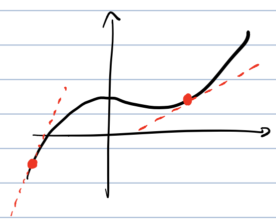
LEZIONE 2 : DERIVATE

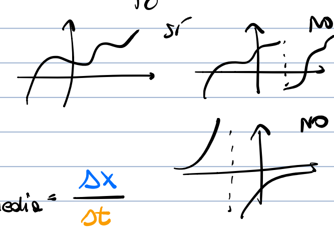
- motivazione / definizione
- derivate funzioni elementari
- regole di derivazione
- monotonia + problemi di ottimizzazione

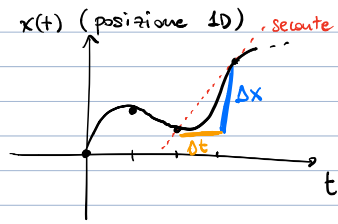
1) DEFINIZIONE

hp: $f: \mathbb{R} \to \mathbb{R}$, f continua

↳ "posso disegnare grafico senza staccare penna dal foglio"

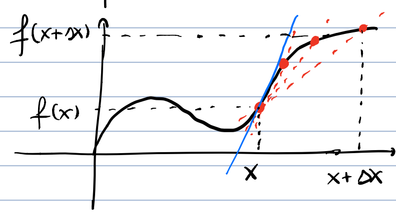






calcolo della velocità istantanea

$$v_{media} = \frac{\Delta x}{\Delta t}$$



secanti nel limite = tangente al grafico

tante velocità medie approssimano velocità istant.

$$f'(x) = \frac{d}{dx} f(x) = \frac{df}{dx}(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

↑ rapporto incrementale

" derivata = limite rapporto incrementale "

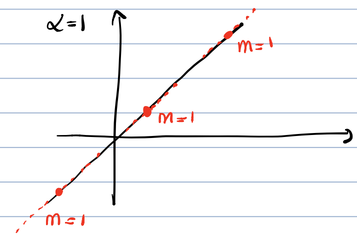
2) DERIVATE FUNZIONI ELEMENTARI

→ • x^α, a^x, $\log_a x$, $\sin x$, $\cos x$, $\operatorname{tg} x$

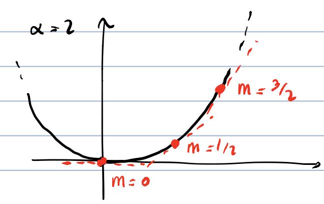
• $\dfrac{P(x)}{Q(x)}$, $\log P(x)$, $\operatorname{tg} P(x)$, $(P(x))^\alpha$ P, Q polinomi

↳ non sono funzioni elementari → regole di derivazioni

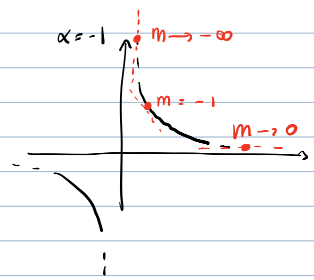
x^α | $f(x) = x^\alpha$ $\boxed{\frac{d}{dx} f(x) = \alpha \cdot x^{\alpha-1}}$ m = pendenza



per $\alpha=1$ $f'(x) = 1 \cdot x^{1-1} = 1$

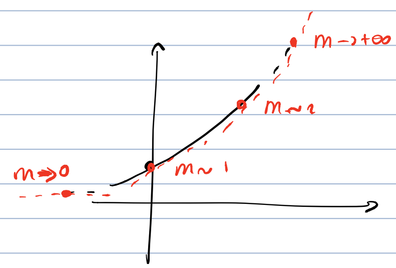


per $\alpha=2$ $f'(x) = 2 \cdot x^{2-1} = 2x^1$ non banale



per $\alpha=-1$ $f'(x) = -1 \cdot x^{-1-1} = -\frac{1}{x^2}$ non banale

a^x | $f(x) = a^x$ $\boxed{f'(x) = a^x \log a}$ $\log = \ln = \log_e$



$a>1$ $f'(x) = a^x \underline{\log a}$

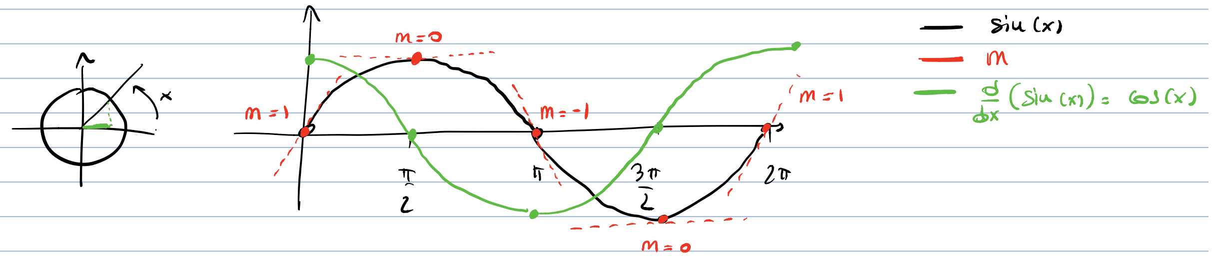
non banale che crescita sia exp

$a = e$ $\boxed{f'(x) = e^x \overbrace{\log e}^{=1} = e^x}$

$\log_a x$ | $\boxed{f'(x) = \frac{1}{x} \frac{1}{\log a} \underset{a=e}{=} \frac{1}{x}}$

NOTA: ecco il motivo per cui vi fanno studiare "e"

sin x | $\quad f'(x) = \cos x$



cos x | $\quad f'(x) = -\sin(x)$

tg x | $\quad f'(x) = \frac{1}{\cos^2 x} = [\sec(x)]^2$ ⟶ COMPITO: GIUSTIFICARE DERIVATA TG CON GRAFICO

$$\left[\begin{array}{l} (\cos x)^2 = (\cos x)\cdot(\cos x) \\ \cos x^2 = \cos(x^2) \\ \cos^2 x = [\cos(x)]^2 \end{array}\right]$$

3) REGOLE DI DERIVAZIONE PER $+, \cdot, \%$

+) $\quad \frac{d}{dx}\left(f(x) \pm g(x)\right) = \frac{df}{dx} \pm \frac{dg}{dx}$ COMPITO: PROVARE A DIM CON LIM. RAPP. INCREM.

es: $\quad \frac{d}{dx}\left(x^7 - \sin x\right) = \frac{d}{dx}(x^7) - \frac{d}{dx}(\sin x) = 7x^6 - \cos x$

• numero) $\quad \frac{d}{dx}\left(\alpha \cdot f(x)\right) = \alpha \frac{df}{dx}$

costante, non dipende da x

es: $\quad \frac{d}{dx}\left(15 \log_3 x\right) = 15 \frac{d}{dx}\left(\log_3 x\right) = 15 \frac{1}{x} \frac{1}{\log 3}$

•) $\frac{d}{dx}\left(f(x)\cdot g(x)\right) \neq \frac{df}{dx}\cdot\frac{dg}{dx}$ NO

$$\frac{d}{dx}\left(f(x)\cdot g(x)\right) = \frac{df}{dx}\cdot g(x) + f(x)\cdot\frac{dg}{dx}$$

es: $\frac{d}{dx}\left(tg(x)\cdot a^x\right) = \frac{1}{\cos^2 x}\cdot a^x + tg(x)\cdot a^x\cdot \log a$

%) $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) \neq \frac{df/dx}{dg/dx}$

$$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{\frac{df}{dx}\cdot g(x) - f(x)\cdot\frac{dg}{dx}}{(g(x))^2}$$

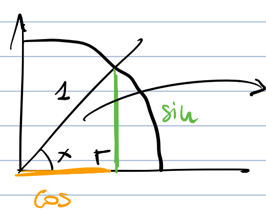
COMPITO: $tg(x) = \frac{\sin(x)}{\cos(x)}$

DIM CHE $\frac{d}{dx}tg(x) = \frac{1}{\cos^2 x}$

es: $\frac{d}{dx}(tg(x)) = \frac{d}{dx}\left(\frac{\sin(x)}{\cos(x)}\right) = \frac{\cos x\cdot\cos x - \sin x\cdot(-\sin x)}{\cos^2 x} =$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

$\sin^2 x + \cos^2 x = 1$



thm Pitagora: $1^2 = (\sin x)^2 + (\cos x)^2$

ESERCIZI: VOL 5 MATEMATICAMENTE.IT (SITO)

↳ pg PDF 181, pg LIBRO 184

101 $f(x) = \frac{3x^2 - 5x}{7x + 1}$

$$f'(x) = \frac{\frac{d}{dx}(3x^2-5x)\cdot(7x+1) - (3x^2-5x)\frac{d}{dx}(7x+1)}{(7x+1)^2} =$$

$\uparrow$

%)

$$= \frac{\left[\frac{d}{dx}(3x^2) - \frac{d}{dx}(5x)\right]\cdot(7x+1) - (3x^2-5x)\left[\frac{d}{dx}(7x) + \overset{=0}{\cancel{\frac{d}{dx}(1)}}\right]}{(7x+1)^2} =$$

$\uparrow$

$\pm$)

$$= \frac{\left[3\frac{d}{dx}(x^2) - 5\frac{d}{dx}(x)\right]\cdot(7x+1) - (3x^2-5x)\cdot 7\cdot\frac{d}{dx}(x)}{(7x+1)^2} =$$

$\uparrow$

• numero)

$$= \frac{\left[3\cdot 2\cdot x - 5\cdot 1\right](7x+1) - (3x^2-5x)\,7\cdot 1}{(7x+1)^2} =$$

$\uparrow$

derivate funz. elementari

$$= \frac{(6x-5)(7x+1) - 7(3x^2-5x)}{(7x+1)^2} = \dots$$ COMPITO

ES) $f(x) = (ax+b)(cx+d) \quad , a,b,c,d \in \mathbb{R}$

$$f'(x) = (ax+b)'\cdot(cx+d) + (ax+b)\cdot(cx+d)' =$$

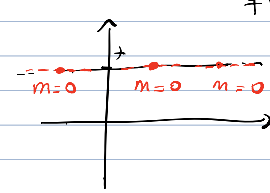
$$= [a\cdot 1+0](cx+d) + (ax+b)[c\cdot 1+0] =$$

$$= a(cx+d) + c(ax+b) =$$

$$= acx + ad + acx + bc =$$

$$= 2acx + ad + bc$$

$f(x) = 7$



4) regola di derivazione di funzioni composte

$$\begin{tikzcd} x \arrow[r, mapsto, "f(x)=x^2"'] \arrow[rr, bend right, "h(x)=a^{x^2}"'] & x^2 \arrow[r, mapsto, "g(x)=a^x"'] & a^{x^2} \end{tikzcd}$$

h è la composizione di f e g (f per prima, g per seconda)

$$h(x) = (\underset{\substack{\uparrow \\ \text{composizione}}}{g \circ f})(x) = g(f(x))$$

$$\frac{d}{dx}\left[(g\circ f)(x)\right] = \frac{d}{dx}\left[g(f(x))\right] = g'(f(x)) \cdot f'(x)$$

es: $f(x) = x^2$, $g(x) = a^x$, $h(x) = (g \circ f)(x) = a^{x^2}$

$$\frac{d}{dx} h(x) = \frac{d}{dx}\left(g(f(x))\right) = \Big[\underbrace{a^x \log a}_{g'}\Big]_{x \to f(x) = x^2} \cdot 2x$$

$$= a^{f(x)} \log a \cdot 2x = a^{x^2} \log a \cdot 2x$$

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$$h(x) = (2x+5)^3$$

$$\begin{tikzcd} x \arrow[r, mapsto, "f(x)=2x+5"'] \arrow[rr, bend right, "h(x)=(2x+5)^3"'] & 2x+5 \arrow[r, mapsto, "g(x)=x^3"'] & (2x+5)^3 \end{tikzcd}$$

$$\frac{d}{dx} h = \frac{d}{dx}\left((g \circ f)(x)\right) = \left(3x^2\right)\Big|_{x \to f(x) = 2x+5} \cdot \frac{d}{dx}(2x+5) =$$

$$= 3(2x+5)^2 \cdot (2 \cdot 1 + 0) = 6(2x+5)^2$$

CASO PARTICOLARE

$$h(x) = [f(x)]^{g(x)} \underset{\substack{\uparrow \\ \text{trucco}}}{=} e^{g(x) \cdot \log(f(x))}$$

$$a = e^{\log a}$$
$$\forall a > 0$$

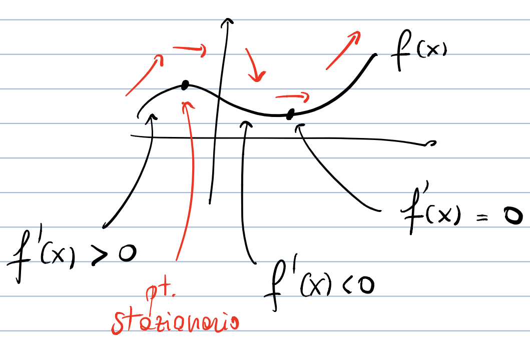
$$x \longmapsto f(x) \longmapsto \log f(x) \longmapsto g(x) \cdot \log f(x) \longmapsto e^{g(x) \log f(x)}$$

COMPITO $\quad (3x+5)^{\sin(x)} \longrightarrow f = e^{\sin(x) \log(3x+5)}$

$$f' = e^{\sin(x) \log(3x+5)} \cdot [\sin(x) \log(3x+5)]'$$

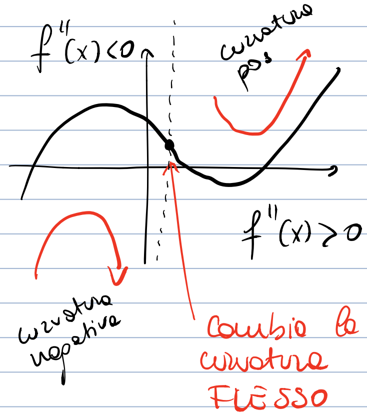
$$\left[\ \log_{g(x)} f(x) \underset{\substack{\uparrow \\ \text{cambio di} \\ \text{variabile}}}{=} \frac{\log f(x)}{\log g(x)}\ \right]$$

5) MASSIMI / MINIMI / MONOTONIA



segno della derivata
(+, −, 0)
determina monotonia

DA CAPIRE



derivata seconda = derivata della derivata di f

$(+, -, 0)$

determina la curvatura

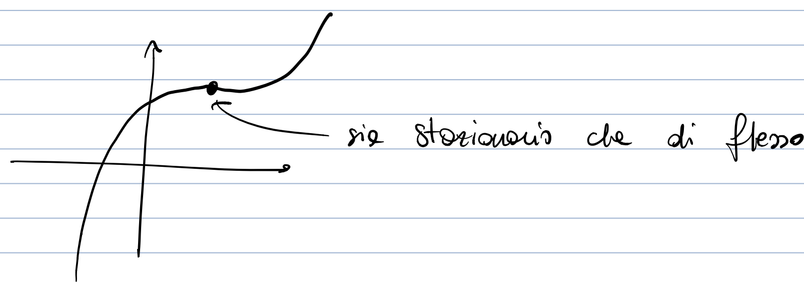
FIDATEVI !!

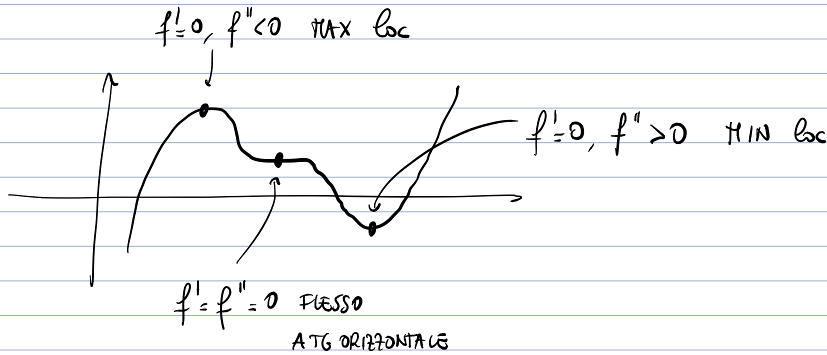
ESEMPI LEZIONE 3

MASSIMI e MINIMI locali sono punti stazionari

$\Rightarrow$ candidati max e min locali sono

i punti in cui $f'(x) = 0$





pdf esercizi ottimizzazione

1) x, y t.c. $x+y=30 \rightsquigarrow y = 30-x$

$\min(x^2+y^2)$

$\hookrightarrow \quad f(x) = x^2 + (30-x)^2 =$

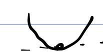
$= x^2 + 900 - 60x + x^2 =$

$= 2x^2 - 60x + 900$

studio $f'(x) = 2 \cdot 2x - 60 \cdot 1 + 0 = 4x - 60$

$f'(x) = 0 \Rightarrow 4x - 60 = 0 \longrightarrow x = 15$

studio $f''(x) = 4 \cdot 1 - 0 = 4$

in $x^* = 15$, $f'(x^*) = 0$, $f''(x^*) = 4 > 0$ 

é MIN LOCALE

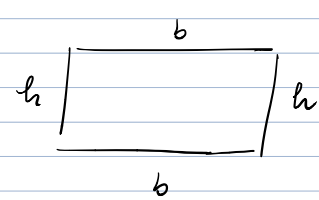
$\Rightarrow \quad x^* = 15$ e $y^* = 30 - x^* = 15$

6) $2p = 40$ cm , $b = h = ?$ per area max

$2(b+h) = 40$

$\hookrightarrow b + h = 20$

$b = 20 - h$



$b, h > 0$

max: $A(h) = (20-h) \cdot h = 20h - h^2$

$\rightarrow A'(h) = 20 \cdot 1 - 2h = 20 - 2h$

$A'(h) = 0 \Rightarrow 20 - 2h = 0 \Rightarrow h = 10$

$\rightarrow A''(h) = -2 \Rightarrow h^* = 10 : A'(h^*) = 0 , A''(h^*) < 0$

$\Rightarrow h^*$ é un MAX locale 

$h^* = 10$ cm e $b^* = 10$ cm