14/06/2023 – M. FRIGERIO

1) Si consideri:

$$\frac{d^2 f}{dt^2} - \beta \frac{df}{dt} = F(t), \text{ con } F \text{ assegnata}$$

$$\text{e } \beta \in \mathbb{C},\ \mathrm{Re}(\beta) \neq 0$$

Allora: $f(t) = (G * F)(t)$

con $G(t)$ soluzione di

$$\frac{d^2 G}{dt^2} - \beta \frac{dG}{dt} = \delta(t) \quad (\star)$$

(verificarlo).

Dimostrare che:

(i) $$\hat{G}(\omega) = -\frac{1}{2\pi}\frac{1}{\omega - i\beta} P\left(\frac{1}{\omega}\right) + A\,\delta(\omega)$$

Mostrare che A è arbitraria perché la sol. di $\star$ è definita a meno di sol. dell'eq. omogenea

Assumendo $\mathrm{Re}(\beta) > 0$, calcolare $G(t)$ come antitrasf. di $\hat{G}(\omega)$

Fissare A t.c. $G(t) = 0 \ \forall t < 0$

(funzione di Green causale)

Soluzione:

Se $f = G * F$, allora

$$\mathcal{F}\left[\frac{d^2(G*F)}{d^2t} - \beta \frac{d(G*F)}{dt} \right] =$$

$$= \sqrt{2\pi}\left(-\omega^2 \hat{G}(\omega) \cdot \hat{F}(\omega) + i\beta\omega \hat{G}(\omega)\hat{F}(\omega)\right) \quad \text{(ii)}$$

Ma siccome $\frac{d^2G}{d^2t} - \beta \frac{dG}{dt} = \delta(t)$

$$\rightarrow (-\omega^2 + i\beta\omega)\hat{G}(\omega) = 1/\sqrt{2\pi}$$

Segue che: (ii)

$$\sqrt{2\pi}\hat{F}(\omega)(-\omega^2 + i\beta\omega)\hat{G}(\omega) = \hat{F}(\omega)$$

quindi f é soluzione e:

$$(-\omega^2 + i\beta\omega)\hat{G}(\omega) = \frac{1}{\sqrt{2\pi}}$$ che é risolta da (i)

Infatti, per verificare (i), sia $\varphi \in \mathcal{D}(\mathbb{R})$:

$$\int_{-\infty}^{+\infty} -\omega(\omega - i\beta)\,\hat{G}(\omega)\,\varphi(\omega)\,d\omega = \int_{-\infty}^{+\infty} \frac{\varphi(\omega)}{\sqrt{2\pi}}\,d\omega$$

chiaram., se $\hat{G}(\omega)$ é soluz.,

$\hat{G}(\omega) + A\delta(\omega)$ é ancora soluz.

Vediamo il pezzo $-\frac{1}{\omega - i\beta} P\left(\frac{1}{\omega}\right)$

$$\int_{-\infty}^{+\infty} \omega P\left(\frac{1}{\omega}\right) \varphi(\omega)\,d\omega :=$$

$$= \lim_{\substack{\varepsilon \to 0 \\ R \to +\infty}} \int_{[-R,-\varepsilon] \cup [\varepsilon, R]} \varphi(\omega)\,d\omega = \int_{\mathbb{R}} \varphi(\omega)\,d\omega \quad \boxed{V}$$

$$G(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left(-\frac{1}{\omega - i\beta} P\left(\frac{1}{\omega}\right) + A\delta(\omega) \right) e^{i\omega t}\,d\omega =$$

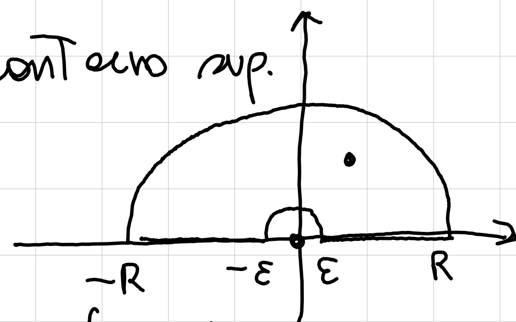
$$= \frac{A}{2\pi} + \lim_{\substack{\varepsilon \to 0^+ \\ R \to +\infty}} \left(\int_{-R}^{-\varepsilon} g(\omega) + \int_{\varepsilon}^{R} g(\omega)\,d\omega \right)$$

dove $g(\omega) = -\dfrac{1}{2\pi}\,\dfrac{e^{i\omega t}}{\omega(\omega - i\beta)}$

$\Rightarrow$ Poli in $\omega = 0$

in $\omega = i\beta \Rightarrow$ semipiano superiore per $Re(\beta) > 0$.

Se $t > 0$, chiudo sul contorno sup.



$$\lim_{\substack{\varepsilon \to 0 \\ R \to +\infty}} \left[\int_{-R}^{-\varepsilon} g(\omega)\,d\omega + \int_{\varepsilon}^{R} g(\omega)\,d\omega \right] + \underbrace{\int_{\gamma_R} g(\omega)\,d\omega}_{\to 0} +$$

$$+ \int_{\gamma_\varepsilon^+} g(\omega)\,d\omega = 2\pi i \operatorname{Res}(f; i\beta).$$

Uso teorema dei piccoli cerchi su γ_ε^+

$$\Rightarrow G(t) - \pi i \operatorname{Res}(f; 0) = \frac{2\pi i}{-2\pi}\,\frac{e^{-\beta t}}{i\beta} = -\frac{e^{-\beta t}}{\beta}$$

$$\Rightarrow G(t) + \frac{\cancel{\pi}\cancel{i}}{2\cancel{\pi}}\,\frac{1}{-\cancel{i}\beta} = -\frac{e^{-\beta t}}{\beta}$$

$$\Rightarrow G(t) = \frac{1}{\beta}\left(\frac{1}{2} - e^{-\beta t}\right) \quad \left(+\frac{A}{2\pi}\right)$$

per $t > 0$

Per $t<0$ chiudo sotto:

$$G(t) + \pi i \operatorname{Res}(f;0) = 0$$

$$\Rightarrow G(t) = -\frac{1}{2\beta} + \frac{A}{2\pi} \quad (t<0)$$

per avere $G(t) = 0 \quad \forall\, t<0$:

$$A = \frac{\pi}{\beta} \Rightarrow G(t) = \frac{\theta(t)}{\beta}\left(1 - e^{-\beta t}\right)$$

2) $$\frac{\sqrt{z-1}\,\sqrt{z}}{z+i}$$

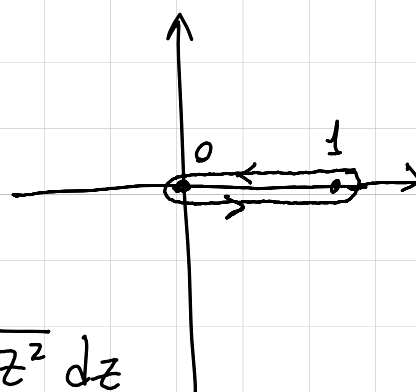
Trovare singol., branch pts. etc

Taglio in $z=0$ e $z=1$

Sing. isolata (polo semplice) in $z=-i$

Olomorfa a $+\infty$

Considerare $\int_\gamma f(z)\,dz$ con



$$\Rightarrow -\int_0^1 i\,\frac{\sqrt{z-z^2}}{z+i} \quad - i\int_0^1 \frac{\sqrt{z-z^2}}{z+i}\,dz$$

$$\mathrm{Re}\int_\gamma f(z)\,dz = -2\int_0^1 \frac{\sqrt{x-x^2}}{x^2+1}\,dx$$

$$\mathrm{Im}\int_\gamma f(z)\,dz = -2\int_0^1 \frac{x\sqrt{x-x^2}}{x^2+1}\,dx$$

$$\int_\gamma f(z)\,dz = -2\pi i\,\mathrm{Res}(f,-i) - 2\pi i\,\mathrm{Res}(f,+\infty)$$

$$\operatorname{Res}(f;-i) = \sqrt{-i}\sqrt{-i-1} = e^{-i\frac{\pi}{4}} 2^{\frac{1}{4}} e^{-\frac{3i\pi}{8}}$$

$$= 2^{1/4} e^{-\frac{5i\pi}{8}} = -2^{1/4}\left(\sin\frac{\pi}{8} + i\cos\frac{\pi}{8}\right)$$

$$\operatorname{Res}(f;+\infty) = \operatorname{Res}\left(-\frac{1}{z^2}f\left(\frac{1}{z}\right),0\right) =$$

$$\neq -\frac{1}{z^2}f\left(\frac{1}{z}\right) = \frac{-z}{iz+1}\,\frac{1}{z^2}\sqrt{\frac{1-z}{z^2}} =$$

$$= -\frac{\sqrt{1-z}}{z^2(iz+1)} = -\frac{\left(1-\frac{z}{2}+O(z^2)\right)\left(1-iz+O(z^2)\right)}{z^2}$$

$$\Rightarrow \operatorname{Res}(f,+\infty) = \frac{1}{2}+i$$

$$I_1 = \pi \operatorname{Re}\left(-i2^{1/4}\left(\sin\frac{\pi}{8}+i\cos\frac{\pi}{8}\right)+\frac{1}{2}+i\right)$$

$$= \pi\left(\frac{1}{2}+2^{1/4}\cos\frac{\pi}{8}\right)$$

$$I_2 = \pi\left(1-2^{1/4}\sin\left(\frac{\pi}{8}\right)\right)$$