31/05/2023 - M. FRIGERIO

1.) In $L^2([-\pi,\pi])$ considerare

$$T: f(x) \longmapsto \alpha f(x) + \beta f(-x), \quad \alpha,\beta \in \mathbb{C}\setminus\{0\}$$

Trovare autovett. e autoval. di T, dire se formano un SONC.

$$T(f_{pari}) = (\alpha+\beta) f_{pari}, \quad T(f_{dispari}) = (\alpha-\beta) f_{dispari}$$

$$\alpha f(x) + \beta f(-x) = \lambda f(x)$$

$$(\alpha-\lambda) f(x) + \beta f(-x) = 0 \qquad f(-x) = \frac{\lambda-\alpha}{\beta} f(x)$$

ma $f(-(-x)) = f(x) \Rightarrow \left(\frac{\lambda-\alpha}{\beta}\right)^2 = 1$

$$(\lambda-\alpha)^2 = \beta^2 \Rightarrow \lambda-\alpha = \pm\beta \qquad \lambda = \alpha \pm \beta$$

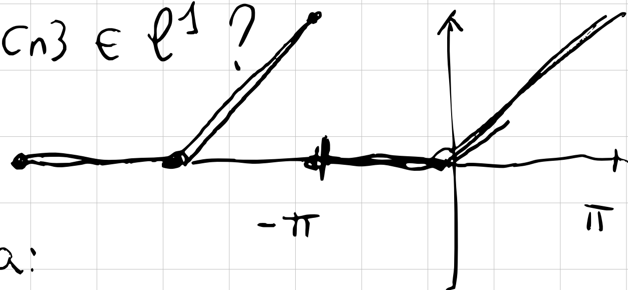
→ sono tutte le autofunz. Contiene un SONC.

$$\|T\|^2_{op.} : \sup_{f\in L^2([-\pi,\pi])} \frac{\|\hat{T} f\|^2_{L^2([-\pi,\pi])}}{\|f\|^2_{L^2}} \leq$$

$$\sup_{f\in L^2} \frac{|\alpha+\beta|^2 \|f_{pari}\|^2 + |\alpha-\beta|^2 \|f\|^2_{dispari}}{\|f\|^2}$$

Quindi $\|\hat{T}\|_{op} = \max\left(|\alpha-\beta|, |\alpha+\beta|\right)$

2.) Serie di Fourier di $\varphi(x) = \frac{x+|x|}{2}$ in $[-\pi,\pi]$ risp a e^{inx}. Converge in π? A cosa?
E in $\frac{5\pi}{2}$? I $\{c_n\} \in \ell^1$?



$\Rightarrow$ In π converge a:

$$\lim_{\varepsilon\to 0^+} \frac{f(\pi+\varepsilon)+f(\pi-\varepsilon)}{2} = \frac{0+\pi}{2} = \frac{\pi}{2}$$

In $\frac{5\pi}{2}$ converge a $\varphi\left(\frac{5\pi}{2}-\pi\right) = \frac{\pi}{2}$

I coeff. non sono in L^1 perché φ è discontinua.

3) $f(t) = \begin{cases} 0 & \text{se } t<0 \\ e^{-t^2} & \text{se } t>0. \end{cases}$

$\mathcal{F}(f(t)) = g(w)$. $g(w) \in L^1(\mathbb{R})$? $\in L^2(\mathbb{R})$?
$\in C^k(\mathbb{R})$ per qualche k? $\lim_{\omega\to\pm\infty} g(\omega) = $?

$g(\omega) \in C^0$ per $f \in L^1$. $g(\omega) \in C^k \; \forall k$

perché $t^k e^{-t^2} \in L^1 \; \forall k$

$\lim_{\omega \to \pm\infty} g(\omega) < +\infty$ per $f \in L^1$.

$g(\omega) \notin L^1$ perché f non é continua.

$$\int_{-\infty}^{+\infty} |g(\omega)|^2 d\omega = 2\pi \int_{-\infty}^{+\infty} |f(t)|^2 dt$$

$$= 2\pi \int_0^{+\infty} e^{-2t^2} dt = \pi \sqrt{\frac{\pi}{2}} \text{ per Parseval}$$

$\mathcal{F}^{-1}(-i\omega g(\omega))$?

$$\mathcal{F}^{-1}(-i\omega g(\omega)) = \frac{d}{dt}\left(\theta(t) e^{-t^2}\right) =$$

$$= \delta(t) - 2t\,\theta(t) e^{-t^2}$$