^2 \delta^4(q - p_n) \tag{27.6}$$

is called **spectral density**. We can easily prove, in the case of a scalar field, that this quantity is Lorentz invariant, in fact

$$\begin{aligned}\rho(q') &= (2\pi)^2 \sum_{n'} |\langle\Omega'| \phi(0) |n'\rangle|^2 \delta^4(q' - p'_n) \\ &= (2\pi)^3 \sum_n \left|\langle\Omega| \Lambda^{-1}\phi(0)\Lambda |n\rangle\right|^2 \delta^4(\Lambda(q - p_n)) \\ &= (2\pi)^3 \sum_n |\langle\Omega| \phi(0) |n\rangle|^2 \frac{1}{\det\Lambda}\delta(q - p_n) = \rho(q) \equiv \rho(q^2) \end{aligned} \tag{27.7}$$

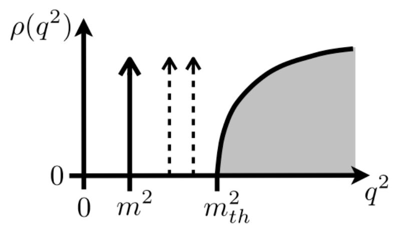


Figure 13. Sketch of the spectral density $\rho(q^2)$ as a function of q^2, which highlights the presence of a δ corresponding to the physical mass associated to the one particle state, of possible additional peaks do to bound states and of a continuum associated to multiparticle states.

Moreover

$$p_n = \sum_{i=1}^{n} p_{n,i} \tag{27.8}$$

where $p_{n,i}$ is the momentum of the i-th particle in the n particle state $|n\rangle$ and therefore $(p_{n,i})^0 > 0$ with $p_{n,i}^2 > 0$; this implies that $p_n^0 > 0$ and $p_n^2 > 0$ and, in turn, that $\rho(q^2)$ vanishes in the backward light cone, so that $\rho(q^2) \to \rho(q^2)\Theta(q^0)$.
We can now insert another identity $\int \mathrm{d}\mu^2\, \delta(q^2 - \mu^2) = 1$ to the propagator, getting

$$\int \mathrm{d}\mu^2 \int \frac{\mathrm{d}q^0\, \mathrm{d}^3 q}{(2\pi)^3} \Big[\Theta(t)e^{-iq^0 t + i\mathbf{q}\cdot\mathbf{x}} + \Theta(-t)e^{iq^0 t - i\mathbf{q}\cdot\mathbf{x}}\Big]\rho(q^2)\Theta(q^0)\delta({q^0}^2 - q^2 - \mu^2) \tag{27.9}$$

Since $E_q^2 = q^2 + \mu^2$ the delta function just becomes $\delta((q^0 - E_q)(q^0 + E_q))$ but, there being a $\Theta(q^0)$, the term $q^0 + E_q$ vanishes, leaving only what follows

$$
\begin{aligned}
&= \int d\mu^2 \int \frac{dq^0\, d^3q}{(2\pi)^3} \Big[\Theta(t)e^{-iq^0t+i\mathbf{q}\cdot\mathbf{x}} + \Theta(-t)e^{iq^0t-i\mathbf{q}\cdot\mathbf{x}}\Big]\rho(q^2)\Theta(q^0)\frac{\delta(q^0 - E_q)}{2E_q} \\
&= \int d\mu^2 \int \frac{d^3q}{2E_q(2\pi)^3} \big[\Theta(t)e^{-iE_qt+i\mathbf{q}\cdot\mathbf{x}} + \Theta(-t)e^{iE_qt-i\mathbf{q}\cdot\mathbf{x}}\big]\rho(q^2)\Theta(E_q) \\
&= \int d\mu^2\, \rho(\mu^2) \int \frac{d^3q}{2E_q(2\pi)^2} \big[\Theta(t)e^{-iE_qt+i\mathbf{q}\cdot\mathbf{x}} + \Theta(-t)e^{iE_qt-i\mathbf{q}\cdot\mathbf{x}}\big]\Theta(E_q) \\
&= \int d\mu^2\, \rho(\mu^2) i\Delta_F^0(x;\mu^2) \qquad (27.10)
\end{aligned}
$$

where $\Delta_F^0(x;\mu^2)$ is the Feynman propagator with mass μ^2 if the same field. We can remove from this integral the one particle states $|1\rangle$

$$
\sum_{n=1} |\langle\Omega|\,\phi(0)\,|n\rangle|^2\delta^4(q-p_n) = \int d^3p |\langle\Omega|\,\phi(0)\,|p\rangle|^2\delta^4(q-p) = Z\Theta(q^0)\frac{\delta(q^2-m^2)}{(2\pi)^3} \qquad (27.11)
$$

which directly comes from the normalization condition of the interacting fields, what we get is

$$
\rho(\mu^2)\Theta(q^0) = Z\delta(\mu^2 - m^2)\Theta(q^0) + (2\pi)^3 \sum_{n>1} |\langle\Omega|\,\phi(0)\,|n\rangle|^2\delta^4(q - p_n) \qquad (27.12)
$$

which inserted in 27.10 gives

$$
\langle\Omega|\,T\phi(x)\phi(0)\,|\Omega\rangle = Zi\Delta_F^0(x;m^2) + \int_{m_{th}^2}^{\infty} d\mu^2\, \rho(\mu^2) i\Delta_F^0(x;\mu) \qquad (27.13)
$$

which provides the Källen-Lehemann spectral representation of the propagator for a scalar theory with interactions. In Fourier space, this representation becomes

$$
G^{(2)}(p^2) = \frac{iZ}{p^2 - m^2 + i\epsilon} + \int_{m_{th}^2}^{\infty} d\mu^2\, \frac{i\rho(\mu^2)}{p^2 - \mu^2 + i\epsilon} \qquad (27.14)
$$

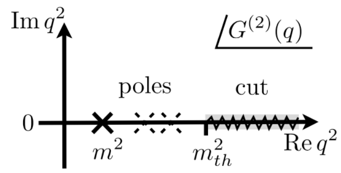


Figure 14. Analytic structure of the propagator $G^{(2)}(q)$ of a scalar field, as a function of $q^2 \in \mathbb{C}$ which highlights the presence of an isolated pole corresponding to the physical mass, possible additional poles due to bound states and a branch cut related to multiparticle states.

As a function of $q^2 \in \mathbb{C}$ te propagator is characterized by an isolated pole for $q^2 = m^2$, with residue i and by a branch cut on the positive real axis, starting from $q^2 = m_{th}^2$ and controlled by the spectral density, with possible additional poles due to bound states. Note in particular that $G^{(2)}(z)$ is analytic in the complex plane away from the real axis. The $i\epsilon$ prescription tells us that the physical sheet lies above the branch-cut. Another relevant relation can be found by taking into account that

$$
\frac{1}{x + i\epsilon} = P\frac{1}{x} - i\pi\delta(x) \qquad \text{for } \epsilon \to 0^+ \qquad (27.15)
$$

in the sense of distributions, where P is the principal part, and therefore

$$
\operatorname{Im} iG^{(2)}(q^2) = \pi\rho(q^2) \qquad (27.16)
$$

The m_{th} factor in the lower bound of the integral depends upon the theory which we're studying. For example, for the $\lambda\phi^4$-theory, we know that only particle states with $2k+1$ particles can be created. So right after the single particle state the only possible multiparticle state that can be creates is $|3\rangle$, which implies that $m_{th} = 3m$.

27.3 Goldstone Theorem

We now state the so awaited Goldstone theorem

Theorem 27.1. Let G be a group of global continuous symmetry with generators t^A, $A = 1, \cdots, \dim G$, acting on some system. By Nöether's theorem, we have $\dim G$ conserved currents $J^A_\mu(x)$ such that $\partial^\mu J^A_\mu(x) = 0$ and the associated charges $Q^a = \int \mathrm{d}^3x J^A_0(x)$. The group G is said to be spontaneously broken if, on the vacuum the generators splits into two sets, labelled by A and B, such that

$$Q^A |0\rangle \neq 0 \qquad Q^B |0\rangle = 0 \tag{27.17}$$

with a non-empty set for A. The unbroken generators labelled by t^B form a subgroup of G, denoted $H \leqslant G$, so that we have $\dim G - \dim H$ broken generators ($A = 1, \cdots, \dim G - \dim H$). Goldstone's theorem states that, independently of the specific pattern of symmetry breaking and physical system we are considering, in the spectrum there will appear one massless and spinless particle for each broken generator. The particle will be scalar or pseudoscalar, depending on the parity pf the associated broken generator. These particles are called Goldstone or Nambu-Goldstonee (NG) bosons.

The proof follows: let ϕ^b be the set of fields responsable for the spontaneous symmetry breaking $G \to H$. The field ϕ^b need not be "fundamental" here; all our remarks apply equally well is ϕ^b is a synthetic object like $\bar{\psi}\Gamma\psi$.
The non invariance implies that the infinitesimal action of the group doesn't leave them invariant on the vacuum

$$\langle 0| \phi^{b\prime}(0) |0\rangle \neq \langle 0| \phi^b(0) |0\rangle \tag{27.18}$$

In other words one must have

$$\langle 0| \delta\phi^b(0) |0\rangle = \langle 0| \left[Q^A, \delta\phi^b(0)\right] |0\rangle \equiv \delta\phi^{A,b} \neq 0 \tag{27.19}$$

for all broken symmetry generators, labelled by $A = 1, \cdots, \dim G - \dim H$. We shall show that if the vacuum is not annihilated by Q^A, so that 27.19 holds, then the theory must involve massless particles.
The place we'll look for zero-mass singularities in the propagator of the conserved currents J^A_μ with the fields ϕ^b

$$\langle 0| T J^a_\mu(x)\phi^b(0) |0\rangle \tag{27.20}$$

Since $\partial^\mu J^A_\mu(x) = 0$

$$\begin{aligned}
\partial^\mu \langle 0| T J^A_\mu(0)\phi^b(0) |0\rangle &= \partial^\mu \langle 0| \left[\Theta(x^0)J^A_\mu(x)\phi^b(0) + \Theta(-x^0)\phi^b(0)J^A_\mu(x)\right] |0\rangle \\
&= \delta(x^0) \langle 0| J^A_\mu(0)\phi^b(0) |0\rangle - \delta(x^0) \langle 0| \phi^b(0)J^A_\mu(x) |0\rangle \\
&= \delta(x^0) \langle 0| J^A_0(0)\phi^b(0) |0\rangle - \delta(x^0) \langle 0| \phi^b(0)J^A_0(x) |0\rangle \\
&= \delta(x^0) \langle 0| \left[J^A_0(x), \phi^b(0)\right] |0\rangle
\end{aligned} \tag{27.21}$$

Let us denote $G^{A,b}_\mu$ the Fourier transform of the 2-point function above

$$\langle 0| T J^A_\mu(x)\phi^b(0) |0\rangle = \int \frac{\mathrm{d}^4p}{(2\pi)^4} G^{A,b}_\mu(p) e^{-ipx} \tag{27.22}$$

By Lorentz invariance

$$G^{A,b}_\mu(p) = ip_\mu H^{A,b}(p^2) \tag{27.23}$$

Therefore, integrating over spacetime the quantity the 4-gradient of the 2-point function one gets

$$\begin{aligned}\int \mathrm{d}^4x\partial^\mu \langle 0| TJ^A_\mu(0)\phi^b(0)|0\rangle &= \int \mathrm{d}^4x\partial^\mu \int \frac{\mathrm{d}^4p}{(2\pi)^4}G^{A,b}_\mu(p)e^{-ipx}\\ &= -i\int \mathrm{d}^4x \int \frac{\mathrm{d}^4p}{(2\pi)^4}{}^\mu G^{A,b}_\mu(p)e^{-ipx}\\ &= \int \mathrm{d}^4x \int \frac{\mathrm{d}^4p}{(2\pi)^4}p^2H^{A,b}(p^2)e^{-ipx}\\ &= \int \mathrm{d}^4p\delta(p)p^2H^{A,b}(p^2) \end{aligned} \tag{27.24}$$

but this, from equation 27.21, is equal to

$$\int \mathrm{d}^3x\delta(x^0)\langle 0|\left[J^A_0(x),\phi^b(0)\right]|0\rangle = \langle 0|\left[Q^A,\phi^b(x)\right]|0\rangle \neq 0 \tag{27.25}$$

which means that

$$\int \mathrm{d}^4p\,\delta(p)p^2H^{A,b}(p^2) = \langle 0|\left[Q^A,\phi^b(0)\right]|0\rangle \neq 0 \tag{27.26}$$

Then, if condition 27.19 holds, the integral of the divergence does not vanish. More precisely we must have

$$H^{A,b}(p^2) = \frac{\delta\phi^{A,b}}{p^2} + \cdots \tag{27.27}$$

which is a pole in 0 of the 2-point function. From the Källen-Lehmann representation this can only be done if there's a massless particle, one for each broken generator A.
If we denote the one particle states $|NG_c\rangle$

$$\langle NG_c|\,\phi^b(0)\,|0\rangle = \delta^b{}_c Z^b{}_c \qquad \langle 0|\,J^A_\mu(0)\,|NG^c\rangle = -ip^\mu f_p^{A,c} \tag{27.28}$$

Using the Källen-Lehmann argument, taking into account the one particle states above, one gets using the usual nor

$$\begin{aligned}\langle 0|TJ^A_\mu(x)\phi^b(0)|0\rangle &= \frac{i(-i)p^\mu f_p^{A,c}\delta^b{}_cZ^b{}_c}{p^2-m_c^2+i\epsilon} + i\int_{M_0^2}^{\infty}\mathrm{d}\mu^2\,\frac{(-ip^\mu)\rho^{A,b}(\mu^2)}{p^2-\mu^2+i\epsilon}\\ &= \frac{p^\mu f_p^{A,c}Z^b{}_c\delta^b{}_c}{p^2-m_c^2+i\epsilon} + p^\mu\int_{M_0^2}^{\infty}\mathrm{d}\mu^2\,\frac{\rho^{A,b}(\mu^2)}{p^2-\mu^2+i\epsilon}\end{aligned} \tag{27.29}$$

where

$$\rho^{A,b}(\mu^2) = (2\pi)^3\sum_{n>1}\langle 0|\,J^A_\mu(0)\,|n\rangle\langle n|\,\phi^b(0)\,|0\rangle\,\delta(p-p_n) \tag{27.30}$$

It's easily found now that

$$\delta\phi^{A,b} = if_p^{A,c}Z^b{}_c\delta^b{}_c = if_p^{A,b}Z^b \tag{27.31}$$

The above proof of the Goldstones theorem does not rely on perturbation theory, indicating that NG particles are expected to arise whenever a global symmetry group is spontaneously broken, no matter whether the theory undergoing the symmetry breaking is weakly or strongly coupled. The fields ϕ^b responsible for the symmetry breaking are also not necessarily elementary fields appearing directly in the Lagrangian at some energy scale, but might be composite fields built with different fields. The most relevant example of this kind is the spontaneous breaking of the $SU(2)$ chiral symmetry in QCD, induced by effective scalar fields ϕ^b constructed out of quark bilinears. In this case, the three NG bosons are spin zero mesons that appear as bound states of the original quarks, the pions π^0, $\pi^{\pm}$. We close this section by noticing that the Goldstones theorem applies for internal symmetries only, namely for those symmetries whose generators commute with the ones of the Poincarè group. Goldstones theorem does not apply for local (i.e. space-time dependent) symmetries.

PART

X

The Photon Propagator

28 The massless photon

We know that we can find the propagator of a given field solving for the Green function problem. In the massless photon case, we know directly from Maxwell's equation, that the free photon field solves

$$(-g^{\mu\nu}\partial_\mu\partial^\mu + \partial^\mu\partial^\nu)A_\mu = 0 \tag{28.1}$$

What the Green function essentially does is to find the inverse of the differential operator. If we search for it in Fourier space, the equation we have to solve is the following

$$(g^{\mu\nu}q^2 - q^\mu q^\nu)D_{\nu\rho} = \delta^\mu{}_\rho \tag{28.2}$$

The problem in inverting the operator $g^{\mu\nu}q^2 - q^\mu q^\nu$ is that this is a projection operator on the transverse direction of the photon field, so it's not invertible. In fact, if we split the photon field into transverse and longitudinal components

$$A_\mu(p) = A_\mu(p) - p_\mu\frac{A_\mu p^\mu}{p^2} + p_\mu\frac{A_\mu p^\mu}{p^2} = A^T_\mu(p) + A^L_\mu(p) \tag{28.3}$$

from this, we see that

$$(g^{\mu\nu}p^2 - p^\mu p^\nu)A^L_\mu(p) = +p^2p^\nu\frac{p_\mu A^\mu}{p^2} - p^2p^\nu\frac{p_\mu A^\mu}{p^2} = 0 \tag{28.4}$$

To solve this problem, which is essentially a consequence of gauge invariance, we put a gauge fixing term in the lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 = -\frac{1}{2}A_\mu O^{\mu\nu}A_\nu \tag{28.5}$$

where

$$O^{\mu\nu} = g^{\mu\nu}\partial_\sigma\partial^\sigma - \left(1 - \frac{1}{\xi}\right)\partial^\mu\partial^\nu \tag{28.6}$$

From this lagrangian, which is not gauge invariant anymore, we find the new equation of motion and therefore the new propagator

$$\left(g^{\mu\nu}q^2 - \left(1 - \frac{1}{\xi}\right)q^\mu q^\nu\right)D_{\nu\rho}(q) = \delta^\mu{}_\rho \tag{28.7}$$

If we choose $D_{\nu\rho} = Ag_{\nu\rho} + Bq_\nu q_\rho$

$$\begin{aligned}\left(g^{\mu\nu}q^2 - \left(1 - \frac{1}{\xi}\right)q^\mu q^\nu\right)(Ag_{\nu\rho} + Bq_\nu q_\rho) &= \delta^\mu{}_\rho \\ Aq^2\delta^\mu{}_\rho + Bq^2q^\mu q_\rho - \left(1 - \frac{1}{\xi}\right)\left(Aq^\mu q_\rho + Bq^2q^\mu q_\rho\right) &= \delta^\nu{}_\rho\end{aligned} \tag{28.8}$$

By inspection of both sides one finds

$$A = \frac{1}{q^2} \qquad B = -\frac{1}{q^2}\frac{1-\xi}{q^2} \tag{28.9}$$

Altogether the propagator becomes, by an arbitrary multiplication by i

$$D_{\mu\nu}(q) = \frac{i}{q^2 - i\epsilon}\left(g_{\mu\nu} + \frac{1-\xi}{q^2}q_\mu q_\nu\right) \tag{28.10}$$

Now we have the freedom to choose the value of the lagrange multiplier ξ. Some useful gauges are the following

$$\begin{aligned} &\xi = 1 \quad \text{Feynman gauge} \quad D_{\mu\nu}(q) = \frac{-ig_{\mu\nu}}{q^2 - i\epsilon} \\ &\xi = 0 \quad \text{Landau gauge} \quad D_{\mu\nu}(q) = \frac{i}{q^2 - i\epsilon}\left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right) \end{aligned} \tag{28.11}$$

Notice that the Landau gauge contains the projector on the transverse state of the photon field.

29 The massive photon propagator

We see then that it's not a simple task to define a massless propagator for the photon. The problem does not arise if in the lagrangian there's a mass term

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}m_A^2 A_\mu A^\mu \tag{29.1}$$

In this case, the Euler-Lagrange's equation for the photon field become

$$\left(-g^{\mu\nu}\partial_\sigma\partial^\sigma + \partial^\mu\partial^\nu - m_A^2\right)A_\mu = 0 \tag{29.2}$$

From this we can find again the propagator in Fourier space

$$\left(g^{\mu\nu}q^2 - q^\mu q^\nu - m_A^2\right)D_{\nu\rho} = \delta^\mu{}_\rho \tag{29.3}$$

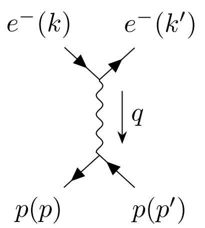
Using again the form for the propagator given before, we find

$$D_{\mu\nu}(q) = \frac{i}{q^2 - m_A^2 + i\epsilon}\left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{m_A^2}\right) \tag{29.4}$$

This was very much easier to find, no problems with the lagrange multiplier or anything. But there's a much subtle problem in this case concerning the term $q_\mu q_\nu / m_A^2$. This massive photon theory is not renormalizable. We'll see now an example of this.

29.1 The renormalizability problem

Consider the following process $e^- p \to e^- p$ at first tree level



$$= \bar{u}(p')(-ie\gamma^\mu)u(p)\frac{i}{q^2 - i\epsilon}\left(-g_{\mu\nu} + (1-\xi)\frac{q_\mu q_\nu}{q^2}\right)\bar{u}(k')(-ie\gamma^\nu)u(k) \tag{29.5}$$

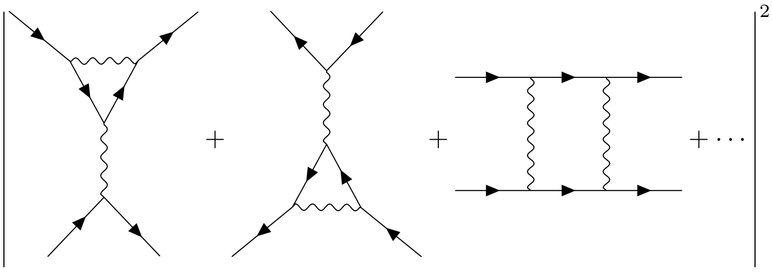
We focus our attention only on the $q_\mu q_\nu$ part and see what happens

$$\bar{u}(p')\gamma^\mu u(p)\frac{q_\mu q_\nu}{q^2}(1-\xi)u(k')\gamma^\nu u(k) = \frac{1}{q^2}\bar{u}(p')\not{q}u(p)(1-\xi)u(k')\not{q}u(k) \tag{29.6}$$

Using the fact that $q = k - k' = p - p'$ and Dirac's equation, we get, from both currents

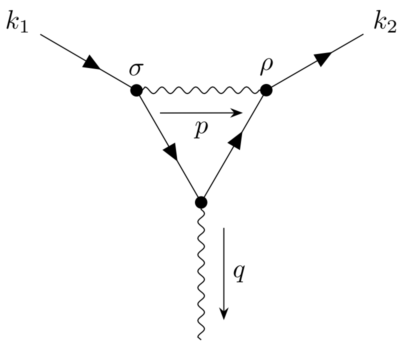
$$\bar{u}(p')(\not{p}' - \not{p})u(p) = (-m_p + m_p)\bar{u}(p')u(p) = 0 \tag{29.7}$$

And so this whole factor is zero. But what happens at the next order in perturbation theory? Some of the diagrams that we have to take into account are



(29.8)

But for our purposes let's take into account only the vertex correction



$$\int \frac{d^4p}{(2\pi)^4}(-ie)^3\bar{u}(k_2)\gamma^\rho \frac{i}{(\not{k}_2 - \not{p}) - m_e^2 + i\epsilon}\gamma^\mu \times$$
$$\times \frac{i}{(\not{k}_1 - \not{p}) - m_e^2 + i\epsilon}\gamma^\sigma u(k_1)\frac{i}{p^2 + i\epsilon}\left(-g_{\rho\sigma} + \frac{1-\xi}{p^2}p_\rho p_\sigma\right)$$

In the limit of big momentum transfer between the electrons $p \to \infty$, the intergal goes as

$$\sim \int^\Lambda \frac{d^4p}{p^4} \propto \log\Lambda \tag{29.9}$$

The integral diverges, but slowly. This divergences can be easily renormalized using measurable quantities like the electric charge.
But if the photon is massive, the integral would go as

$$\sim \int^\Lambda \frac{d^4p}{p^2m^2} \propto \frac{\Lambda^2}{m^2} \tag{29.10}$$

This divergence explodes much more rapidly than the first one. A theory with such divergences is much more difficult to renormalize. This consideration creates a big problem in the construction of a theory of weak interactions since we want it to be a gauge theory but at the same time it needs to be small ranged, since experimental evidence gives us the range of weak interaction of the order of 10^-16 m. This requires a massive gauge boson mediator.
The problem to this was solved by Higgs, Englert and Brout with the know known Higgs mechanism.

PART

XI

The Higgs Mechanism

The Higgs mechanism is the happy marriage between gauge invariance and spontaneous symmetry breaking and helps us give a mass to the photon field without putting explicitly the mass term, which we have seen to be quite problematic.
It might seem strange that we would like to give mass to the photon, but the same mechanism which we'll see to give mass to gauge bosons was before known in statistical physics in a process studied in detail by Anderson. Superconductors are materials that at very low temperature repel magnetic field lines and have zero electrical resistance. This phenomenon is dominated by the interaction between conduction electrons and phonons that gives rise to the possibility of creating electron-electron bound state since the phonon attraction exceeds Coulomb repulsion. This bound state behaves like a boson. Then we can imagine to send some light on the system and see how it behaves. What we observe is that light will penetrate the material but up to a characteristic distance λ, which means that the field decays as $e^{-z/\lambda}$. What this means is that the photon acquires a mass from its interaction with the electron-electron pairs. Its Compton wavelength is in fact $\lambda = 1/M$.

30 How to give a mass to the photon

30.1 The miracle of the Higgs mechanism

Let us consider a general global $U(1)$ lagrangian

$$\mathcal{L} = |\partial_\mu \phi|^2 - V(|\phi|^2) \tag{30.1}$$

We know that to make this theory a local one, we need to introduce the gauge covariant derivative, where the connection is given by the photon field A_μ

$$D_\mu = \partial_\mu - ieA_\mu \tag{30.2}$$

where the $-i$ is there since the derivative needs to be antihermitian. From requirement of local invariance, there also appears the kinetic term of the photon field, such that the final local $U(1)$ lagrangian becomes

$$\mathcal{L} = |D_\mu \phi|^2 - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - V(|\phi|^2) \tag{30.3}$$

Expanding the potential and the gauge covariant derivative, we'll get the full lagrangian comprised of the kinetic terms of both scalar boson and vector boson fields, plus the interactions between them

$$\mathcal{L} = |\partial_\mu \phi|^2 - m^2|\phi|^2 - \lambda(|\phi|^2)^2 - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - ieA^\mu \phi^\dagger \overleftrightarrow{\partial}_\mu \phi + e^2 A_\mu A^\mu |\phi|^2 \tag{30.4}$$

In this theory we have therefore four degrees of freedom: two massive scalar bosons with opposite charge and 2 polarization of a massless photon. We could easily use two real fields, in the same way as we've done for the Gell-Mann Levy model

$$\phi = \frac{\sigma + i\pi}{\sqrt{2}} \qquad \phi^\dagger = \frac{\sigma - i\pi}{\sqrt{2}} \tag{30.5}$$

Again, as done before, we flip the sign of the mass term $m^2 \to -\mu^2$ and break the symmetry in a nonlinear way

$$\phi(x) = \frac{v+\sigma(x)}{\sqrt{2}} e^{i\pi(x)/v} \qquad v = \frac{\mu}{\sqrt{\lambda}} \tag{30.6}$$

If this realization seems strange, let us just point out that it's not much different from the one used in the linear sigma model, in fact by expanding the exponential one finds

$$\phi(x) \approx \frac{v+\sigma(x)}{\sqrt{2}} \left(1 + i\frac{\pi(x)}{v} + o(\pi^2)\right) = \frac{v+\sigma(x)+i\pi(x)}{\sqrt{2}} \tag{30.7}$$

Clearly by putting 30.6 in the lagrangian 30.4, the potential won't depend on π and we can rewrite it as

$$V(|\phi|^2) = -\mu^2(\phi^\dagger\phi) + \lambda(\phi^\dagger\phi)^2 = \lambda\left(\phi^\dagger\phi - \frac{v^2}{2}\right) \tag{30.8}$$

Therefore

$$\mathcal{L} = (\partial_\mu + ieA_\mu)\left(\frac{v+\sigma}{\sqrt{2}}\right)e^{-i\frac{\pi(x)}{v}}(\partial_\mu + ieA_\mu)\left(\frac{v+\sigma}{\sqrt{2}}\right)e^{i\frac{\pi(x)}{v}} - \frac{\lambda}{4}\left((v^+\sigma)^2 - v^2\right) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{30.9}$$

What we can now do, and this is the important step, is to get rid of the exponential factor, and so of the soon to be Goldstone boson, by means of the following gauge transformation, called **unitary gauge**

$$\begin{cases} \phi'(x) = e^{-i\alpha(x)}\phi(x) \\ \phi^{\dagger\prime}(x) = \phi^\dagger(x)e^{i\alpha(x)} \\ A'_\mu(x) = A_\mu(x) + \frac{1}{e}\partial_\mu\alpha(x) \end{cases} \tag{30.10}$$

for which we have to choose $\alpha(x) = \dfrac{\pi(x)}{v}$. By using it in the lagrangian, this will give

$$\begin{aligned} \mathcal{L} &= (\partial_\mu + ieA_\mu)\left(\frac{v+\sigma}{\sqrt{2}}\right)(\partial_\mu - ieA_\mu)\left(\frac{v+\sigma}{\sqrt{2}}\right) - \frac{\lambda}{4}((v+\sigma)^2 - v^2) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \\ &= \frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma + \frac{1}{2}e^2A_\mu A^\mu(v+\sigma)^2 - \frac{\lambda}{4}((v+\sigma)^2 - v^2) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \end{aligned} \tag{30.11}$$

What we get is then, beside interaction and some constant terms from the vacuum expectation value, is

$$\begin{aligned} \mathcal{L}_\phi &= \frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma - \frac{m_\sigma^2}{2}\sigma^2 \qquad m\sigma = \sqrt{2}\mu \\ \mathcal{L}_{A_\mu} &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{m_A^2}{2}A_\mu A^\mu \qquad m_A = ev \end{aligned} \tag{30.12}$$

The photon propagator in this theory is given by

$$\frac{-g^{\mu\nu}}{q^2} + \frac{-g^{\mu\sigma}}{q^2}(ievq_\sigma)\frac{1}{q^2}(ievq_\rho)\frac{-g^{\rho\nu}}{q^2} + \cdots = -ig^{\mu\nu}\frac{1}{q^2 - e^2v^2} \tag{30.13}$$

which exactly gives back the massive propagator with mass $m_A = ev$.
We found a mass to the photon which is proportional to the coupling. The degrees of freedom of this theory, which bare in mind is the same as the initial lagrangian, are still four since we now have one massive scalar particle and three polarization of the photon,

since it's now a massive boson. Please understand: the initial lagrangian with the opposite sign mass and the final lagrangian describe exactly the same physical system; all we have down is to select a convenient gauge and rewrite the fields in terms of fluctuations about a particular ground state. We have sacrificed manifest symmetry in favour of notation that makes the physical content more transparent, and allows us to extract the Feynman rules more directly. The extra degree of freedom of the photon field came from the Goldstone boson π which the photon "ate" when we choose a particular gauge.
As a little recap, we have in the end the following

Field	Mass term	Propagator
σ	$m_\sigma^2 = 2\lambda v^2 = 2\mu^2$	$\frac{i}{p^2 - m_\sigma^2 + i\epsilon}$
π	disappeared	
A_μ	$m_A^2 = e^2 v^2$	$\frac{i}{q^2 - m_A^2 + i\epsilon}\left(g_{\mu\nu} + \frac{q_\mu q_\nu}{m_A^2}\right)$

(30.14)

But wait, we got again a propagator which threatens the renormalizability of the theory! To see that in reality the problem now is different, since we didn't explicitly add a mass term but it came from gauge invariance, we now see that the matrix elements of the S-matrix are the same as the ones that we would get with the usage of the t'Hooft gauge.

30.2 The t'Hooft gauge

Consider now the following gauge fixed lagrangian

$$\mathcal{L} = |D_\mu \phi|^2 + \mu^2 \phi^\dagger \phi - \lambda(\phi^\dagger\phi)^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}\left(\partial_\mu A^\mu - \frac{iev}{\sqrt{2}}\xi(\phi - \phi^\dagger)\right)^2 \tag{30.15}$$

Since $\phi - \phi^\dagger = i\sqrt{2}\pi$

$$-\frac{1}{2\xi}(\partial_\mu A^\mu + ev\xi\pi)^2 = -\frac{1}{2\xi}\left((\partial_\mu A^\mu)^2 + e^2v^2\xi^2\pi^2 + 2ev\xi\pi\partial_\mu A^\mu\right) \tag{30.16}$$

This means that in this gauge, the longitudinal part of the photon field is the π field. In the broken phase the lagrangian becomes

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\sigma)^2 - \frac{m_\sigma^2}{2}\sigma^2 + \frac{1}{2}(\partial_\mu\pi)^2 - \frac{e^2v^2\xi}{2}\pi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{e^2v^2}{2}A_\mu A^\mu - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 + \text{ int.} \tag{30.17}$$

Hence we obtain what follows

Field	Mass term	Propagator
σ	m_σ^2	$\frac{i}{p^2 - m_\sigma^2 + i\epsilon}$
π	$m_\pi^2 = v^2e^2\xi$	$\frac{i}{p^2 - m_A^2\xi + i\epsilon}$

(30.18)

For the photon part, we have to find the propagator in the same manner as done before, given the lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m_A^2 A_\mu A^\mu - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 \tag{30.19}$$

The equation of motion is given by

$$\left[(\partial_\mu\partial^\mu - m_A^2)g^{\mu\nu} - \left(1 - \frac{1}{\xi}\right)\partial^\mu\partial^\nu\right]A_\mu = 0 \tag{30.20}$$

Again, by imposing $D_{\mu\nu} = Ag_{\mu\nu} + Bq_\mu q_\nu$ what we find is

$$D_{\mu\nu}(q) = \frac{-i}{q^2 - m_A^2 + i\epsilon}\left(g_{\mu\nu} - (1-\xi)\frac{q_\mu q_\nu}{q^2 - \xi m_A^2}\right) \tag{30.21}$$

Note that when $v = 0 \implies m_A = 0$ we get back the covariant gauge. Now the problem with the ultraviolet divergence is solved since we don't have the term $q_\mu q_\nu / m_A^2$ term.

31 Building up the theory

Now we have to build a lagrangian for the fermionic part so that, upon finding the Feynman rules, we can evaluate the S-matrix of the process $f + \bar{f} \to f + \bar{f}$ and prove that the results we get from the t'Hooft gauge and the unitary gauge are the same, and so no divergent problems appear.

31.1 The fermion lagrangian

It's easy to build up the lagrangian for the fermion with the interaction of the scalar field

$$\mathcal{L} = \bar{\psi}_T(i\not{D})\psi_L + \bar{\psi}_R(i\not{D})\psi_R - Y\left[\bar{\psi}_R\phi^\dagger\psi_L + \bar{\psi}_L\phi\psi_R\right] \tag{31.1}$$

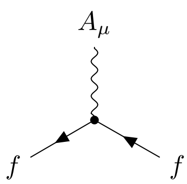
The Yukawa coupling terms are such since ϕ and ψ_L rotate in the same manner. In terms of the σ and π fields, the interaction term becomes

$$\begin{aligned}\mathcal{L} &= -Y\left[\bar{\psi}\frac{1-\gamma_5}{2}\left(\frac{v+\sigma-i\pi}{\sqrt{2}}\right)\frac{1-\gamma_5}{2}\psi + \bar{\psi}\frac{1+\gamma_5}{2}\left(\frac{v+\sigma+i\pi}{\sqrt{2}}\right)\frac{1+\gamma_5}{2}\psi\right] \\ &= -Y\bar{\psi}\left[\frac{v+\sigma-i\pi}{\sqrt{2}}\frac{1-\gamma_5}{2} + \frac{v+\sigma+i\pi}{\sqrt{2}}\frac{1+\gamma_5}{2}\right]\psi \\ &= -Y\bar{\psi}\left[\frac{v+\sigma}{\sqrt{2}} + \frac{i\pi}{\sqrt{2}}\gamma_5\right]\psi = -\frac{Yv}{\sqrt{2}}\bar{\psi}\psi - \frac{Y}{\sqrt{2}}\bar{\psi}\sigma\psi - \frac{iY}{\sqrt{2}}\bar{\psi}\gamma_5\psi\pi\end{aligned} \tag{31.2}$$

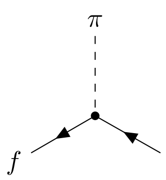
In the end we got, as expected, a mass term for the fermions which depends on the coupling $m_f = Yv/\sqrt{2}$. In the end, in the lagrangian we have

$$\mathcal{L} = \bar{\psi}(i\not{D})\pi - m_f\bar{\psi}\psi + eA_\mu\bar{\psi}\gamma^\mu\frac{1-\gamma_5}{2}\psi - \frac{Y}{\sqrt{2}}\bar{\psi}\sigma\psi - \frac{iY}{\sqrt{2}}\bar{\psi}\gamma_5\psi\pi \tag{31.3}$$

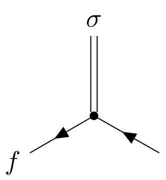
which gives us directly the Feynman rules



$= -ie\gamma_\mu\frac{1-\gamma_5}{2}$



$= -\frac{Y}{\sqrt{2}}\gamma_5$



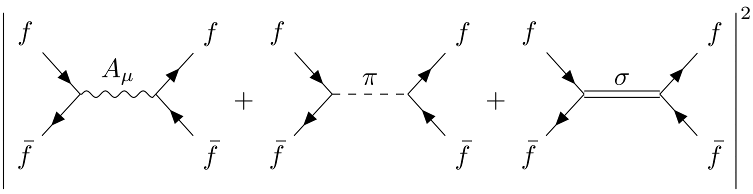
$= i\frac{Y}{\sqrt{2}}$

(31.4)

The second vertex disappears in the unitary gauge.

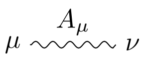
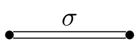
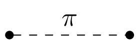
32 The $f\bar{f}$ matrix element

We now focus our attention on the process $f\bar{f} \to f\bar{f}$. The matrix element of this process will be given at tree level, in general, by the following diagrams

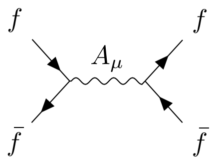


(32.1)

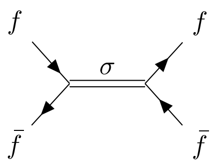
The propagators in the two gauges are summarized here

t′Hooft gauge		**Unitary gauge**
$\dfrac{-i}{q^2 - m_A^2 + i\epsilon}\left(g_{\mu\nu} - (1-\xi)\dfrac{q_\mu q_\nu}{q^2 - \xi m_A^2}\right)$		$\dfrac{-i}{q^2 - m_A^2 + i\epsilon}\left(g_{\mu\nu} - \dfrac{q_\mu q_\nu}{m_A^2}\right)$
$\dfrac{i(\not{p} + m_f)}{p^2 - m_f^2 + i\epsilon}$		$\dfrac{i(\not{p} + m_f)}{p^2 - m_f^2 + i\epsilon}$
$\dfrac{i}{p^2 - m_\sigma^2 + i\epsilon}$		$\dfrac{i}{p^2 - m_\sigma^2 + i\epsilon}$
$\dfrac{i}{p^2 - \xi m_A^2 + i\epsilon}$		ABSENT

We now compute the various matrix elements in the two gauges.
First we start from the unitary gauge, in which the pion term is absent, and get what follows

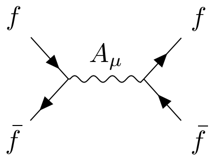


$$= \bar{u}(k_1)(-ie\gamma_\mu)v(k_2)\frac{-i}{q^2 - m_A^2 + i\epsilon}\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{m_A^2}\right)\bar{v}(p_2)(-ie\gamma_\nu)u(p_1)$$

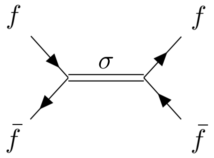


$$= \bar{u}(k_1)\frac{-iY}{\sqrt{2}}v(k_2)\frac{i}{p^2 - m_\sigma^2 + i\epsilon}\bar{u}(p_2)\frac{-iY}{\sqrt{2}}u(p_1) \tag{32.2}$$

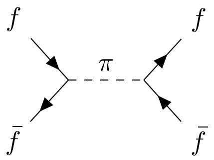
In the t'Hooft gauge instead we get



$$= \bar{u}(k_1)(-ie\gamma_\mu)v(k_2)\frac{-i}{q^2 - m_A^2 + i\epsilon}\left(g_{\mu\nu} - (1-\xi)\frac{q_\mu q_\nu}{q^2 - m_A^2\xi}\right)\bar{v}(p_2)(-ie\gamma_\nu)u(p_1)$$



$$= \bar{u}(k_1)\frac{-iY}{\sqrt{2}}v(k_2)\frac{i}{p^2 - m_\sigma^2 + i\epsilon}\bar{u}(p_2)\frac{-iY}{\sqrt{2}}u(p_1)$$



$$= \bar{u}(k_1)\frac{-Y\gamma_5}{\sqrt{2}}v(k_2)\frac{i}{p^2 - m_A^2\xi + i\epsilon}\bar{v}(p_2)\frac{-Y\gamma_5}{\sqrt{2}}u(p_1) \tag{32.3}$$

The main difference, beside the pion part, is in the photon. By simple comutation we can see that the t'Hooft term is just given by the unitary one plus another term

$$\text{tH} = \text{UG} + \left(\frac{-e^2}{4}\right)\bar{u}(k_1)\not{q}(1-\gamma_5)v(k_2)\frac{i\left(\frac{1-\xi}{q^2 - m_A^2\xi} - \frac{1}{m_A^2}\right)}{q^2 - m_A^2}\bar{v}(p_2)\not{q}(1-\gamma_5)u(p_1) \tag{32.4}$$

Let's take a better look at the additional term. The middle term is just

$$\frac{1}{q^2 - m_A^2}\left(\frac{m_A^2 - m_A^2\xi - q^2 + m_A^2\xi}{(q^2 - m_A^2\xi)m_A^2}\right) = -\frac{1}{m_A^2(q^2 - m_A^2\xi)} \tag{32.5}$$

Then

$$\text{tH} = \text{UG} + \frac{e^2}{4}\bar{u}(k_1)\not{q}(1-\gamma_5)v(k_2)\frac{1}{m_A^2}\frac{1}{q^2 - m_A^2\xi}\bar{u}(p_2)\not{q}(1-\gamma_5)u(p_1) \tag{32.6}$$

In the s-channel $q = k_1 + k_2 = p_1 + p_2$ and so

$$\begin{aligned}&\bar{u}(k_1)(\not{k}_1 + \not{k}_2)(1-\gamma_5)v(k_2) = \bar{u}(k_1)\not{k}_1(1-\gamma_5)v(k_2) + \bar{u}(k_1)(1+\gamma_5)\not{k}_2 v(k_2)\\ &= (m_f - m_f)\bar{u}(k_1)v(k_2) - 2m_f\bar{u}(k_1)\gamma_5 v(k_2)\end{aligned} \tag{32.7}$$

and the same goes, with the opposite sign for the other half of the term. In the end what we get is

$$\begin{aligned}\text{tH} &= \text{UG} - \frac{e^2}{4}\frac{4m_f^2}{m_A^2}\bar{u}(k_1)\gamma_5 v(k_2)\frac{1}{q^2 - m_A^2\xi}\bar{v}(p_2)\gamma_5 u(p_1)\\ &= \text{UG} - \frac{e^2}{4}\frac{4Y^2v^2}{2v^2e^2}\bar{u}(k_1)\gamma_5 v(k_2)\frac{1}{q^2 - m_A^2\xi}\bar{v}(p_2)\gamma_5 u(p_1)\\ &= \text{UG} - \bar{u}(k_1)\frac{Y}{\sqrt{2}}\gamma_5 v(k_2)\frac{1}{q^2 - m_A^2\xi}\bar{v}(p_2)\frac{Y}{\sqrt{2}}\gamma_5 u(p_1)\end{aligned} \tag{32.8}$$

Eureka, the additional term in the t'Hooft gauge gives exactly the unitary gauge plus the pion term but with opposite sign, so that cancels out to give exactly the same result in the two gauges.

Therefore, given the fact that the t'Hooft gauge has no ultraviolet divergences and gives the same result as the unitary gauge, found through the help of Higgs mechanics, then even the unitary gauge has no ultraviolet divergences. The problem of renormalizability has been solved.

PART

XII

Yang-Mills Theories

33 Abelian and non-abelian gauge invariance

We have seen multiple times that the requirement that a theory has to be **locally** invariant under a $U(1)$ symmetry group, requires the construction of a covariant derivative and directly from that, there arises a new field with it's specific kinetic term.
What we want now is to give a better understanding of the underlying geometric nature of the covariant derivative and the field which pops up whenever we require local gauge invariance.
The simplest case we encountered of local gauge invariance is the local $U(1)$ symmetry group of QED. This symmetry is a very special one since is an **abelian** gauge symmetry. Limiting ourseves to only abelian groups is clearly not enough and so we'll apply the same reasoning that we have seen for the local $U(1)$ symmetries to more complex symmetry groups and see what happens.

33.1 The geometry of gauge invariance

As we have seen, local symmetries endanger causality in field theory. So a better way of talking about symmetries is by their local counterpart. In the case of QED the symmetry group is $U(1)$ which means that the fermion fields transform as

$$\psi \to e^{ig\alpha(x)}\psi \qquad \bar{\psi} \to \bar{\psi}e^{-ig\alpha(x)} \tag{33.1}$$

If we want to take the derivative in the diraction n^μ of a spinor field $n^\mu \partial_\mu \psi$ we're implicitly doing a wrong calculation since

$$n^\mu \partial_\mu \psi = \lim_{\epsilon \to 0} \frac{\psi(x+n\epsilon) - \psi(x)}{\epsilon} \tag{33.2}$$

The subtraction is ill defined! We are comparing two fields which transform in completely different ways. If we want to compare two quantities which transform in a given manner in two different points in spacetime, we need a *comparator* $U(y,x)$ that transforms in the following manner

$$U(y,x) \to e^{ig\alpha(y)}U(y,x)e^{-ig\alpha(x)} \tag{33.3}$$

so that $\psi(y)$ and $U(y,x)\psi(x)$ transform in the same manner

$$\begin{aligned} &\psi(y) \to e^{ig\alpha(y)}\psi(y) \\ &U(y,x)\psi(x) \to e^{ig\alpha(y)}U(y,x)e^{-ig\alpha(x)}e^{ig\alpha(x)}\psi(x) = e^{ig\alpha(y)}U(y,x)\psi(x) \end{aligned} \tag{33.4}$$

Moreover, a sensible requirement is that $U(y,y) = 1$. Without loss of generality, the comparator can be seen as a pure phase $e^{i\phi(y,x)}$.[19] With this we can construct a well defined derivative D_μ in spacetime

$$n^\mu D_\mu \psi = \lim_{\epsilon \to 0} \frac{\psi(x-n\epsilon) - U(x+n\epsilon,x)\psi(x)}{\epsilon} \tag{33.5}$$

[19] A more general formulation is based on the usage of Wilson loops.

Infinitasimally the comparator becomes

$$U(x+n\epsilon,x) = 1 + ig\epsilon n_\mu \Gamma^\mu + \mathcal{O}(\epsilon^2) \tag{33.6}$$

so that

$$\begin{aligned} n^\mu D_\mu \psi &= \lim_{\epsilon\to 0} \frac{1}{\epsilon}[\psi(x+n\epsilon) - \psi(x) - ig\epsilon n^\mu \Gamma_\mu \psi(x)] \\ &= \lim_{\epsilon\to 0} \frac{1}{\epsilon}[\psi(x+n\epsilon) - \psi(x)] - ign^\mu \Gamma_\mu \psi(x) \end{aligned} \tag{33.7}$$

which gives us the final form of our covariant derivative in the abelian case

$$D_\mu = \partial_\mu - ig\Gamma_\mu \tag{33.8}$$

The quantity Γ_μ is called **connection** in differential geometry and it's what permits us to transport vectors along curves in a manifold.
The transformation rule for the comparator let's us find the transformation rule for the connection

$$\Gamma_\mu \to (1+ig\alpha(x))\Gamma_\mu(1-ig\alpha(x)) - \frac{i}{g}(\partial_\mu\alpha(x))(1-ig\alpha(x)) = \Gamma_\mu - g\partial_\mu\alpha(x) \tag{33.9}$$

From this we see that the transformation rule is just the one of the photon field A_μ under a gauge transformation. So for a global $U(1)$ symmetry, we identify the connection with the photon field and the covariant derivative becomes

$$D_\mu = \partial_\mu - ieA_\mu \tag{33.10}$$

where we put back the charge of the electron instead of the general charge g.

From a simple geometrical construction we can easily find the field tensor. To do this we must simply ask the following question: given the infinitesimal form of the comparator, which transports a gauge-dependent field an infinitesimal distance in spacetime, what would happen if we transported the field along an infinitesimal parallelogram?

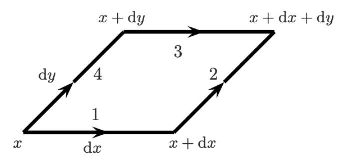


Figure 15. Parallelogram used to calculate the rotation of a test field ϕ moved along a closed loop in the presence of a non-zero gauge field A^μ

Let us denote $U_{\mathrm{d}x}(x) = 1 - igA_\mu(x)\,\mathrm{d}x^\mu$ the infinitesimal action of the comparator we just saw which moves from the point x an infinitesimal amount $\mathrm{d}x$. If we want to move along the path 2 in the figure 15

$$\begin{aligned} U_{\mathrm{d}y}(x+\mathrm{d}x) &= 1 - igA_\nu(x+\mathrm{d}x)\,\mathrm{d}y^\nu \\ &= 1 - igA_\nu(x)\,\mathrm{d}y^\nu - ig\partial_\mu A_\nu(x)\,\mathrm{d}x^\mu\,\mathrm{d}y^\nu \end{aligned} \tag{33.11}$$

Combining paths 1 and 2 we get

$$\begin{aligned} U_{\mathrm{d}x}(x+\mathrm{d}y)U_{\mathrm{d}y}(x) = 1 &- igA_\nu(x)\mathrm{d}y^\nu - igA_\mu(x)\mathrm{d}x^\mu - ig\partial_\nu A_\mu(x)\mathrm{d}x^\nu\mathrm{d}y^\mu \\ &- g^2 A_\mu(x)A_\nu(x)\mathrm{d}x^\mu\mathrm{d}y^\nu \end{aligned} \tag{33.12}$$

Instead of performing now a round trip $1\to 2\to 3\to 4$, we evaluate next $4\to 3$ which we then subtract from the just found $1\to 2$. In this way we can reuse the same result found for $1\to 2$ just by exchanging $A_\mu \mathrm{d}x^\mu$ with $A_\nu \mathrm{d}y^\nu$ and viceversa. Whit this we get

$$\begin{aligned} U_{\mathrm{d}x}(x+\mathrm{d}y)U_{\mathrm{d}y}(x) = 1 &- igA_\nu(x)\mathrm{d}y^\nu - igA_\mu(x)\mathrm{d}x^\mu - ig\partial_\nu A_\mu(x)\mathrm{d}x^\mu\mathrm{d}y^\nu \\ &- g^2 A_\mu(x)A_\nu(x)\mathrm{d}x^\mu\mathrm{d}y^\nu \end{aligned} \tag{33.13}$$

so that for the whole trip one gets

$$-ig(\partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu])\mathrm{d}x^\mu \mathrm{d}y^\nu \tag{33.14}$$

Thus, using the commutator relation for the photon field, the final expression for the comparator is given by

$$U(x,y) = 1 - ig(\partial_\mu A_\nu - \partial_\nu A_\mu)\mathrm{d}x^\mu \mathrm{d}y^\nu = 1 - ig\epsilon^2 F_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}y^\nu \tag{33.15}$$

We found geometically a locally invariant quantity which depends on the derivatives of the connection, and therefore can give a kinetic term to the photon field.
A more general way of finding the field strength tensor is by requiring that the second covariant derivative transform as the fields. By some simple calculations one finds a general definition for the field tensor

$$[D_\mu, D_\nu] = igF_{\mu\nu} \tag{33.16}$$

This definition will be very useful in the case of non-abelian theories.
Formulated in this way, the electromagnetic tensor is nothing but the curvature tensor. If you've met the concept of curvature previously, then you probably know that the way we ordinarily conceive of Minkowski space it is a flat space.
However, the idea of Yang-Mills theory is to throw out the ordinary concept of flat Minkowski space. What the connection, the vector potential in this case, does is to introduce a twist into Minkowski space. The curvature F then measures the extent to which this twist causes a deviation from the ordinary flat geometry of Minkowski space.

33.2 Non abelian case

What if now the field does not transform under the abelian $U(1)$ group but under a more general group? Under a general lie group the field ψ transforms as

$$\psi(x) \to e^{ig\alpha_A(x)\lambda_A}\psi(x) \qquad \psi^\dagger(x) \to \psi^\dagger(x)^{-ig\alpha_A(x)\lambda_A} \tag{33.17}$$

where λ_A, with $A = 1, \cdots, \dim G$, are the generators of the underlying symmetry group which respect the following orthogonality relation

$$\mathrm{Tr}\,\lambda^A\lambda^B = \frac{1}{2}\delta^{AB} \tag{33.18}$$

Moreover, for any semi-simple Lie group we have

$$\left[\lambda^A, \lambda^B\right] = if^{ABC}\lambda^C \tag{33.19}$$

With this in mind, we can see how the connection transforms since from that we can find the transformation rule for the gauge field

$$\begin{aligned}\Gamma'_\mu &= -\frac{i}{g}[\partial_\mu(1 + ig\alpha_A(x)\lambda_A)]\,(1 - ig\alpha_B(x)\lambda_B) + (1 + ig\alpha_A(x)\lambda_A)\Gamma_\mu(1 - ig\alpha_B(x)\lambda_B)\\ &= -\frac{i}{g}(ig\partial_\mu\alpha_A(x)\lambda_A)(1 - ig\alpha_B(x)\lambda_B) + \Gamma_\mu + ig\alpha_A(x)\lambda_A\Gamma_\mu - ig\Gamma_\mu\alpha_B(x)\lambda_B\\ &= \Gamma_\mu + (\partial_\mu\alpha_A(x))\lambda_A + ig\alpha_A(x)[\lambda_A, \Gamma_\mu]\end{aligned} \tag{33.20}$$

Since the generators in the adjoint representation furnish a base for the Lie group, we can surely expand the connection on them as

$$\Gamma_\mu = \sum_{B=1}^{\dim G} G_\mu^B \lambda^B \tag{33.21}$$

so to get the transformation of the connection in terms of the gauge fields G_μ^A

$$\begin{aligned} G_\mu^{A\prime}\lambda^A &= G_\mu^A \lambda^A + \partial_\mu \alpha_A(x)\lambda_A + ig\alpha_A \sum_{B=1}^{\dim G} G_\mu^B \left[\lambda^A, \lambda^B\right] \\ &= G_\mu^A \lambda^A + \partial_\mu \alpha^A \lambda^A - f^{ABC}\alpha^A G_\mu^B \lambda^C \end{aligned} \tag{33.22}$$

By projecting using the trace condition $\operatorname{Tr}\lambda^A\lambda^B = \delta^{AB}/2$ one finds the transformation rule of the gauge field

$$\left(G_\mu^A\right)' = G_\mu^A + \partial_\mu \alpha^A(x) - f^{ABC}\alpha^B G_\mu^C \tag{33.23}$$

Now that we have the transformation rule for the gauge field, we can construct the field tensor using the definition 33.16 with covariant derivative $D_\mu = \partial_\mu - igG_\mu^A\lambda^A$ summed over the rep index

$$[D_\mu, D_\nu] = -igG_{\mu\nu}^A \lambda^A \tag{33.24}$$

What we get is the following

$$\begin{aligned} [D_\mu, D_\nu] &= [\partial_\mu, \partial_\nu] - ig\left(\left[\partial_\mu, G_\nu^A\lambda^A\right] - \left[\partial_\nu, G_\mu^B\lambda^B\right]\right) + g^2\left[G_\mu^A\lambda^A, G_\nu^B\lambda^B\right] \\ &= -ig\left(\partial_\mu G_\nu^A - \partial_\nu G_\mu^A\right)\lambda^A + ig^2 G_\mu^A G_\nu^B f^{ABC}\lambda^C \end{aligned} \tag{33.25}$$

where in the last step we used the commutator relation for the generator of the Lie algebra. Given this result, one finds

$$\begin{aligned} -igG_{\mu\nu}^A\lambda^A &= -ig\left(\partial_\mu G_\nu^A - \partial_\nu G_\mu^A\right)\lambda^A + ig^2 G_\mu^A G_\nu^B f^{ABC}\lambda^C \\ \Longrightarrow\ G_{\mu\nu}^A\lambda^A &= \left(\partial_\mu G_\nu^A - \partial_\nu G_\mu^A\right)\lambda^A - gG_\mu^A G_\nu^B f^{ABC}\lambda^C \end{aligned} \tag{33.26}$$

if we now project using the trace condition found earlier

$$\begin{aligned} G_{\mu\nu}^A \operatorname{Tr}\lambda^A\lambda^D &= \left(\partial_\mu G_\nu^A - \partial_\nu G_\mu^A\right)\operatorname{Tr}\lambda^A\lambda^D - gG_\mu^A G_\nu^B f^{ABC}\operatorname{Tr}\lambda^C\lambda^D \\ \Longrightarrow\ G_{\mu\nu}^D &= \left(\partial_\mu G_\nu^D - \partial_\nu G_\mu^D\right) - gf^{ABD}G_\mu^A G_\nu^B \\ &= \left(\partial_\mu G_\nu^D - \partial_\nu G_\mu^D\right) + gf^{DAB}G_\mu^A G_\nu^B \end{aligned} \tag{33.27}$$

where in the last step we used the antisymmetric property of the structure constants. In the end we have our definition of the general field tensor associated to a certain gauge field of a symmetry group

$$G_{\mu\nu}^A = \partial_\mu G_\nu^A - \partial_\nu G_\mu^A + gf^{ABC}G_\mu^B G_\nu^C \tag{33.28}$$

The same result could have been found by imposing the invariance of the gauge tensor field $(G_{\mu\nu}^A)' = G_{\mu\nu}^A$ by first defining it like the EM-tensor $G_{\mu\nu}^A = \partial_\mu G_\nu^A - \partial_\nu G_\mu^A$ and then noticing that, for the invariance to hold, there has to be an additional term which will turn out to be the $gf^{ABC}G_\mu^B G_\nu^C$ term. This new term is the price to pay for the non-abelian nature of the theory.

34 Non-abelian gauge theory

We now have all the tools we need to write down the lagrangian for a generic gauge field associated to a given local symmetry group with all the possible interactions of the gauge field with matter fields such as fermions.
In turn this will give us a theory which we can quantise and we know, thanks to 't Hooft and Veltman[20], to be a renormalizable theory which is a fundamental requirement for the construction of a physical theory such as the standard model.

[20] They won the nobel prize in 1999 for this.

34.1 Form the EM lagrangian to the general case

We know that the lagrangian for the electromagnetic field is given in terms of the EM-field tensor $F_{\mu\nu}$

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = -\frac{1}{4}F^2 \tag{34.1}$$

and that the interaction with the matter field is introduced by means of the $U(1)$ gauge covariant derivative $D_\mu = \partial_\mu - ieA_\mu$ where e is the $U(1)$ coupling constant and A_μ is the photon field.
If we take QED for example, the total lagrangian for the theory is given by

$$\mathcal{L}_{QED} = -\frac{1}{4}F^1 + \bar{\psi}(i\not{D} - m)\psi = -\frac{1}{4}F^2 + \bar{\psi}(i\not{\partial} - m)\psi - ie\bar{\psi}\not{A}\psi \tag{34.2}$$

Or in case of scalar electrodynamics, where now the matter field is given by a complex scalar field

$$\mathcal{L}_{SQED} = -\frac{1}{4}F^2 + |D_\mu\phi|^2 - m^2|\phi|^2 \tag{34.3}$$

The interactions are contained into the covariant derivative where we have the coupling between the gauge field and the matter field. With the interaction lagrangian we can build up the well known Feynman rules for the said theories.

With this in mind, we can extend what we just said to a general gauge theory. From now on we'll only consider $SU(N)$ gauge theories and, specifically, $SU(2)$ and $SU(3)$. Following the definition 34.1 we write, for the generic gauge field $G^A_{\mu\nu}$

$$\mathcal{L} = -\frac{1}{4}G^A_{\mu\nu}(G^{\mu\nu})^A = -\frac{1}{4}\,\mathrm{Tr}(G_{\mu\nu}G^{\mu\nu}) \tag{34.4}$$

The trace is over the rep index since we have the implicit summation. The main difference between the simple abelian $U(1)$ gauge theory and its non-abelian counterpart is that, while in QED we have only one gauge field, the photon, in a general $SU(N)$ theory we'll have N^2-1 "photons" since the gauge field belong in the adjoint rep. So for QCD, which is based on the $SU(3)$ Lie group, we'll have 8 gauge fields, the gluons. For the $SU(2)$ part of the electroweak theory we'll have 3 gauge fields which, as we'll see later, will be related to the three vector bosons which mediate the weak interaction $W^\pm, Z^0$.

Having at our disposal the covariant derivative associated to a given symmetry group and the related field tensor, we can construct any theory we want, coupling the gauge field to some matter field. Suppose that we want to write down the lagrangian of a field theory with a generic $SU(N)$ gauge field, a complex scalar field and a fermion field. It's really easy to do[21]

$$\mathcal{L} = |D_\mu\phi|^2 + \bar{\psi}(i\not{D} - m)\psi - \frac{1}{4}G^A_{\mu\nu}G^{\mu\nu}_A - V(|\phi|^2) \tag{34.5}$$

[21] Sometimes you'll see that the rep index will be down, or up. That's just so that the indices are not too crowded, but really the rep index is just a "numbering" index, there's no difference between up and down.

The coupling of the matter fields with the gauge field is hidden, for now, in the covariant derivative. An example of a non-abelian gauge theory is the quantum chromodynamics QCD, that describes the strong interaction

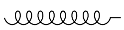
$$\mathcal{L}_{QCD} = -\frac{1}{4}G^A_{\mu\nu}G^{\mu\nu}_A + \bar{\psi}_q(i\not{D} - m_q)\psi_q \tag{34.6}$$

Remark. In the limit $g = 0$, the equation 34.4 will describe free particles, as many as there are generators of the group, each one characterised by a lagrangian identical to the free EM-lagrangian, which describes the quantum of the field with spin 1 and **massless**. This is the main reason of why the Yang-Mills theory was abandoned until the studies on the spontaneous symmetry breaking.

34.2 Quantization

Constructing the Feynman rules for a Yang-Mills theory is not so easy as in the abelian-case. We'll now give the explicit results which in some cases are easy to see while in others, specifically the 3-gluon[22] and 4-gluon interaction, are not so easy and can be found using functional quantization which is beyond the scope of this notes. Without further ado, let's begin to study the Feynman rules of the theory 34.5 where there's no symmetry breaking and all the gluons are massless. The propagators are easily found by the one we know from before

[22] We use gluon as a generic term for the non-abelian gauge field.

μ, A  ν, B $= \dfrac{i}{q^2 + i\epsilon}\left(-g_{\mu\nu} + (1-\xi)\dfrac{q_\mu q_\nu}{q^2}\right)\delta^{AB}$ $\qquad A, B = 1, \cdots, \dim G$

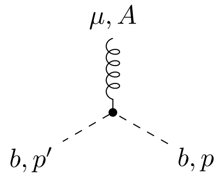
a b $= \dfrac{i}{q^2 - m^2_\phi + i\epsilon}\delta_{ab}$ $\qquad a, b = 1, \cdots, \dim \text{rep}$

a, α  b, β $= \dfrac{i(\not{q} + m)_{\alpha\beta}}{q^2 - m^2_f + i\epsilon}\delta_{ab}$ $\qquad a, b = 1, \cdots, \dim \text{rep}$

It's important to remember where the various indices run since, more often than not, they are hidden and not explicitly wrote down.
Now for the various interaction vertices: first the scalar-gluon interactions which we get by expanding the covariant derivative on the complex scalar field

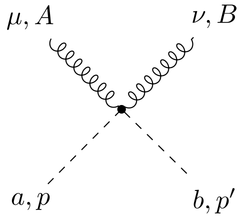
$$\begin{aligned} |D_\mu\phi|^2 &= (D_\mu\phi)^\dagger(D^\mu\phi) = (\partial_\mu + igG^A_\mu\lambda^A)\phi^\dagger(\partial^\mu - igG^\mu_A\lambda_A)\phi \\ &= |\partial_\mu\phi|^2 - igG^A_\mu\lambda^A\phi^\dagger\overleftrightarrow{\partial^\mu}\phi + g^2G^A_\mu G^\mu_B\lambda^A\lambda_B\phi^\dagger\phi \end{aligned} \tag{34.7}$$

- $igG^A_\mu \lambda^A \phi^\dagger \overleftrightarrow{\partial^\mu} \phi$



$= g(p_\mu + p'_\mu)\,(\lambda_A)^a{}_b$

- $g^2 G^A_\mu G^\mu_B \lambda^A \lambda_B \phi^\dagger \phi$

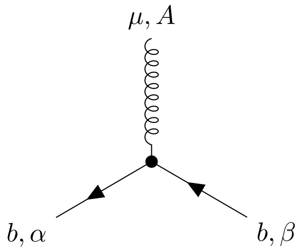


$= -ig^2 g_{\mu\nu} \left(\lambda^A \lambda^B\right)^a{}_b$

then the fermion-gluon interaction

$$i\bar{\psi}\not{D}\psi = i\bar{\psi}\gamma^\mu(\partial_\mu - igG^A_\mu \lambda^A)\psi = i\bar{\psi}\not{\partial}\psi + g\bar{\psi}\gamma^\mu G^A_\mu \lambda^A \psi \tag{34.8}$$

- $g\bar{\psi}\gamma^\mu G^A_\mu \lambda^A \psi$



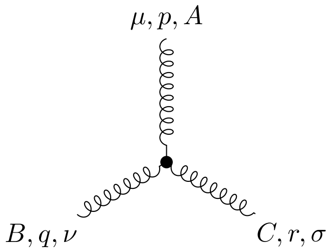
$= -ig\,(\gamma^\mu)^\alpha{}_\beta \left(\lambda^A\right)^a{}_b$

and then the interesting bit, which wasn't present in the abelian case, the gluon-gluon interactions

$$\begin{aligned}
-\frac{1}{4}G^A_{\mu\nu}G^{\mu\nu}_A &= -\frac{1}{4}\left(\partial_\mu G^A_\nu - \partial_\nu G^A_\mu + gf^{ABC}G^B_\mu G^C_\nu\right)\left(\partial^\mu G^\nu_A - \partial^\nu G^\mu_A + gf^{ADE}G^\mu_D G^\nu_E\right) \\
&= -\frac{1}{4}\left[+gf^{ADE}\left(\partial_\mu G^A_\nu\right)G^\mu_D G^\nu_E - gf^{ADE}\left(\partial_\nu G^A_\mu\right)G^\mu_D G^\nu_E + \cdots\right. \\
&\left.\cdots + gf^{ABC}\left(\partial^\mu G^\nu_A\right)G^B_\mu G^C_\nu - gf^{ABC}\left(\partial^\nu G^\mu_A\right)G^\mu_B G^\nu_C + g^2 f^{ABC}f^{ADE}G^B_\mu G^C_\nu G^\mu_D G^\nu_E\right]
\end{aligned} \tag{34.9}$$

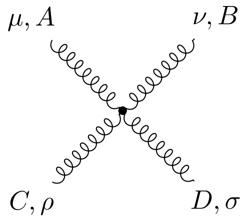
which gives rise to two possible vertices, one with a 3-gluon interaction and one with 4-gluon interation, whose Feynman rules are

- $-\frac{g}{4} f^{ABC} \left[\left(\partial_\mu G^A_\nu \right) G^\mu_B G^\nu_c + \text{permutations} \right]$



$$= g f^{ABC} \left(g^{\mu\nu}(p-q)^\sigma + g^{\nu\sigma}(q-r)^\mu + g^{\sigma\mu}(r-p)^\nu \right)$$

- $g^2 f^{ABC} f^{ADE} G^B_\mu G^C_\nu G^\mu_D G^\nu_E$



$$= -ig^2 \left[f^{ABE} f^{CDE} \left(g^{\mu\rho} g^{\nu\sigma} - g^{\mu\sigma} g^{\nu\rho} \right) + \text{permutations} \right]$$

Again, it's important to note that the non-abelian nature of the theory adds new interaction vertices between the gauge fields. In QED two photons cannot interact at tree level. The lowest order interaction for two photon scattering in QED is given by the box diagram.[23]

34.3 Ghosts and other strange things

Whenever we try to quantize a theory we do so by using functional quantization, i.e. starting from the Feynman path integral for some theory we can build up the Feynman rules using functional derivatives.

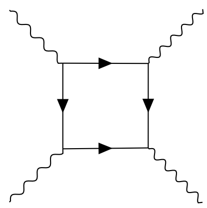
Whenever working with gauge theories there's a catch: the gauge freedom makes the functional integral ill defined. This can be simply understood since, whenever evaluating a functional integral, one integrates aver all the possible field configurations. Given that for gauge theories we have infinitely many equivalent field configurations we have to integrate over an infinite quantity[24] making the integral ill defined.

To overcome this problem one uses the so called Faddeev-Popov method which, in some sense, fixes the gauge so to make the functional integral convergent. By doing so, tho, we introduce a new fictitious field, **ghost field**. This field cannot be associated to a real particle since it changes under a change of gauge and it can be eliminated by a suitable choice of the gauge. Nevertheless one can do calculation with such fields as if they where real particles only to realize that all physical quantities, being independent on the choice of the gauge, won't have any dependence on ghost fields.

Ghosts don't appear in abelian theories since they come up directly from the additional terms which only non-abelian theories have.

A peculiarity of ghosts is that they behave like a fermion and like a boson at the same time. This further adds to the fact that this fields cannot be associated to any real particle.

23



Box diagram of the two photon process in QED.

24 This is not entirely true since even without gauge freedom, the degrees of freedom of a field are again infinite. But just to give a taste of the problem, this semplification suffices.

PART XIII

Glashow-Weinberg-Salam theory

We now have sufficient knowledge to formulate the GSW theory of weak and electromagnetic interactions among leptons and quarks and to study its properties. Let us first state the starting point and the aim of our study

1. There exist charged and neutral currents.
2. The charged currents contain only couplings between left-handed fermions. This result is given by Fermi theory of weak interactions which, as we'll see, is the low energy limit of the GSW theory.
3. The bosons $W^{\pm}, Z^0$ mediating the weak interaction must be very massive.
4. Nevertheless we'll begin with massless bosons which then receive masses through the Higgs mechanism. At that point we want to simultaneously include the photon field.

Given this list of properties, we can begin to build up the first part of the SM which accounts for the electroweak sector.

35 The GWS Lagrangian

35.1 Symmetry breaking

Let's begin, as we always must, to find the symmetry group of the theory. We know that at least there must be one gauge boson for the photon. Moreover there must be another two vector bosons for the $W^{\pm}$ fields. With this we need at least the $SU(2)$ symmetry group since it has 3 generators. But it turns out that this group is too small since it only accounts for left-handed interactions but we know that the electromagnetism is perfectly symmetric between left and right-handed fermions.
What Glashow proposed was the following minimal group

$$SU(2)_W \otimes U(1)_Y \tag{35.1}$$

where the reps are defined by the **isospin** symmetry and the **hypercharge**. The $U(1)_Y$ is not to be confused with the $U(1)$ symmetry group of electromagnetism, that will come later after symmetry breaking. Based on this symmetry group, the existence of a fourth gauge boson was theorized since the group has 4 generators. It will turn out that the additional gauge boson is, in fact, the Z^0 which mediates the weak neutral currents.

Since we have that the total symmetry group is the product of two groups, we need two different coupling constants g, g'. The kinetic part of the lagrangian will be then

$$\mathcal{L} = -\frac{1}{4}W^A_{\mu\nu}W^{\mu\nu}_A - \frac{1}{4}B_{\mu\nu}B^{\mu\nu} \tag{35.2}$$

where

$$W_{\mu}\nu^A = \partial_\mu W^A_\nu - \partial_\nu W^A_\mu + g\epsilon^{ABC}W^B_\mu W^C_\nu \qquad B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu \tag{35.3}$$

are the gauge tensors respectively for the $SU(2)$ and $U(1)$ part of the symmetry group.

Given the local nature of the interactions, we need to give mass to the bosons. On the other hand, the photon will be given by a linear combination of the simmetry generators which remain unbroken under the action of the Higgs mechanism.
To induce the symmetry breaking, we introduce a complex isospin doublet with hypercharge 1 called the **Higgs doublet**

$$H = \begin{pmatrix} H^+ \\ H^0 \end{pmatrix} \tag{35.4}$$

The hypercharge is set by the Gell-Mann Nishijima formula $Q = \tau_3 + Y/2$

$$\begin{aligned} H^+ \qquad & 1 = \frac{1}{2} + \frac{Y}{2} \implies Y = 1 \\ H^0 \qquad & 0 = -\frac{1}{2} + \frac{Y}{2} \implies Y = 1 \end{aligned}$$

To classify the fields in the Standard Model we'll heavily use the following notation $\Big(\text{dim } \text{rep}_{SU(2)}, \text{dim } \text{rep}_{SU(3)}\Big)_Y$, and so the Higgs field belongs in the rep

$$H \sim (2, 1)_1 \tag{35.5}$$

A field Φ^a of the form $(R, 1)_1$ will transform as

$$\Phi^a \to \exp\left(i\alpha^A(x)\frac{\sigma^A}{2} + i\beta 1\right)^a_{\ b} \Phi^b \tag{35.6}$$

The lagrangian for the Higgs field is given by

$$\mathcal{L} = |D_\mu H|^2 - V(|H|) \tag{35.7}$$

where $V(|H|)$ is some symmetry breaking potential and the covariant derivatives are given by

$$D_\mu = \partial_\mu - ig'\frac{Y}{2}B_\mu - igW^A_\mu\lambda^A = \partial_\mu - i\frac{g'}{2}B_\mu - igW^a_\mu\tau^a \tag{35.8}$$

By choosing the potential as

$$V(H) = -\mu^2|H|^2 + \lambda|H|^4 \tag{35.9}$$

we induce a vev[25] for H, which can be taken to be real and in the lower component. Thus using the Higgs mechanism we choose

[25] vev stands for vacuum expectation value

$$H = \exp\left(\frac{i}{2}\pi^A\sigma^A\right)\begin{pmatrix} 0 \\ \dfrac{h+v}{\sqrt{2}} \end{pmatrix} \tag{35.10}$$

where $v = \frac{\mu}{\sqrt{\lambda}}$ and $\sigma^A/2$ are the normalized generators of $SU(2)$. With the vev fixed it's easy to find that the broken generators, i.e. the ones for which $T\langle H\rangle \neq 0$ are given by

$$\tau_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \tau_2 = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \tau_3 - \frac{Y}{2} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{35.11}$$

and the unbroken generator is given by

$$\tau_3 + \frac{Y}{2} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{35.12}$$

which is exactly the electric charge as given by the Gell-Mann Nishijima formula! The symmetry breaking pattern is therefore

$$SU(2)_W \otimes U(1)_Y \to U(1)_Q \tag{35.13}$$

and we expect, thanks to Goldstone theorem and the Higgs mechanism, three out of four vector bosons to be massive while one remains massless (spoiler: the only vector boson without mass will be the photon!).

Putting the vev in the kinetic part of the Higgs, we get

$$\begin{aligned} |D_\mu H|^2 &= \frac{v^2}{8}\begin{pmatrix} 0 & 1 \end{pmatrix}\begin{pmatrix} g'B_\mu + gW^3_\mu & g(W^1_\mu - iW^2_\mu) \\ g(W^1_\mu + iW^2_\mu) & g'B_\mu - gW^3_\mu \end{pmatrix}\begin{pmatrix} g'B_\mu + gW^3_\mu & g(W^1_\mu - iW^2_\mu) \\ g(W^1_\mu + iW^2_\mu) & g'B_\mu - gW^3_\mu \end{pmatrix}\begin{pmatrix} 0 & 1 \end{pmatrix} \\ &= g^2\frac{v^2}{8}\left[\left(W^1_\mu\right)^2 + \left(W^2_\mu\right)^2 + \left(\frac{g'}{g}B_\mu - W^3_\mu\right)^2\right] \end{aligned} \tag{35.14}$$

The W^1, W^2 terms are degenerate in mass

$$M_W^2 = \frac{v^2 g^2}{4}$$

The remaining terms are given by

$$\frac{v^2g^2}{4}(W^3_\mu)^2 + \frac{v^2g'^2}{4}B^2_\mu - \frac{2gg'v^2}{4}B^\mu W^3_\mu = \frac{v^2}{4}\begin{pmatrix} B_\mu & W^3_\mu \end{pmatrix}\begin{pmatrix} g^2 & gg' \\ gg' & g'^2 \end{pmatrix}\begin{pmatrix} B^\mu \\ W^\mu_3 \end{pmatrix} \tag{35.15}$$

It's clear that the initial basis is not the basis given by the mass eigenstates. We can therefore go to the latter by diagonalizing 35.15

$$\begin{aligned} \det\begin{pmatrix} g^2 - m & gg' \\ gg' & g'^2 - m \end{pmatrix} &= (g^2 - m)(g'^2 - m) - (gg')^2 = 0 \\ &= m^2 + (gg')^2 - m(g^2 + g'^2) - (gg') \\ &= m(m - g^2 - g'^2) = 0 \\ m = 0 \qquad & m = g^2 + g'^2 \end{aligned} \tag{35.16}$$

The two solutions give us what we wanted: a masses mode and a massive one. Looking for the eigenvectors will give us linear combinations of the B_μ and W^3_μ fields which will turn out to be the massless photon field and the massive Z^0 gauge boson field.
By means of the following reparametrization

$$\sin\theta_W = \frac{g'}{\sqrt{g^2 + g'^2}} \qquad \cos\theta_W = \frac{g^2}{\sqrt{g^2 + g'^2}} \tag{35.17}$$

where θ_W is called the **Weinberg angle**. One can easily show that this rotation gives us

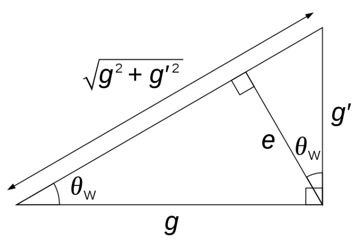


Figure 16. Graphic visualization of the Weinber angle

indeed the linear combination that we need

$$\begin{pmatrix} Z^0_\mu \\ A_\mu \end{pmatrix} = \begin{pmatrix} \cos\theta_W & -\sin\theta_W \\ \sin\theta_W & \cos\theta_W \end{pmatrix} \begin{pmatrix} W^3_\mu \\ B_\mu \end{pmatrix} \implies \begin{cases} Z^0_\mu = \cos\theta_W W^3_\mu - \sin\theta_W B_\mu \\ A_\mu = \sin\theta_W W^3_\mu + \sin\theta_W B_\mu \end{cases} \tag{35.18}$$

Now notice the following

$$W^A_\mu \tau^A = \frac{1}{\sqrt{2}}\left(W^+_\mu \tau^+ + W^-_\mu \tau^-\right) + W^3_\mu \tau^3 \tag{35.19}$$

where

$$\tau^\pm = \tau^1 \pm i\tau^2 \tag{35.20}$$

and so the definite charge gauge fields, in the same way as the pion fields, are given by

$$W^+_\mu = \frac{1}{\sqrt{2}}(W^1_\mu + iW^2_\mu) \qquad W^-_\mu = \frac{1}{\sqrt{2}}(W^1_\mu - iW^2_\mu) \tag{35.21}$$

Therefore what we have in the hand are the following fields

$$\begin{aligned} &W^\pm_\mu && m_W = \frac{vg}{2} \\ &Z^0_\mu && m_Z = \frac{1}{2\cos\theta_W} gv = \frac{v}{2}\sqrt{g^2 + g'^2} = \frac{m_W}{\cos\theta_W} \\ &A_\mu && m_A = 0 \end{aligned} \tag{35.22}$$

Already there's an unambiguous prediction: the W bosons should be lighter than the Z boson, opening a possible decay channel if there's a suitable term in the lagrangian. We'll see that this term is indeed there.
Moreover we find that, at tree level the following result should hold

$$\frac{m_W^2}{\cos^2\theta_W m_Z^2} = 1 \tag{35.23}$$

This is the result of another hidden symmetry of the Standard Model, the **custodial symmetry**.

35.2 Gauge sector

Putting together what we found, we can write down the kinetic term in the lagrangian for the Z and A bosons after symmetry breaking

$$\mathcal{L}_K = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}Z_{\mu\nu}Z^{\mu\nu} + \frac{1}{2}m_Z^2 Z_\mu Z^\mu \tag{35.24}$$

where

$$Z_{\mu\nu} = \partial_\mu Z_\nu - \partial_\nu Z_\mu \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{35.25}$$

Since the gauge bosons transform in the adjoint rep, their interactions are given by commutators and in particular, the W^3_μ part of the photon field gives us the known coupling

$$g\left[A_\mu, W^A_\nu \tau^A\right] = g\sin\theta_W W^3_\mu W^A_\nu \left[\tau^3, \tau^A\right] \implies e = g\sin\theta_W = g'\cos\theta_W \tag{35.26}$$

With this in mind, the $W^\pm$ combinations will have ± 1 charge in units of e, which is what we want.

Without giving the full calculation, one can find that the full gauge lagrangian is

$$\begin{aligned}
\mathcal{L} &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}Z_{\mu\nu}Z^{\mu\nu} - \frac{1}{2}(\partial_\mu W^+_\nu - \partial_\nu W^+_\mu)(\partial^\mu (W^-)^\nu - \partial^\nu (W^-)^\mu) \\
&+ \frac{1}{2}m_Z^2 Z_\mu Z^\mu - m_W^2 W^+_\mu (W^-)^\mu \\
&+ ie\cot\theta_W \left[Z^{\mu\nu}W^+_\mu W^-_\nu - (\partial_\mu W^+_\nu - \partial_\nu W^+_\mu)Z^\mu (W^-)^\nu + (\partial_\mu W^-_\nu - \partial_\nu W^-_\mu)Z^\mu (W^+)^\nu\right] \\
&+ ie\left[F^{\mu\nu}W^+_\mu W^-_\nu - (\partial_\mu W^+_\nu - \partial_\nu W^+_\mu)A^\mu (W^-)^\nu + (\partial_\mu W^-_\nu - \partial_\nu W^-_\mu)A^\mu (W^+)^\nu\right] \\
&+ \frac{1}{2}\frac{e^2}{\sin^2\theta_W}\left(W^+_\mu (W^+)^\mu W^-_\nu (W^-)^\nu - W^+_\mu (W^-)^\mu W^+_\nu (W^-)^\nu\right) \\
&+ e^2\left(A^\mu W^+_\mu A^\nu W^-_\nu - A_\mu A^\mu W^+_\nu (W^-)^\nu\right) + e^2\cot\theta_W\left(Z^\mu W^+_\mu Z^\nu W^-_\nu - Z_\mu Z^\mu W^+_\nu (W^-)^\nu\right) \\
&+ e^2\cot\theta_W\left(W^+_\mu W^-_\nu A^\mu Z^\nu + W^-_\mu W^+_\nu A^\mu Z^\nu - 2W^+_\mu (W^-)^\mu A^\nu Z_\nu\right)
\end{aligned} \tag{35.27}$$

35.3 Higgs Sector

We can now return to the field h, the **Higgs Boson**. This boson remains in the spectrum of the theory even after the choice of the unitary gauge $\pi = 0$ in the Higgs mechanism. While the initial Higgs doublet was charged under the weak and hypercharge groups, the Higgs boson h is not.
The part of the lagrangian which gives us the dynamics of the Higgs field is given by the expansion of the covariant derivative after symmetry breaking

$$\begin{aligned}
\mathcal{L}_H &= \frac{1}{2}(\partial_\mu h)(\partial^\mu h) - \frac{m_h^{2\,2}}{2} - g\frac{m_h^2}{4m_W}h^3 - \frac{g^2 m_h^2}{32 m_W^2}h^4 + 2\frac{h}{v}\left(m_W^2 W^+_\mu (W^-)^\mu + \frac{1}{2}m_Z^2 Z^\mu Z_\mu\right) + \\
&+ \frac{h^2}{v^2}\left(m_W^2 W^+_\mu (W^-)^\mu + \frac{1}{2}m_Z^2 Z^\mu Z_\mu\right)
\end{aligned} \tag{35.28}$$

where $m_h = \sqrt{2}\,\mu$ and μ is the initial symmetry breaking parameter in the unbroken Higgs doublet lagrangian and v is the induced vev.

As we can see from 35.28 that the Higgs field interacts with itself in cubic and quartic interactions and with the other gauge bosons, again, with a cubic and a quartic interaction.

To summarize, we started with four parameters from the initial lagrangian: μ, g, g', λ and ended up with four new parameters e, θ_W, m_h and m_W. Using the experimental values $\alpha_{EM}(m_e) \approx 1/137$, $m_Z = 91.2$ GeV, $m_W = 80.4$ GeV and $m_h = 126$ GeV, we find

$$e = 0.303 \quad \sin^2\theta_W = 0.223 \quad g = \frac{e}{\sin\theta_W} = 0.64 \quad g' = \frac{e}{\cos\theta_W} = 0.34 \quad v = \frac{2m_W}{g} = 251 \text{ GeV} \tag{35.29}$$

35.4 Lepton Sector

Let's study the interactions between the electroweak gauge bosons and the leptons. Before starting, we have to classify the left handed and right handed leptons

$$L_e = \begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L \qquad e_R \tag{35.30}$$

We see that the left handed field shows up as an isospin doublet, whereas the right handed field as singlet. The hypercharge is always set by the Gell-Mann Nishijima formula

$$\begin{aligned}
&\nu_{e\,L} && 0 = \frac{1}{2} + \frac{Y}{2} \implies Y = -1 \\
&e^-_L && -1 = -\frac{1}{2} + \frac{Y}{2} \implies Y = -1 \\
&e^-_R && -1 = \frac{Y}{2} \implies Y = -2
\end{aligned}$$

At the end the classification will be

$$\begin{aligned} L_e &= (2,1)_{-1} \\ e_R &= (1,1)_{-2} \end{aligned} \tag{35.31}$$

Moreover, in the Standard Model, there are three **generations** of $SU(2)$ doublet pairs of quark and leptons. The quarks will be studied later, for now we see that the three generations of leptons are

$$L^i = \begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L, \begin{pmatrix} \nu_\mu \\ \mu^- \end{pmatrix}_L, \begin{pmatrix} \nu_\tau \\ \tau^- \end{pmatrix}_L \qquad i = 1,2,3 \tag{35.32}$$

These all transform as a left handed Weyl spinor[26]. The right handed fermions for all three generations are

$$e^i_R = \{e_R, \mu_R, \tau_R\} \qquad i = 1,2,3 \tag{35.33}$$

[26] They transform in the $\left(\frac{1}{2}, 0\right)$ representation of the Lorentz group

It's important to note that right-handed neutrinos have not been yet observed and we won't include them, but we easily could in case they do exist.

The coupling between the leptons and the gauge boson is given by the covariant derivative in the fermion lagrangian

$$\mathcal{L} = i\bar{L}_i \not{D} L_i + i\bar{e}^i_R \not{D} e^i_R \tag{35.34}$$

where the covariant derivatives are different between the left handed part and the right handed one. All leptons couple to the hypercharge gauge boson as we stated in 35.31. We denote Y_L the left handed hypercharge and Y_R the right handed one. So the expanded lagrangian will be

$$\mathcal{L} = i\bar{L}_i \left(\not{\partial} - ig\not{W}^A \tau^A - i\frac{g'}{2} Y_L \not{B}\right) L_i + i\bar{e}^i_R \left(\not{\partial} - i\frac{g'}{2} Y_L \not{B}\right) e^i_R \tag{35.35}$$

To be clear, the L or R subscript in the lagrangian are just for convenience, since they indicate the implicit chirality of the field. But since all leptons are all left or right-handed Weyl spinors, it would be technically correct to replace

$$\begin{aligned} \bar{L}_R \not{\partial} L &\to L^\dagger \bar{\sigma}^\mu \partial_\mu L \\ \bar{e}_R \not{\partial} e_R &\to e^\dagger_R \sigma^\mu \partial_\mu e_R \end{aligned} \tag{35.36}$$

However, since we'll almost always deal with the fields in the broken phase, where the left and right-handed spinors combine into Dirac spinors, for semplicity we'll always write everything in the Dirac rep where

$$\bar{L}\partial\!\!\!/ L = L^\dagger \gamma^0 \gamma^\mu \partial_\mu \frac{1-\gamma_5}{2} L$$
$$\bar{e}_R \partial\!\!\!/ e_R = e_R \gamma^0 \gamma^\mu \partial_\mu \frac{1+\gamma_5}{2} e_R \tag{35.37}$$

As it's clear, there are still no masses for the leptons. To find them we have to build the Yukawa sector of the lagrangian where the fields interact with the Higgs doublet. This will give mass to the leptons afer symmetry breaking.
From the transformation rule of the lepton fields and the Higgs doublet, it's easy to see that the only scalar quantities we can construct are the following[27]

$$\mathcal{L} = Y\left[\bar{L}_e H e_R + \bar{e}_R H^\dagger L_e\right] \tag{35.38}$$

[27] We now focus only on one generation, the process can be easily generalized to all three.

After symmetry breaking, this part will give us the mass for the electrons with the following term

$$-m_e\left(\bar{e}_L e_R + \bar{e}_R e_L\right) \qquad m_e = \frac{Y}{\sqrt{2}} v \tag{35.39}$$

After electroweak symmetry breaking, together with the diagonalization of the masses for the gauge bosons, the lagrangian 35.35 becomes

$$\mathcal{L} = \bar{L}_e\left[g\tau^3\left(Z_\mu \cos\theta_W + A_\mu \sin\theta_W\right) + \frac{g'}{2} Y_L\left(-Z_\mu \sin\theta_W + A_\mu \cos\theta_W\right)\right]\gamma^\mu L_e +$$
$$+ Y_R \frac{g'}{2} \bar{e}_R\left(-Z_\mu \sin\theta_W + A_\mu \cos\theta_W\right)\gamma^\mu e_R \tag{35.40}$$

The terms proportional to the photon field are

$$A_\mu\left[\bar{L}_e\left(g\tau^3 \sin\theta_W + \frac{g'}{2} Y_L \cos\theta_W\right) L_e + \left(\frac{g'}{2}\cos\theta_W\right) Y_R\left(\bar{e}_R \gamma^\mu e_R\right)\right] \tag{35.41}$$

and using the fact that $g\sin\theta_W = g'\cos\theta_W = gg'/\sqrt{g^2+g'^2}$ we get to the expected result for the QED interaction between photons and charged leptons

$$A_\mu g \sin\theta_W\left[\bar{L}_e \gamma^\mu\left(\tau^3 + \frac{Y_L}{2}\right) L_e + \frac{Y_R}{2}\left(\bar{e}_R \gamma^\mu e_R\right)\right]$$
$$= A_\mu g \sin\theta_W\left[-\bar{e}_L \gamma^\mu e_L - \bar{e}_R \gamma^\mu e_R\right]$$
$$= g\sin\theta_W A_\mu J^\mu_{EM} \tag{35.42}$$

where we used the unbroken generator in 35.12 and

$$J^\mu_{EM} = Q_e(\bar{e}\gamma^\mu e) \tag{35.43}$$

with $Q_e = e = g\sin\theta_W$. As we discussed, the electromagnetic interaction does not differentiate between left and right handed chirality.

The terms proprtional to the Z^0 boson are

$$Z_\mu\left[g\cos\theta_W \bar{L}_e \gamma^\mu \tau^3 L_e - \frac{Y_L}{2} g' \sin\theta_W \bar{L}_e \gamma^\mu L_e - \frac{Y_R}{2} g' \sin\theta_W \bar{e}_R \gamma^\mu e_R\right]$$
$$= Z_\mu\left[(g\cos\theta_W + g'\sin\theta_W)\bar{L}_e \gamma^\mu \tau^3 L_e - g'\sin\theta_W q J^\mu_{EM}\right]$$
$$= Z_\mu \frac{g}{\cos\theta_W}\left(J^\mu_3 - q\sin^2\theta_W J^\mu_{EM}\right) \tag{35.44}$$

Therefore the Z^0 boson not only couples to the EM current but even with an axial current

$$J_3^\mu = \bar{L}_e \gamma^\mu \tau^3 L_e \tag{35.45}$$

There remain only the interaction terms between the leptons and the $W^\pm$ bosons. Recalling 35.19 we get, from the lagrangian 35.35

$$gW_\mu^A \bar{L}_e \gamma^\mu \tau^A L_e = g\left[\frac{1}{\sqrt{2}} W_\mu^+ \bar{L}_e \gamma^\mu \tau^- L_e + \frac{1}{\sqrt{2}} W_\mu^- \bar{L}_e \gamma^\mu \tau^+ L_e + W_\mu^3 \bar{L}_e \gamma^\mu \tau^3 L_e\right] \tag{35.46}$$

and we can directly see that the charged currents are

$$\begin{aligned} J_\mu^+ = \bar{L}_e \gamma_\mu \tau^+ L_e &= \begin{pmatrix} \bar{\nu}_e & \bar{e}^- \end{pmatrix}_L \gamma_\mu \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L \\ &= \begin{pmatrix} \bar{\nu}_e & \bar{e}^- \end{pmatrix}_L \gamma_\mu \begin{pmatrix} e^- \\ 0 \end{pmatrix}_L = \bar{\nu}_{eL} \gamma_\mu e_L^- \\ &= \bar{\nu}_e \gamma_\mu \frac{1-\gamma_5}{2} e^- \end{aligned} \tag{35.47}$$

and

$$J_\mu^- = (J_\mu^+)^\dagger = \bar{e} \gamma_\mu \frac{1-\gamma_5}{2} \nu_e \tag{35.48}$$

The axial part for W^3 goes into the photon and Z_0 boson.

The full interaction lagrangian between the leptons and the gauge boson after electroweak symmetry breaking becomes

$$\mathcal{L} = qeA_\mu J_{EM}^\mu + \frac{g}{\cos\theta_W} Z_\mu \left(J_3^\mu - q\sin\theta_W J_{EM}^\mu\right) + \frac{g}{\sqrt{2}}\left(W_\mu^+ J^{\mu -} + W_\mu^- J^{\mu +}\right) \tag{35.49}$$

PART

XIV

Quarks and the Standard Model

Now that we talked about the electroweak sector

$$SU(2)_W \times U(1)_Y \to U(1)_Q \tag{35.50}$$

of the Standard Model, we're ready to add one of the missing part: the quarks.
We will not talk about QCD which is the remaining $SU(3)$ the full symmetry of the Standard Model, but only how quarks enter in the electroweak theory and how we can give masses to them with the help of the Higgs mechanism.
It will turn out that whenever we try to diagonalize the mass spectrum, we'll introduce some kind of mixing between the quarks which will be mediated by the electroweak gauge bosons.

36 The Quarks

Whenever talking about particles we should give the representation in which they're in based on the full $SU(3) \times SU(2) \times U(1)$ symmetry.

To be more specific, quarks come in three flavours, just like leptons, and appear in the theory in their chiral basis

$$\begin{aligned} Q_L^i &= \begin{pmatrix} u \\ d \end{pmatrix}_L, \begin{pmatrix} c \\ s \end{pmatrix}_L, \begin{pmatrix} t \\ b \end{pmatrix}_L \\ u_R^i &= \{u_R, c_R, t_R\} \qquad d_R = \{d_R, s_R, b_R\} \end{aligned} \tag{36.1}$$

Their name are: up, down, charm, strange, top and bottom quarks. Whenever using the notation u^i we'll mostly mean the up row of quarks in the SM which are up and have electric charge, in units of e, $2/3$, charm and top quarks, the others are the d^is which have electric charge $-1/3$.
Their irrep in the full SM gauge group is

$$Q_L \sim (2,3)_{\frac{1}{3}} \quad u_R \sim (1,3)_{\frac{4}{3}} \quad d_R \sim (1,3)_{-\frac{2}{3}} \tag{36.2}$$

Quarks carry a lot of indices: one index for the isospin charge, one index for the color charge, one family index and a Lorentz index. We'll omit them, as per usual, since the notation would be too cluttered with them. But remember still that to construct invariant quantities all indices must be saturated in such a way to have a singlet for any of the possible symmetries.

36.1 Interactions and lagrangian

In the same way as we did with the leptons, we start from the following lagrangian

$$\begin{aligned}\mathcal{L} = i\bar{Q}_L \left(\not\partial - ig\not W^A \tau^A - i\frac{g'}{2} Y_{Q_L} \not B\right) Q_L \\ + i\bar{u}_R \left(\not\partial - i\frac{g'}{2} Y_{u_R} \not B\right) u_R + i\bar{d}_R \left(\not\partial - i\frac{g'}{2} Y_{d_R} \not B\right) d_R \end{aligned} \tag{36.3}$$

By direct comparison with the leptons, we can easily see that the calculations will be the same and so we give directly the results for the various currents that one expects to find, coupled to the respective gauge bosons.
The full fermion currents will be the following

$$J^\mu_{EM} = -\bar{e}\gamma^\mu e + \frac{2}{3}\bar{u}^a \gamma^\mu u_a - \frac{1}{3}\bar{d}^a \gamma^\mu d_a \tag{36.4}$$

is the EM current coupled to the photon[28]

$$J^3_\mu = \bar{\nu}_e \gamma_\mu \frac{1-\gamma_5}{2}\nu_e - \bar{e}\gamma_\mu \frac{1-\gamma_5}{2} e + \bar{u}^a \gamma_\mu \frac{1-\gamma_5}{2} u_a + \bar{d}^a \gamma_\mu \frac{1-\gamma_5}{2} d_a \tag{36.5}$$

is the axial current coupled to the Z^0 boson, and the charged ones

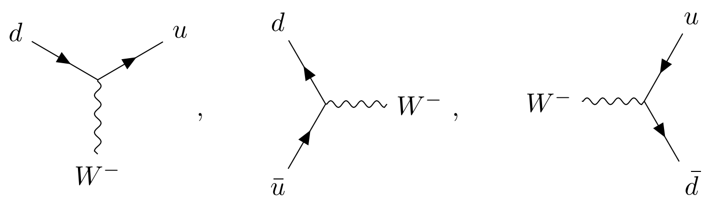
$$\begin{aligned} J^+_\mu &= \bar{L}_e \gamma_\mu \tau^+ L_e + \bar{Q}^a \gamma_\mu \tau^+ Q_a \\ J^-_\mu &= \bar{L}_e \gamma_\mu \tau^- L_e + \bar{Q}^a \gamma_\mu \tau^- Q_a \end{aligned} \tag{36.6}$$

which are coupled to the charged $W^\pm$ bosons.

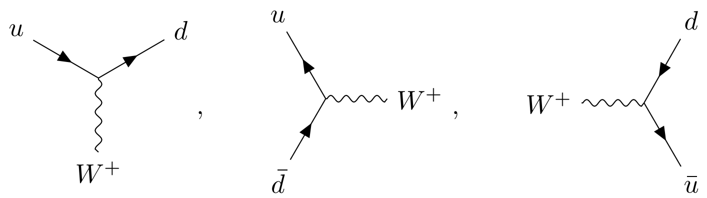
[28] Note the color index $a = 1, 2, 3$ on the quarks which is saturated.

36.2 Some interaction vertices

In the charged currents we see that an up quark can go to a down quark by emitting a W^+ and viceversa. The interaction vertices can be easily seen from the lagrangian given above and are



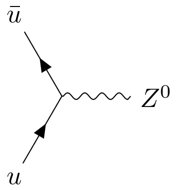
$$\equiv -i\frac{g}{\sqrt{2}}\gamma_\mu \frac{1-\gamma_5}{2}$$



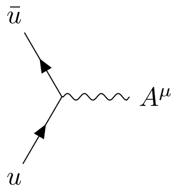
$$\equiv -i\frac{g}{\sqrt{2}}\gamma_\mu \frac{1-\gamma_5}{2}$$

This summerize the charged currents interactions. Remember that the u stands for every quark in the upper region of the SM and d for every quark in the down region. Don't get too attached to this interaction vertices since in some moments we'll change the basis of our theory for the quarks in order to diagonalize the mass matrix. The Feynman rules in this case are in the, so called, **current basis**.

Then there are the neutral currents



$$\equiv -i\frac{g}{\cos\theta_W}\gamma_\mu\left(\frac{1}{2}\tau^3 - Q_f\sin^2\theta_W - \frac{1}{2}\tau^3\gamma_5\right)$$



$$\equiv -ieQ_f\gamma_\mu$$

37 Quark masses and mixing angles

We're now ready to study how the quarks gain masses. To do so we'll need to reintroduce all the families since, as we'll see, diagonalizing the mass metrix will inevitably mix the current eigenstates quarks from different families.

37.1 Yukawa sector

From the irreps 36.2 and the Higgs we need to construct all the possible renormalizable scalar quantities. We'll need however another form of the Higgs field since H^* won't cut it. The field we'll use is the charge conjugate of H defined by

$$\tilde{H} = i\sigma^2 H^* = \begin{pmatrix} H^{0*} \\ -H^- \end{pmatrix} \tag{37.1}$$

Now let's see what kind of scalars we can build up. If we start from $\bar{Q}_L H$ we can easily see that this we'll be

$$\bar{Q}_L H \sim (\bar{2}, \bar{3}3)_{-\frac{1}{3}}(2,1)_1 = (\bar{2}\times 2, \bar{3}\times 3)_{1-\frac{1}{3}} \tag{37.2}$$

We know that $\bar{2}\times 2$ and $\bar{3}\times 3$ both contain a singlet state. What's missing is the hypercharge singlet since $1 - \frac{1}{3} = 2/3$. If we search in 36.2 for a suitable quantity, we see that the d_R quark serves our purpose and so a suitable renormalizable operator for our Yukawa sector will be

$$\bar{Q}_L H d_R + \text{h.c.} = \bar{Q}_R H d_R + \bar{d}_R H^\dagger Q_L \tag{37.3}$$

where we added the hermitian conjugate, as always, to include the reality of the lagrangian. And this settles down the down part of the lagrangian. For the up part we'll use the charge conjugate Higgs since, if you try, we cannot construct scalar quantities between up quarks with the normal Higgs doublet.
It's easy to see that the only renormalizable scalar quantity we can construct using u_R is

$$\bar{Q}_L \tilde{H} u_R + \bar{u}_R \tilde{H}^\dagger Q_L \tag{37.4}$$

Therefore, if we now put in all the familes and the Yukawa coupling we get the Yukawa sector for quarks

$$\mathcal{L}_Y = Y_U^{ij}\left(\bar{Q}_L^i H d_R^j + \bar{d}_R^j H^\dagger Q_L^i\right) + Y_D^{ij}\left(\bar{Q}_L^i \tilde{H} u_R^j + \bar{u}_R^j \tilde{H}^\dagger Q_L^i\right) \tag{37.5}$$

37.2 Symmetry breaking

Given the Yukawa sector we can use symmetry breaking and, by going in the unitary gauge[29] we get, for example for the down quarks

$$\begin{pmatrix}\bar{u}_L & \bar{d}_L\end{pmatrix}\begin{pmatrix}0 \\ \frac{v+h}{\sqrt{2}}\end{pmatrix} d_R = \bar{d}_L d_R \left(\frac{v+h}{\sqrt{2}}\right) \tag{37.6}$$

[29] Remember that whenever we speak about unitary gauge we're implying that we set the Goldstone boson "to zero", which is a way of saying that the gauge field eats the Goldston boson gaining a new degree of freedom.

And so the mass for the down quarks is given by a $SU(2)$ symmetry breaking Dirac term, as we expect

$$Y_D^{ij}\frac{v}{\sqrt{2}}\bar{d}_L^i d_R^j \implies M_D^{ij} = \frac{v}{\sqrt{2}} Y_D^{ij} \tag{37.7}$$

For the up quarks is the same but the mass matrix is given in terms of the Yukawa of the up quarks

$$M_U^{ij} = \frac{v}{\sqrt{2}} Y_U^{ij} \tag{37.8}$$

With this we see that the mass terms in the lagrangian for the quarks are

$$\mathcal{L} = \bar{d}_L^i M_{ij}^D d_R^j + \bar{d}_R^j M_{ij}^{D\dagger} d_L^i + \bar{u}_L^i M_{ij}^U u_R^j + \bar{u}_R^j M_{ij}^{U\dagger} u_L^i \tag{37.9}$$

but nobody assures us that the mass matrices will be diagonal, but we would like them to be diagonal since the mass of a given quark is an experimentally. Since we don't have any constraint on the specific form of the mass matrix we just found, we don't know how, and if it exist, to diagonalize it.

37.3 On the diagonalization of matrices

We now prove that there exist a method through which we can diagonalize any matrix that we want. This process is called **singular value decomposition** and it provides to matrices $\tilde{L}, \tilde{R}$ unitary such that

$$L^\dagger M R = \hat{M}$$

where we'll use the hatted matrix to say that it's the diagonalized form of M.
From the generic matrix M we can construct two hermitian matrices

$$MM^\dagger \qquad M^\dagger M \tag{37.10}$$

which in general do not commute. We can easily prove that this matrices have the same eigenvalues

$$\begin{aligned} P_{MM^\dagger} &= \det\big(MM^\dagger - \lambda\big) = \det\{M\}\det\big(M^\dagger - \lambda M^{-1}\big) \\ &= \det\big(M^\dagger - \lambda M^{-1}\big)\det M = \det\big(M^\dagger M - \lambda\big) = P_{M^\dagger M} \end{aligned} \tag{37.11}$$

and since both matrices have the same characteristic polynomial, they'll have the same eigenvalues. Being both hermitian, we know that they can be diagonalized thanks to the spectral theorem and so there exist two matrices L and R which diagonalize the matrices to the same diagonal form since they have both the same eigenvalues

$$L(MM^\dagger)L^\dagger = \hat{D} = R(M^\dagger M)R^\dagger \tag{37.12}$$

Starting from this we define the following

$$M' = L^\dagger M R \qquad (M')^\dagger = R^\dagger M^\dagger L \tag{37.13}$$

and it's easy to see that

$$M'(M')^\dagger = (M')^\dagger M' = \hat{D} \tag{37.14}$$

We know that we can always decompose a matrix into two hermitian matrices as

$$M' = \left(\frac{M' + M'^\dagger}{2}\right) + i\left(\frac{M' - M'^\dagger}{2}\right) = H_1 + iH_2 \tag{37.15}$$

Thw two matrices we just defined H_1, H_2 are obviously diagonalizable since they hare hermitian but we would like them by be diagonalizable to the same unitary matrix. From quantum mechanics we know that this is possible if the two matrices commute! And it's easy to see that

$$[H_1, H_2] = \frac{1}{4i}\left[M' + M'^\dagger, M' - M'^\dagger\right] = \frac{1}{2i}\left(M'M'^\dagger - M'^\dagger M'\right) = 0 \tag{37.16}$$

Therefore there exists a unitary matrix W such that $W^\dagger M' W = \hat{M}'$ is a complex diagonal matrix and therefore, being complex diagonal we can put it in the form

$$\hat{M}' = \hat{M}\hat{\varphi} \tag{37.17}$$

where $\hat{\varphi}$ is a matrix og phases. Moreover

$$W^\dagger M' W = \hat{M}\hat{\varphi} = W^\dagger L^\dagger M R W \tag{37.18}$$

and therefore if we define

$$\tilde{L} = LW \qquad \tilde{R} = RW\hat{\varphi}^{-1} \tag{37.19}$$

we found the matrices that diagonalize M.

37.4 The CKM matrix

Now that we know a way diagonalizing any matrix, we can use it to diagonalize the mass matrix for the quarks. Take the up quarks for example

$$\bar{u}^i_L \hat{m}^u_{ij} u^j_R = \bar{u}^i_L (U^\dagger_{u_L})_{ik} M^U_{kl} (U_{u_R})_{kj} u^j_R \tag{37.20}$$

where

$$\hat{m}^u = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_c & 0 \\ 0 & 0 & m_t \end{pmatrix} \tag{37.21}$$

and the new mass eigenstates are written in term of the old ones as

$$u^i_L = (U_{u_L})_{ij} u^j_L \qquad u^i_R = (U_{u_R})_{ij} u^j_R \tag{37.22}$$

Same thing goes for the down quarks where the diagonal form of the mass matrix will be

$$\hat{m}^d = \begin{pmatrix} m_d & 0 & 0 \\ 0 & m_s & 0 \\ 0 & 0 & m_b \end{pmatrix} \tag{37.23}$$

and the mass eigenstates

$$d_L^i = (U_{d_L})_{ij} d_L^j \qquad d_R^i = (U_{d_R})_{ij} d_R^j \tag{37.24}$$

The kinetic terms are also modified by this change of basis. The gauge boson interactions do not mix the families in the original current basis where the lagrangian is

$$\begin{aligned}\mathcal{L} =& \begin{pmatrix}\bar{u}_L & \bar{d}_L\end{pmatrix}^i \left[i\partial\!\!\!/ + \gamma_\mu \begin{pmatrix} \frac{g'}{6} B_\mu + \frac{g}{2} W_\mu^3 & \frac{g}{\sqrt{2}} W_\mu^+ \\ -\frac{g}{\sqrt{2}} W_\mu^- & \frac{g'}{6} B_\mu - \frac{g}{2} W'_\mu \end{pmatrix} \right] \begin{pmatrix} u_L \\ d_L \end{pmatrix}^i \\ &+ \bar{u}_R^i \left(i\partial\!\!\!/ + g' \frac{2}{3} B\!\!\!/ \right) u_R^i + \bar{d}_R^i \left(i\partial\!\!\!/ - g' \frac{1}{3} B\!\!\!/ \right) d_R^i \\ &- \frac{v}{\sqrt{2}} \left[\bar{d}_L^i \left(U_{d_L} M_d U_{d_R}^\dagger \right)_{ij} d_R^j + \bar{u}_L^i \left(U_{u_L} M_u U_{u_R}^\dagger \right)_{ij} u_R^j + h.c. \right] \end{aligned} \tag{37.25}$$

where i, j are flavour indices. When we do the change of basis the unitarity of the transformation makes the matrices drop out since the hypercharge interactions are generation diagonal

$$i \sum_i \bar{u}_R^i \partial\!\!\!/ u_R^i \underset{\text{flavour}}{\equiv} \bar{u}_R \mathbb{1} u_R \to \bar{u}_R \underbrace{U_{u_R}^\dagger \mathbb{1} U_{u_R}}_{\mathbb{1}} u_R = \bar{u}_R \mathbb{1} u_R \tag{37.26}$$

Moreover the same happens on the B_μ and W_μ^3 terms since these do not mix up and down-type quarks. This means that the interaction. This in turn makes the interaction with the photon unchanged.

The interesting bit comes out from the isospin doublet, the left part, where the two components change with different unitary matrices

$$Q_L^i = \begin{pmatrix} u_L^i \\ d_L^i \end{pmatrix} \to \begin{pmatrix} U_{u_L}^{ij} u_L^j \\ U_{U_R}^{ij} d_L^j \end{pmatrix} \tag{37.27}$$

whenever the interaction mixes the to quark types. This happens with the $W^\pm$ couplings

$$\frac{g}{\sqrt{2}} W_\mu^+ \bar{u}_L \gamma^\mu \mathbb{1} d_L \underset{\text{flavour}}{\equiv} \bar{u}_L \mathbb{1} d_L = \bar{u}_L \underbrace{U_{u_L}^\dagger \mathbb{1} U_{d_L}}_{V_{CKM}} d_L \tag{37.28}$$

where a new matric in flavour space appears since we cannot use unitarity to reduce the new term to the identity. This matrix is known as the **Cabibbo-Kobayashi-Maskawa (CKM) matrix**. The CKM matrix is a complex unitary matrix, and thus has nine real degrees of freedom, or three compelx degrees of freedom. If V_{CKM} were real, it would be a $O(3)$ matrix, i.e. with three degrees of freedom. This means that out of the nine parameters of the complex CKM, three are angles and six are phases. However since the quark fields as mass eigenstates have a residual $U^6(1)$ symmetry

$$d_L^i = e^{i\alpha_i} d_L^i \quad d_R^i = e^{i\alpha_i} d_R^i \quad u_L^i = e^{i\beta_i} u_L^i \quad u_R^i = e^{i\beta_i} u_R^i \tag{37.29}$$

we can use this freedom to set some phases to zero. Under these transformations, V_{CKM} generally transforms. However, if the two rotations are the same $\alpha_i = \beta_i$, the matrix remains unchanges. Therefore out of the 6 possible phases we could have set to zero, there remain only 5 possible combinations that effectivly change the CKM matrix. Therefore there remain only one free phase in the CKM. The total remaining degrees of freedom are: three angles $\theta_{12}, \theta_{23}, \theta_{13}$, corresponding to rotations in the ij-flavour planes, and a phase δ. The angle θ_{12} is called **Cabibbo angle** θ_C.

One possible representation of the CKM matrix is the following

$$\begin{aligned} V &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_{23} & \sin\theta_{23} \\ 0 & -\sin\theta_{23} & \cos\theta_{23} \end{pmatrix} \\ &\times \begin{pmatrix} \cos\theta_{13} & 0 & \sin\theta_{13}e^{i\delta} \\ 0 & 1 & 0 \\ -\sin\theta_{13}e^{i\delta} & 0 & \cos\theta_{13} \end{pmatrix} \begin{pmatrix} \cos\theta_{12} & \sin\theta_{12} & 0 \\ -\sin\theta_{12} & \cos\theta_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix} \\ &= \begin{pmatrix} V_{ud} & V_{uc} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \end{aligned} \tag{37.30}$$

The presence of the phase reflects the CP violation of the weak charged currents[30].

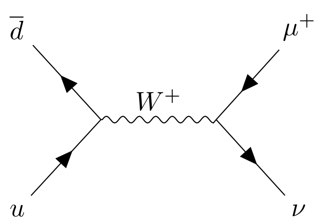
[30] In reality is more complicated than this, but we won't go into details.

38 What's so special about the CKM matrix?

38.1 Interaction vertices

Now that we have a complete theory of weak interactions we can start constructing Feynman diagrams and evaluating some measurable quantities. It turns out that whenever we have a flavour changing current we'll need now to insert in the interaction vertex one of the possible elements of the CKM matrix.

Let's take for example the pion decay. As we argued in previous chapters, the pion decay hamiltonian is given by 25.1 and, with this, we found the decay width for the charged pion 25.14. But now that we now how it really works, we can build up the following tree level diagram for the pion decay



(38.1)

and find out the amplitude of it

$$\frac{G_F}{\sqrt{2}} V_{ud}^* \bar{d}\gamma^\mu(1-\gamma_5)u\bar{\mu}\gamma_\mu(1-\gamma_5)\nu \tag{38.2}$$

where an additional V_{ud} term appears with respect to the initial amplitude. This factor can greatly suppress some processes for which the CKM matrix element is very small.

One useful representation of the CKM matrix that helps us have a better understanding of the order of magnitude of the various coupling parameters is the **Wolfstein** representation. It's based on the approximate parametrization in terms of $\lambda = \sin\theta_C \approx 0.22$

$$V \approx \begin{pmatrix} 1 & \lambda & \lambda^3 \\ \lambda & 1 & \lambda^2 \\ \lambda^3 & \lambda^2 & 1 \end{pmatrix} \tag{38.3}$$

38.2 How to measure the CKM elements

The CKM matrix elements can be measured from leptonic and semileptonic decays. With the top quarks the processes are a little bit more difficoult since it's mass prevents it from forming bound states with other quarks. In that case we use hadron mixing, we won't go into much detail about it.
Here we give some example: for leptonic decays we can have

$$\begin{aligned} &\pi^+ \to \mu^+\nu_\mu \qquad V_{ud} \\ &K^+ \to \mu^+\nu_\mu \qquad V_{us} \\ &D^+ \to \mu^+\nu_\mu \qquad V_{cd} \\ &B^+ \to \tau^+\nu_\tau \qquad V_{ub} \end{aligned} \tag{38.4}$$

for semileptonic decays

$$\begin{aligned} &n \to pe^-\bar{\nu}_e \qquad V_{ud} \\ &K^+ \to \pi^0 e^+\nu_e \qquad V_{us} \\ &K^0 \to \pi^- e^+\nu_e \qquad V_{us} \\ &D^+ \to K^0\mu^+\nu_\mu \qquad V_{cs} \end{aligned} \tag{38.5}$$

and even non-leptonic decays like

$$K^+ \to \pi^+\pi^- \qquad V_{us}^* V_{ud} \tag{38.6}$$

Discrete Symmetries

PART XV

Until now we have studied the continuous transformations that can be constructed by starting from infinitesimal transformations close to unity. If a theory is invariant under such transformation it will possess a Noether current and thus there will be a conservation law. In addition, however, there is the class of discrete symmetries, which have to be described differently. Discrete symmetries can be employed to relate the behavior of different physical systems, for example, those that differ by an interchange of particles and antiparticles. New conserved quantities (e.g. parity, charge parity) and selection rules can be generated by discrete symmetries. We will study three types of discrete transformations which are of fundamental importance: parity $\mathcal{P}$, charge conjugation $\mathcal{C}$, and time reversal $\mathcal{T}$. After determining all their properties[31], we'll see that not all the interactions are invariant under these kind of transformations. A prominent example for this is the famous maximal violation of parity in weak interactions. Also the time reversal symmetry is (very slightly) violated in nature as witnessed by the decay of the neutral kaon mesons.

[31] We'll exploit some properties of the gamma matrices, like

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$$

$$\{\gamma^\mu, \gamma_5\} = 0$$
$$\left(\gamma^0\right)^2 = 1$$
$$\left(\gamma^i\right)^2 = -1$$
$$\gamma^0\gamma^\mu\gamma^0 = (\gamma^\mu)^\dagger$$

39 Parity $\mathcal{P}$

The parity is an operation that inverts the spatial coordinates

$$\mathcal{P}: \begin{pmatrix} x \\ y \\ z \end{pmatrix} \longrightarrow \begin{pmatrix} -x \\ -y \\ -z \end{pmatrix} \tag{39.1}$$

and has the property to come back to the original condition if we apply the operator another time, so $\mathcal{P}^2 = 1$. We can define a parity transformation on a field in the following way

$$\psi'_\alpha = \left(\mathcal{P}\,\psi(\mathbf{x}, t)\mathcal{P}^\dagger\right)_\alpha = \mathcal{P}_{\alpha\beta}\psi_\beta(-\mathbf{x}, t) \tag{39.2}$$

From this we get

$$\begin{cases} \psi' = \mathcal{P}\psi \\ \psi'^\dagger = \psi^\dagger\mathcal{P}^\dagger \end{cases} \tag{39.3}$$

39.1 Determination of the operator

We are interested in determining the parity operator as a function of gamma matrices. In order to do this we exploit the bilinear covariant $\bar{\psi}'\gamma^\mu\psi'$, requiring the following conditions

$$\bar{\psi}'\gamma^0\psi' = \bar{\psi}\gamma^0\psi \tag{39.4}$$
$$\bar{\psi}'\gamma^i\psi' = -\bar{\psi}\gamma^i\psi \qquad i = 1, 2, 3 \tag{39.5}$$

which is equivalent to write

$$\bar{\psi}'\gamma^\mu\psi' = g^{\mu\mu}\bar{\psi}\gamma^\mu\psi \tag{39.6}$$

Developing these conditions, we get

$$\begin{cases} \psi^\dagger\mathcal{P}^\dagger\gamma^0\gamma^0\mathcal{P}\psi = \psi^\dagger\gamma^0\gamma^0\psi \\ \psi^\dagger\mathcal{P}^\dagger\gamma^0\gamma^i\mathcal{P}\psi = -\psi^\dagger\gamma^0\gamma^i\psi \end{cases} \longrightarrow \qquad \begin{cases} \mathcal{P}^\dagger\mathcal{P} = 1 \\ \mathcal{P}^\dagger\gamma^0\gamma^i\mathcal{P} = -\gamma^0\gamma^i \end{cases} \tag{39.7}$$

so we can say that parity is a hermitian operator. The system is solved if the parity operator is

$$\mathcal{P} = C\gamma^0 \tag{39.8}$$

where C is a constant such that $|C|^2 = 1$, so the possible values are ± 1. Henceforth we are going to assume $C = 1$ without losing generality.

39.2 Bilinear covariant transformations

Now let's see how the bilinear covariants transform under a parity transformation.

- A **scalar** quantity will transform without changing sign

$$\bar{\psi}'\psi' = \psi^\dagger \mathcal{P}^\dagger \gamma^0 \mathcal{P}\psi = \psi^\dagger \gamma^0 \psi = \bar{\psi}\psi \tag{39.9}$$

- A **pseudoscalar** quantity will transform changing sign

$$\bar{\psi}'\gamma_5\psi' = \psi^\dagger \mathcal{P}^\dagger \gamma^0 \gamma_5 \mathcal{P}\psi = -\psi^\dagger \gamma^0 \gamma_5 \psi = -\bar{\psi}\gamma_5\psi \tag{39.10}$$

- A **vector** quantity will transform like the Minkowski tensor without changing sign, as we have seen in equation 39.6.

- A **pseudovector** quantity will transform like the Minkowski tensor changing sign

$$\bar{\psi}'\gamma^\mu\gamma_5\psi' = \psi^\dagger \mathcal{P}^\dagger \gamma^0 \gamma^\mu \gamma_5 \mathcal{P}\psi = \psi^\dagger \gamma^\mu \gamma_5 \gamma^0 \psi$$

$\mu = 0 \implies \psi^\dagger \gamma^0 \gamma_5 \gamma^0 \psi = -\bar{\psi}\gamma^0\gamma_5\psi$

$\mu = 1, 2, 3 \implies \psi^\dagger \gamma^i \gamma_5 \gamma^0 \psi = \bar{\psi}\gamma^i\gamma_5\psi$

So in the end we can write

$$\bar{\psi}'\gamma^\mu\gamma_5\psi' = -g^{\mu\mu}\bar{\psi}\gamma^\mu\gamma_5\psi \tag{39.11}$$

- A **tensor** quantity, similarly to the vector case, will transform like the Minkowski tensor without changing sign

$$\begin{aligned}\bar{\psi}'\gamma^\mu\gamma^\nu\psi' &= \psi^\dagger \mathcal{P}^\dagger \gamma^0 \gamma^\mu \gamma^\nu \mathcal{P}\psi = \bar{\psi}\gamma^0\gamma^\mu\gamma^0\gamma^0\gamma^\nu\gamma^0\psi \\ &= \bar{\psi}(\gamma^\mu)^\dagger(\gamma^\nu)^\dagger\psi \end{aligned} \tag{39.12}$$

$\mu = \nu = 0 \implies \bar{\psi}\left(\gamma^0\right)^\dagger\left(\gamma^0\right)^\dagger\psi = \bar{\psi}\gamma^0\gamma^0\psi$

$\mu, \nu \neq 0 \implies \bar{\psi}\left(\gamma^i\right)^\dagger\left(\gamma^j\right)^\dagger\psi = \bar{\psi}\left(-\gamma^i\right)\left(-\gamma^j\right)\psi = \bar{\psi}\gamma^i\gamma^j\psi$

We can write it in a compact way using the Minkowski tensor, so in the end

$$\bar{\psi}'\gamma^\mu\gamma^\nu\psi' = g^{\mu\mu}g^{\nu\nu}\bar{\psi}\gamma^\mu\gamma^\nu\psi \tag{39.13}$$

From these properties we can say that QCD and QED lagrangians, which correspond to strong and electromagnetic interactions, are invariant under a parity transformation, whereas the Fermi lagrangian, which corresponds to weak interaction, is not.

40 Charge conjugation $\mathcal{C}$

A transformation of charge conjugation allows us to exchange the charge of two particles. Formally the charge conjugation is a transformation that exchanges a particle with its antiparticle, so if we take for instance the solution for the Dirac equation

$$\psi(x) = \sum_{r=1}^{2} \sum_{p} \sqrt{\frac{m}{VE(p)}} \left[a_r(p) e^{-ipx} u_r(p) + b_r^\dagger(p) e^{ipx} v_r(p) \right] \tag{40.1}$$

where the particle is destroyed and its antiparticle is created, we can define its charge conjugation as

$$\psi_c(x) = \sum_{r=1}^{2} \sum_{p} \sqrt{\frac{m}{VE(p)}} \left[a_r^\dagger(p) e^{ipx} v_r(p) + b_r(p) e^{-ipx} u_r(p) \right] \tag{40.2}$$

in which the antiparticle is destroyed whereas the particle is created.
As we did for the parity, we can give a definition of a charge transformation on a field as

$$(\psi_c)_\alpha = \left(\mathcal{C}\, \psi(\mathbf{x}, t) \mathcal{C}^\dagger \right)_\alpha = \mathcal{C}_{\alpha\beta} \psi_\beta^\dagger(\mathbf{x}, t) \tag{40.3}$$

From this we get

$$\begin{cases} \psi_c = \mathcal{C}\psi^\dagger \\ \psi_c^\dagger = \psi \mathcal{C}^\dagger \end{cases} \tag{40.4}$$

Similarly to the parity transformation, it has the property to come back to the original condition if we apply the operator another time, so $\mathcal{C}^2 = 1$.

40.1 Determination of the operator

In order to determine the charge operator, we do a similar work as we did for the parity, requiring that the following conditions are respected

$$\bar{\psi}_c \psi_c = \bar{\psi}\psi \tag{40.5}$$

$$\bar{\psi}_c \not{\partial} \psi_c = \bar{\psi} \not{\partial} \psi \tag{40.6}$$

From the first one we get[32]

$$\begin{aligned} \left(\psi_c^\dagger\right)_\alpha \left(\gamma^0\right)_{\alpha\beta} (\psi_c)_\beta &= \psi_\gamma \left(\mathcal{C}^\dagger\right)_{\gamma\alpha} \left(\gamma^0\right)_{\alpha\beta} (\mathcal{C})_{\beta\sigma} \psi_\sigma^\dagger = -\psi_\sigma^\dagger \left(\mathcal{C}^\dagger \gamma^0 \mathcal{C}\right)_{\gamma\sigma} \psi_\gamma \\ &= \psi_\sigma^\dagger \left(\gamma^0\right)_{\sigma\gamma} \psi_\gamma = \psi_\sigma^\dagger \left(\gamma^0\right)_{\gamma\sigma}^T \psi_\gamma \end{aligned} \tag{40.7}$$

[32] From this point we exploit the fact that Dirac fields follow an anticommutation rule

from which we can extract

$$\mathcal{C}^\dagger \gamma^0 \mathcal{C} = -\gamma^0 \tag{40.8}$$

From the second condition we get[33]

$$\begin{aligned} \left(\psi_c^\dagger\right)_\alpha \left(\gamma^0\gamma^\mu\right)_{\alpha\beta} (\partial_\mu \psi_c)_\beta &= \psi_\sigma \left(\mathcal{C}^\dagger\right)_{\sigma\alpha} \left(\gamma^0\gamma^\mu\right)_{\alpha\beta} (\mathcal{C})_{\beta\gamma} \left(\partial_\mu \psi^\dagger\right)_\gamma \\ &= \psi_\gamma^\dagger \left(\gamma^0\gamma^\mu\right)_{\gamma\sigma} (\partial_\mu \psi)_\sigma \end{aligned} \tag{40.9}$$

[33] In this point we use the Leibniz rule

$$\partial_\mu \left(\bar{\psi}\gamma^\mu\psi\right) = \left(\partial_\mu \bar{\psi}\right)\gamma^\mu\psi + \bar{\psi}\gamma^\mu(\partial_\mu\psi) = 0$$

put equal to 0 because the 4-divergence of a Noether current is always conserved

By comparison we can say

$$\mathcal{C}^\dagger\left(\gamma^0\gamma^\mu\right)\mathcal{C} = \left(\gamma^0\gamma^\mu\right)^T = \left(\gamma^\mu\right)^T\gamma^0 \tag{40.10}$$

For $\mu = 0$ it's easy to show from equation 40.10 that the operator $\mathcal{C}$ is hermitian.
Adding the unitary quantity $\gamma^0\gamma^0$, from equation 40.10 we get

$$\mathcal{C}\left(\gamma^0\gamma^\mu\gamma^0\gamma^0\right)\mathcal{C} = \mathcal{C}\gamma^{\mu\dagger}\gamma^0\mathcal{C} = \left(\gamma^\mu\right)^T\gamma^0 \tag{40.11}$$

Now we multiply both the sides by γ^0 and we exploit the anticommutation between $\mathcal{C}$ and γ^0, since the equation 40.8 is true, so we get

$$\begin{aligned}
\mathcal{C}\gamma^{\mu\dagger}\gamma^0\mathcal{C}\gamma^0 &= \left(\gamma^\mu\right)^T \\
-\mathcal{C}\gamma^{\mu\dagger}\mathcal{C} &= \left(\gamma^\mu\right)^T \\
\mathcal{C}\gamma^\mu\mathcal{C} &= -\gamma^{\mu *}
\end{aligned} \tag{40.12}$$

The only thing to do now is determining the operator. We know that γ^0, γ^1, γ^3 are real and simmetric (like also γ_5) whereas γ^2 is imaginary. Since $\mathcal{C}$ anticommutes with γ^0, γ^1, γ^3 and commutes with γ^2, imposing the condition $\mathcal{C}^2 = 1$ we can define the operator up to a sign

$$\mathcal{C} = i\gamma^2 \tag{40.13}$$

40.2 Bilinear covariant transformations

Now let's see the bilinear covariants like we did for the parity.

- A **scalar** quantity will transform without changing sign, as we know from equation 40.5.

- A **pseudoscalar** quantity will transform without changing sign

$$\begin{aligned}
\bar{\psi}_c\gamma_5\psi_c &= \psi\,\mathcal{C}\gamma^0\gamma_5\,\mathcal{C}\psi^\dagger = -\psi^\dagger\left(\mathcal{C}\gamma^0\gamma_5\,\mathcal{C}\right)^T\psi \\
&= -\psi^\dagger\mathcal{C}^T\gamma_5^T\left(\gamma^0\right)^T\mathcal{C}^T\psi = -\psi^\dagger\left(\mathcal{C}\mathcal{C}\right)^T\gamma_5\gamma^0\psi \\
&= -\psi^\dagger\gamma_5\gamma^0\psi = \bar{\psi}\gamma_5\psi
\end{aligned} \tag{40.14}$$

- A **vector** quantity will transform changing sign

$$\begin{aligned}
\bar{\psi}_c\gamma^\mu\psi_c &= \psi\,\mathcal{C}\gamma^0\gamma^\mu\mathcal{C}\psi^\dagger = -\psi^\dagger\left(\mathcal{C}\gamma^0\gamma^\mu\mathcal{C}\right)^T\psi \\
&= -\psi^\dagger\mathcal{C}^T\left(\gamma^\mu\right)^T\left(\gamma^0\right)^T\mathcal{C}^T\psi = \psi^\dagger\mathcal{C}^T\left(\gamma^\mu\right)^T\mathcal{C}^T\gamma^0\psi \\
&= \psi^\dagger\left(\mathcal{C}\gamma^\mu\mathcal{C}\right)^T\gamma^0\psi = -\psi^\dagger\left(\gamma^\mu\right)^\dagger\gamma^0\psi
\end{aligned} \tag{40.15}$$

$\mu = 0 \implies -\psi^\dagger\gamma^0\gamma^0\psi = -\bar{\psi}\gamma^0\psi$

$\mu = 1,2,3 \implies \psi^\dagger\gamma^i\gamma^0\psi = -\bar{\psi}\gamma^i\psi$

So we can write

$$\bar{\psi}_c\gamma^\mu\psi_c = -\bar{\psi}\gamma^\mu\psi \tag{40.16}$$

- A **pseudovector** quantity will transform without changing sign

$$\begin{aligned}\bar{\psi}_c\gamma^\mu\gamma_5\psi_c &= \psi\,\mathcal{C}\gamma^0\gamma^\mu\gamma_5\,\mathcal{C}\psi^\dagger = -\psi^\dagger\left(\mathcal{C}\gamma^0\gamma^\mu\gamma_5\,\mathcal{C}\right)^T\psi \\ &= -\psi^\dagger\mathcal{C}^T\gamma_5^T(\gamma^\mu)^T\left(\gamma^0\right)^T\mathcal{C}^T\psi = -\psi^\dagger\gamma_5(\mathcal{C}\gamma^\mu\mathcal{C})^T\gamma^0\psi \\ &= \psi^\dagger\gamma_5(\gamma^\mu)^\dagger\gamma^0\psi \end{aligned} \tag{40.17}$$

$\mu = 0 \implies \psi^\dagger\gamma_5\gamma^0\gamma^0\psi = \bar{\psi}\gamma^0\gamma_5\psi$

$\mu = 1,2,3 \implies -\psi^\dagger\gamma_5\gamma^i\gamma^0\psi = -\bar{\psi}\gamma_5\gamma^i\psi = \bar{\psi}\gamma^i\gamma_5\psi$

We can write

$$\bar{\psi}_c\gamma^\mu\gamma_5\psi_c = \bar{\psi}\gamma^\mu\gamma_5\psi \tag{40.18}$$

- A **tensor** quantity, similarly the vector case, will transform changing sign

$$\begin{aligned}\bar{\psi}_c\gamma^\mu\gamma^\nu\psi_c &= \psi\,\mathcal{C}\gamma^0\gamma^\mu\gamma^\nu\mathcal{C}\psi^\dagger = -\psi^\dagger\left(\mathcal{C}\gamma^0\gamma^\mu\gamma^\nu\mathcal{C}\right)^T\psi \\ &= -\psi^\dagger\mathcal{C}^T(\gamma^\nu)^T(\gamma^\mu)^T\gamma^0\mathcal{C}^T\psi = \psi^\dagger(\mathcal{C}\gamma^\mu\gamma^\nu\mathcal{C})^T\gamma^0\psi \\ &= \psi^\dagger(\mathcal{C}\gamma^\nu\mathcal{C})^T(\mathcal{C}\gamma^\mu\mathcal{C})^T\gamma^0\psi = -\psi^\dagger(\gamma^\mu)^\dagger(\gamma^\nu)^\dagger\gamma^0\psi \end{aligned} \tag{40.19}$$

$\mu = \nu = 0 \implies -\psi^\dagger\gamma^0\gamma^0\gamma^0\psi = -\bar{\psi}\psi$

$\mu,\nu \neq 0 \implies -\psi^\dagger\gamma^i\gamma^j\gamma^0\psi = -\bar{\psi}\gamma^i\gamma^j\psi$

We can write

$$\bar{\psi}_c\gamma^\mu\gamma^\nu\psi_c = -\bar{\psi}\gamma^\mu\gamma^\nu\psi \tag{40.20}$$

A transformation under charge conjugation of the electromagnetic and Yang-Mills fields is the direct consequence of the fact that these fields are generated by the charge of the particles so, if we switch particles with antiparticles, the sign changes

$$\mathcal{C}A^\mu\mathcal{C} = -A^\mu \tag{40.21}$$

$$\mathcal{C}F^{\mu\nu}\mathcal{C} = -F^{\mu\nu} \tag{40.22}$$

Once we got that, from the bilinear covariants we can say that QCD and QED lagrangians are invariant under a charge conjugation transformation, whereas the Fermi lagrangian is not.

41 Time reversal $\mathcal{T}$

If we consider the Classical Mechanics, it's possible to see how a generic system is invariant applying the transformation $t \to -t$: the new system retraces all the configurations backwards from which the original system passed. We can say that Classical Mechanics is invariant under a time reversal $\mathcal{T}$ transformation.
Passing to Quantum Mechanics, things are more complicated. If we have a generic observable at a time $t = 0$ in the initial state i, we can study its time evolution in the final state f using an operator. In order to see a time reversal transformation we have to use the operator $\mathcal{T}$ such that

$$|A_T; t\rangle = \mathcal{T}\,|A; -t\rangle \tag{41.1}$$

The main consequence of this operation is that $\mathcal{T}$ is not a linear operator. Let's consider the time evolution of a wave function

$$|E; t\rangle = e^{-iEt}\,|E; t = 0\rangle \tag{41.2}$$

If we apply the time reversal operator

$$|E_T; t\rangle = e^{-iEt}\,|E_T; t = 0\rangle = \mathcal{T}\Big[e^{-iE(-t)}\,|E; t = 0\rangle\Big] = \mathcal{T}\big[e^{+iEt}\,|E; t = 0\rangle\big] \tag{41.3}$$

the sign of the exponential changes. We can solve this problem defining the operator $\mathcal{T}$ as an antiunity operator; in this way we will have the exponential with the right sign.
An antiunity operator is an antilinear and unity operator, so if we take two states A, B, they have to satisfy two conditions

$$\mathcal{T}(\alpha\,|A\rangle + \beta\,|B\rangle) = \alpha^*\mathcal{T}\,|A\rangle + \beta^*\mathcal{T}\,|B\rangle \tag{41.4}$$

$$\langle B_T|A_T\rangle = \langle A|B\rangle^* \tag{41.5}$$

41.1 Determination of the operator

Once we defined the time reversal operator as an antiunity operator, we can determine it as a function of gamma matrices. This operator acts on a field in the following way

$$(\psi_T)_\alpha = \big(\mathcal{T}\,\psi(\mathbf{x}, t)\mathcal{T}^\dagger\big)_\alpha = \mathcal{T}_{\alpha\beta}\psi^*_\beta(\mathbf{x}, -t) \tag{41.6}$$

From this we get

$$\begin{cases} \psi_T = \mathcal{T}\psi^* \\ \psi_T^\dagger = (\psi^*)^\dagger\mathcal{T}^\dagger \end{cases} \tag{41.7}$$

The system has to come back to the original condition if we apply the operator another time, so $\mathcal{T}^2 = 1$.
In order to determine the operator, let's consider the Dirac equation

$$(i\partial\!\!\!/ - m)\psi = 0 \tag{41.8}$$

$$\big(i\gamma^0\partial^0 + i\underline{\gamma}\,\underline{\partial} - m\big)\psi = 0 \tag{41.9}$$

Doing the complex conjugate and taking $\mathcal{T}^2 = 1$, we get

$$\big(-i\gamma^0\partial^0 - i\underline{\gamma}^*\underline{\partial} - m\big)\mathcal{T}\mathcal{T}\psi^* = 0 \tag{41.10}$$

$$\big(-i\gamma^0\partial^0 - i\underline{\gamma}^*\underline{\partial} - m\big)\mathcal{T}\psi_T = 0 \tag{41.11}$$

Multiplying on the left for $\mathcal{T}^\dagger$ we will have

$$\mathcal{T}^\dagger\big(-i\gamma^0\partial^0 - i\underline{\gamma}^*\underline{\partial} - m\big)\mathcal{T}\psi_T = 0 \tag{41.12}$$

If we want that equation 41.12 is equal to

$$\big(-i\gamma^0\partial^0 - i\underline{\gamma}^*\underline{\partial} - m\big)\psi_T = 0 \tag{41.13}$$

the only possible solution, up to a sign, is

$$\mathcal{T} = i\gamma^1\gamma^3 \tag{41.14}$$

We can see that the time reversal operator is a hermitian operator. Moreover, the transformation with this operator leaves γ^0 invariant and changes sign of γ^i, so we get the condition

$$\mathcal{T}(\gamma^\mu)^*\mathcal{T} = (\mathcal{T}\gamma^\mu\mathcal{T})^* = g^{\mu\mu}\gamma^\mu \tag{41.15}$$

41.2 Bilinear covariant transformations

In conclusion, let's see the bilinear covariants.

- A **scalar** quantity will transform without changing sign

$$\bar{\psi}_T\psi_T = (\psi^*)^\dagger\mathcal{T}\gamma^0\,\mathcal{T}\psi^* = \psi^\dagger\big(\mathcal{T}\gamma^0\mathcal{T}\big)^*\psi = \bar{\psi}\psi \tag{41.16}$$

- A **pseudoscalar** quantity will transform without changing sign

$$\begin{aligned}\bar{\psi}_T\gamma_5\psi_T &= (\psi^*)^\dagger\mathcal{T}\gamma^0\gamma_5\,\mathcal{T}\psi^* = \psi^\dagger\big(\mathcal{T}\gamma^0\gamma_5\,\mathcal{T}\big)^*\psi \\ &= \psi^\dagger\big(\mathcal{T}\gamma^0\mathcal{T}\big)^*\gamma_5\psi = \psi^\dagger\gamma^0\gamma_5\,\psi = \bar{\psi}\gamma_5\psi\end{aligned} \tag{41.17}$$

- A **vector** quantity will transform like the Minkowski tensor without changing sign

$$\begin{aligned}\bar{\psi}_T\gamma^\mu\psi_T &= (\psi^*)^\dagger\mathcal{T}\gamma^0\gamma^\mu\mathcal{T}\psi^* = \psi^\dagger\big(\mathcal{T}\gamma^0\gamma^\mu\mathcal{T}\big)^*\psi \\ &= \psi^\dagger\gamma^0(\mathcal{T}\gamma^\mu\mathcal{T})^*\psi = g^{\mu\mu}\bar{\psi}\gamma^\mu\psi\end{aligned} \tag{41.18}$$

- A **pseudovector** quantity will transform like the Minkowski tensor without changing sign

$$\begin{aligned}\bar{\psi}_T\gamma^\mu\gamma_5\psi_T &= (\psi^*)^\dagger\mathcal{T}\gamma^0\gamma^\mu\gamma_5\,\mathcal{T}\psi^* = \psi^\dagger\big(\mathcal{T}\gamma^0\gamma^\mu\gamma_5\,\mathcal{T}\big)^*\psi \\ &= \psi^\dagger\gamma^0(\mathcal{T}\gamma^\mu\mathcal{T})^*\gamma_5\psi = g^{\mu\mu}\bar{\psi}\gamma^\mu\gamma_5\psi\end{aligned} \tag{41.19}$$

- A **tensor** quantity, similarly the vector case, will transform like the Minkowski tensor changing sign

$$\begin{aligned}\bar{\psi}_T\gamma^\mu\gamma^\nu\psi_T &= (\psi^*)^\dagger\mathcal{T}\gamma^0\gamma^\mu\gamma^\nu\mathcal{T}\psi^* = \psi^\dagger\big(\mathcal{T}\gamma^0\gamma^\mu\gamma^\nu\mathcal{T}\big)^*\psi \\ &= \psi^\dagger\gamma^0(\mathcal{T}\gamma^\mu\mathcal{T}\mathcal{T}\gamma^\nu\mathcal{T})^*\psi = \bar{\psi}(\mathcal{T}\gamma^\nu\mathcal{T})^*(\mathcal{T}\gamma^\mu\mathcal{T})^*\psi \\ &= g^{\mu\mu}g^{\nu\nu}\bar{\psi}\gamma^\nu\gamma^\mu\psi = -g^{\mu\mu}g^{\nu\nu}\bar{\psi}\gamma^\mu\gamma^\nu\psi\end{aligned} \tag{41.20}$$