5/04/2023 – M. FRIGERIO

1.) $\displaystyle\int_\gamma (z^2 - \bar{z}^2)\,dz = ?$ dove γ segmento di retta da $z_0 = 1$ a $z_1 = i$

$$\gamma : [0,1] \ni t \longmapsto \gamma(t) = (1-t) + it$$

$$\int_\gamma f(z)\,dz = \int_0^1 f(\gamma(t))\,\gamma'(t)\,dt$$

$$= \int_0^1 \left((1-t+it)^2 - (1-t-it)^2 \right)(-1+i)\,dt$$

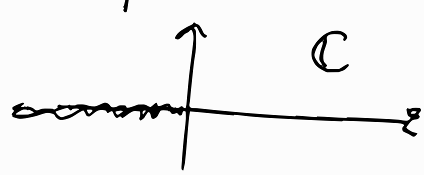
$$= \int_0^1 4it(1-t)(-1+i)\,dt = 4(1+i)\int_0^1 (t^2-t)\,dt$$

$$= 4(1+i)\left[\frac{t^3}{3} - \frac{t^2}{2}\right]_0^1 = -\frac{2(1+i)}{3}$$

2.) Calcolare $\displaystyle\int_\gamma z^z(1+\log z)\,dz$ supponendo ramo princ.

e con $\gamma : \theta \in (-\pi, \pi) \longmapsto \gamma(\theta) = Re^{i\theta}$ $(R > 0)$

$z^z = e^{z \log z}$: definita dove é definito il log



$$\frac{d}{dz} z^z = \frac{d}{dz} e^{z \log z} = \underbrace{z^z (1+\log z)}_{\text{è l'integranda}}$$

Dunque, dato $\varepsilon > 0$

$$\int_\gamma z^z (1+\log z)\, dz = \lim_{\varepsilon \to 0} z^z \Big|_{z = Re^{-i(\pi-\varepsilon)}}^{z = Re^{i(\pi-\varepsilon)}} =$$

$$= e^{-R(\ln R + i\pi)} + e^{-R(\ln R - i\pi)} =$$

$$= -2i R^{-R} \sin(\pi R) = -\frac{2i \sin(\pi R)}{R^R}$$

3.) $\displaystyle\int_{\gamma_R} \frac{\log z}{z^2}\, dz$ dove $\gamma_R : \theta \in (-\pi, \pi) \longrightarrow Re^{i\theta}$ $(R>0)$

$$\int_{-\pi}^{\pi} \underbrace{\frac{\ln R + i\theta}{R^2 e^{2i\theta}}}_{f(\gamma(\theta))} \overbrace{iRe^{i\theta}}^{\gamma'(\theta)} d\theta = \int_{-\pi}^{\pi} i \frac{(\ln R + i\theta)}{R} e^{-i\theta} d\theta =$$

$$= i \frac{\ln R}{R} \frac{e^{-i\theta}}{-i} \Big|_{-\pi}^{\pi} - \frac{1}{R} e^{-i\theta} (1+i\theta) \Big|_{-\pi}^{\pi} =$$

$$= 0 - \frac{1}{R} (\cancel{-1} - i\pi \cancel{+1} - i\pi) = \frac{2i\pi}{R}$$