3/05/2023 – M. FRIGERIO

1) Calcolare la Trasf. di Fourier della p.d.f. Lorentziana $p(x) = \frac{1}{\pi} \frac{1}{1+x^2}$

N.B. $\int_{-\infty}^{+\infty} p(x)\,dx = 1$, $\int_{-\infty}^{+\infty} x\,p(x)\,dx = 0$

ma $\int_{-\infty}^{+\infty} x^2 p(x)\,dx = +\infty \Rightarrow$ p.d.f. con varianza <u>infinita</u> !!!

$$\int_{-\infty}^{+\infty} \frac{e^{ikx}}{1+x^2}\,dx = I(k)$$

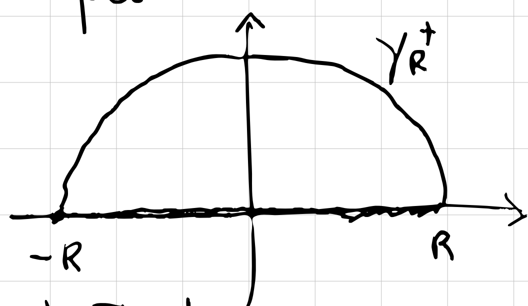
$(z = x+iy)$

Se $k>0$, allora $ikz = ikx - ky \Rightarrow$ e^{ikz} decresce esponenzialm. nel semipiano superiore di $\mathbb{C}$.

$$\Rightarrow I(k) = 2\pi i \,\mathrm{Res}\left(\frac{e^{ikz}}{1+z^2}, i\right) =$$

$$= 2\pi i \,\frac{e^{-k}}{2i} = \pi e^{-k} \quad \text{per } k>0$$

Ho usato il contorno



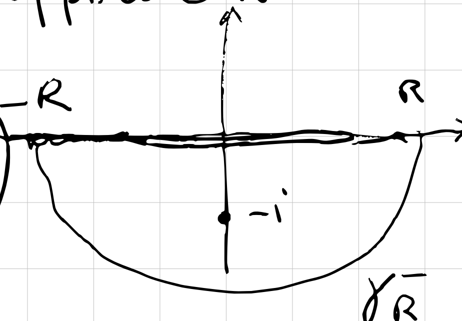
per $R \to +\infty$

$\int_{\gamma_R^+} f(z)\,dz \to 0$ per lemma di Jordan

Infatti $\frac{1}{1+z^2} \to 0$ per $|z| \to +\infty$

e $e^{ikz} \to 0$ per $k>0$, $|z| \to +\infty$ e $\mathrm{Im}(z) > 0$

Per $k<0$, devo chiudere il contorno nel semipiano inferiore per applicare il lemma di Jordan:



$$I(k) = \underset{\text{contorno in verso orario!}}{-}\, 2\pi i \,\mathrm{Res}\left(\frac{e^{ikz}}{1+z^2}, -i\right)$$

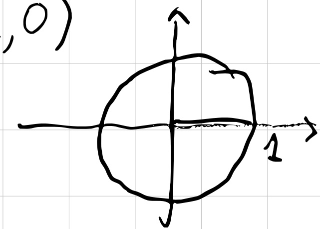
$$= \frac{-2\pi i\, e^{k}}{-2i} = \pi e^{k} = \pi e^{-|k|} \quad \text{per } k>0$$

In totale, $I(k) = \pi e^{-|k|} \quad \forall\, k \in \mathbb{R}$

(é facile verificare che é corretto anche per $k=0$)

2) $A = \int_0^{2\pi} \sin\left(e^{e^{i\theta}}\right) d\theta \qquad z = e^{i\theta}$

$\gamma = \mathcal{C}(1,0)$



$dz = i e^{i\theta} d\theta \Rightarrow d\theta = \frac{dz}{iz}$

$\Rightarrow A = \int_{\mathcal{C}} \frac{\sin(e^z)}{iz} dz$

$\sin(e^z) \to$ analitica perché compos di f analitiche

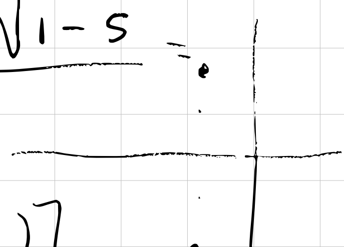
$\to \operatorname{Res}(f,0) = \frac{\sin(e^0)}{i} = \frac{\sin(1)}{i}$

$\Rightarrow 2\pi \sin(1)$

$$\int_{-\infty}^{+\infty} \frac{x \sin(kx)}{x^2+2x+5} = \operatorname{Im}\left[\lim_{R\to+\infty}\left(\int_{\lambda_R} + \int_{\gamma_R^{\pm}} \frac{z e^{ikz}}{z^2+2z+5} dz\right)\right]$$

$z^2 + 2z + 5 = 0 \Rightarrow z = \frac{-1 \pm \sqrt{1-5}}{1} =$

$= -1 \pm 2i \qquad$ If $k > 0$



$\Rightarrow \operatorname{Im}\left[2\pi i \operatorname{Res}\left(\frac{z e^{ikz}}{z^2+2z+5}, -1+2i\right)\right]$

$$= \mathrm{Im}\left[2\pi i \, \frac{(-1+2i)\, e^{-ik-2k}}{(-1+2i+1+2i)}\right]$$

$$= \mathrm{Im}\left[\frac{\pi}{2}(-1+2i)\, e^{-2k}\left(\cos(k) - i\sin(k)\right)\right]$$

$$= \frac{\pi}{2} e^{-2k}\left(\sin k + 2\cos k\right)$$

If $k<0$,

$$\mathrm{Im}\left[-2\pi i \,\mathrm{Res}\left(f_1, -1-2i\right)\right] =$$

$$= \mathrm{Im}\left[\cancel{-2}\pi \cancel{i}\, \frac{(-1-2i)}{\cancel{-4i}\; 2}\, e^{-ik+2k}\right]$$

$$= \frac{\pi}{2} e^{-2|k|}\left(\sin(k) - 2\cos k\right)$$

$$= \frac{\pi}{2} e^{-2|k|}\left(\sin(k) + 2(-1)^{\mathrm{sign}(k)} \cos k\right)$$

3) $PV\displaystyle\int_{-\infty}^{+\infty} \frac{e^{ikx}}{x}\,dx = I_k$

$k>0$: $\displaystyle\int_{\frown} \frac{e^{ikz}}{z}\,dz = I(k) + \lim_{\varepsilon\to 0}\int_{\gamma_\varepsilon^+} \frac{e^{ikz}}{z}\,dz$

$= I(k) - \cancel{2}\pi i \operatorname{Res}\left(\frac{e^{ikz}}{z}, 0\right) =$

$= I(k) - i\pi = 0 \quad \Rightarrow I(k) = i\pi$

$k<0$ $\quad I(k) + i\pi = 0 \quad I(k) = -i\pi$

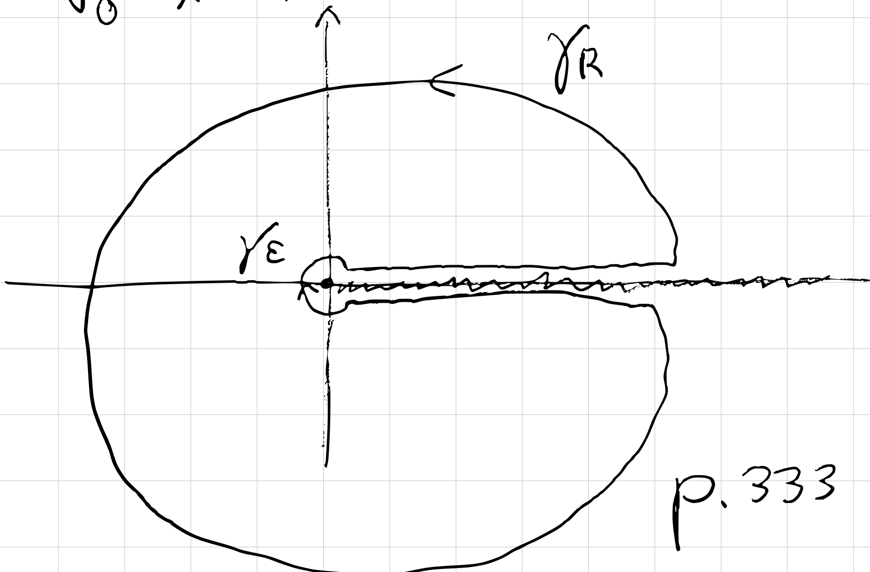
$k=0 \Rightarrow$ <u>undefined</u> !!!

$\displaystyle\lim_{R\to+\infty}\lim_{\varepsilon\to 0} \int_{\varepsilon}^{R} + \int_{-R}^{-\varepsilon} \frac{1}{x}\,dx$

$\displaystyle= \lim_{R\to+\infty}\lim_{\varepsilon\to 0} \ln R - \ln\varepsilon - (\ln\varepsilon - \ln R)$

$\displaystyle= \lim_{R\to+\infty}\lim_{\varepsilon\to 0} 2\ln\frac{R}{\varepsilon}$ undefined

4) $\displaystyle\int_0^{+\infty} \frac{\ln x}{x^2+4}\,dx = I$



p. 333

$$\frac{1}{x^2+4} = \frac{1}{4}\left(1 - \frac{x^2}{4} + \dots\right)$$

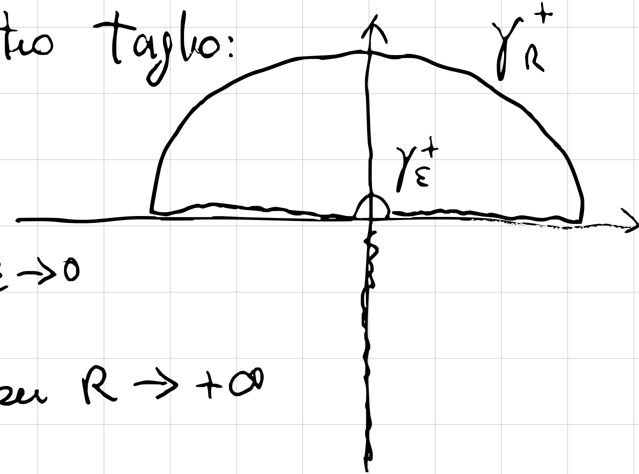
γ_ε: $\displaystyle\frac{\varepsilon\cdot(\ln\varepsilon + i\theta)}{\varepsilon^2+4} \longrightarrow 0$ as $\varepsilon \to 0$

γ_R: $\displaystyle\frac{R(\ln R + i\theta)}{R^2+4} \longrightarrow 0$ as $R \to +\infty$

$$I + \int_{\gamma_R} f(z)\,dz + \int_{\gamma_\varepsilon} f(z)\,dz +$$

$\Rightarrow$ i contributi lungo $\mathbb{R}^+$ si <u>cancellano</u> !!

Devo usare un altro taglio:



Ancora $\displaystyle\int_{\gamma_\varepsilon^+} f(z)\,dz \to 0$ per $\varepsilon \to 0$

e $\displaystyle\int_{\gamma_R^+} f(z)\,dz \to 0$ per $R \to +\infty$

Adesso però:

$$2\pi i \,\mathrm{Res}\left(\frac{\ln x}{x^2+4}, +2i\right) =$$

$$= \lim_{\substack{\varepsilon \to 0^+ \\ R \to +\infty}} \left[\int_{\gamma_\varepsilon^+ + \gamma_R^+} f(z)\,dz + \underbrace{\int_{[\varepsilon, R]} f(z)\,dz}_{\text{tende a } I} + \int_{[-R,-\varepsilon]} f(z)\,dz\right] =$$

$$= I + \int_{-\infty}^{0} \frac{\ln x}{x^2+4}\,dx = I + \int_0^{+\infty} \frac{\ln x + i\pi}{x^2+4}\,dx$$

$$\left(x = |x| e^{i\pi} \text{ su } \mathbb{R}^-\right)$$

$$= 2I + iA \quad \text{dove } A = \pi \int_0^{+\infty} \frac{1}{x^2+4}\,dx \in \mathbb{R}$$

Quindi $2\pi i \,\mathrm{Res}\left(\frac{\ln x}{x^2+4}, 2i\right) =$

$$= 2\pi i \frac{\ln 2 + i\frac{\pi}{2}}{4i} = \underbrace{\frac{\pi}{2}\ln 2 + i\frac{\pi^2}{4}}$$

$$= 2I + iA \Rightarrow I = \frac{1}{2}\mathrm{Re}(\dots) =$$

$$= \frac{\pi}{4}\ln 2$$